Ανισότητες ΙΙΙ

Andreescu   Nesbitt

Βαγγέλης Ψύχας

Ανισότητα Bergtröm ή Titu Andreescu

Εφαρμογή B-C-S ✧ $$\left(x_1^2+x_2^2+\cdots+x_\nu^2\right)\left(y_1^2+y_2^2+\cdots+y_\nu^2\right)\geq$$

$$\geq\left(x_1y_1+x_2y_2+\cdots+x_\nu y_\nu\right)^2$$

Αν $x_i=\dfrac{\alpha_i}{\sqrt{\beta_i}}$ και $y_i=\sqrt{\beta_i}\quad \beta_i>0\ldots$

✧ $$\left(\left(\frac{\alpha_1}{\sqrt{\beta_1}}\right)^2+\left(\frac{\alpha_2}{\sqrt{\beta_2}}\right)^2+\cdots+\left(\frac{\alpha_\nu}{\sqrt{\beta_\nu}}\right)^2\right)\left(\left(\sqrt{\beta_1}\right)^2+\left(\sqrt{\beta_2}\right)^2+\cdots+\left(\sqrt{\beta_\nu}\right)^2\right)\geq$$

$$\geq\left(\frac{\alpha_1}{\sqrt{\beta_1}}\cdot\sqrt{\beta_1}+\frac{\alpha_2}{\sqrt{\beta_2}}\cdot\sqrt{\beta_2}+\cdots+\frac{\alpha_\nu}{\sqrt{\beta_\nu}}\cdot\sqrt{\beta_\nu}\right)^2.$$

Ανισότητα Bergtröm ή Titu Andreescu

◊ $(x_1^2 + x_2^2 + \cdots + x_\nu^2)(y_1^2 + y_2^2 + \cdots + y_\nu^2) \geq (x_1 y_1 + x_2 y_2 + \cdots + x_\nu y_\nu)^2$

Αν $x_i = \dfrac{\alpha_i}{\sqrt{\beta_i}}$ και $y_i = \sqrt{\beta_i}$ $\quad \beta_i > 0$ ...

Εφαρμογή
B-C-S

◊ $$\left(\left(\frac{\alpha_1}{\sqrt{\beta_1}}\right)^2 + \left(\frac{\alpha_2}{\sqrt{\beta_2}}\right)^2 + \cdots + \left(\frac{\alpha_\nu}{\sqrt{\beta_\nu}}\right)^2\right)\left(\left(\sqrt{\beta_1}\right)^2 + \left(\sqrt{\beta_2}\right)^2 + \cdots + \left(\sqrt{\beta_\nu}\right)^2\right) \geq$$

$$\geq \left(\frac{\alpha_1}{\sqrt{\beta_1}} \cdot \sqrt{\beta_1} + \frac{\alpha_2}{\sqrt{\beta_2}} \cdot \sqrt{\beta_2} + \cdots + \frac{\alpha_\nu}{\sqrt{\beta_\nu}} \cdot \sqrt{\beta_\nu}\right)^2.$$

◊ $$\frac{\alpha_1^2}{\beta_1} + \frac{\alpha_2^2}{\beta_2} + \cdots + \frac{\alpha_\nu^2}{\beta_\nu} \geq \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_\nu)^2}{\beta_1 + \beta_2 + \cdots + \beta_\nu}$$

Ανισότητα Nesbitt

✧ Αν $a, b, c > 0$, να αποδείξετε ότι:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

✧ $$\frac{\alpha_1^2}{\beta_1} + \frac{\alpha_2^2}{\beta_2} + \frac{\alpha_3^2}{\beta_3} \geq \frac{(\alpha_1 + \alpha_2 + \alpha_3)^2}{\beta_1 + \beta_2 + \beta_3} \quad (\text{I})$$

✧ $$(\alpha + \beta + \gamma)^2 \geq 3(\alpha\beta + \beta\gamma + \gamma\alpha) \quad (\text{II})$$

$$\diamond\ \frac{\alpha_1^2}{\beta_1}+\frac{\alpha_2^2}{\beta_2}+\frac{\alpha_3^2}{\beta_3}\geq\frac{(\alpha_1+\alpha_2+\alpha_3)^2}{\beta_1+\beta_2+\beta_3}\quad \text{(I)}$$

$$\diamond\ (\alpha+\beta+\gamma)^2\geq 3(\alpha\beta+\beta\gamma+\gamma\alpha)\quad \text{(II)}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a^2}{a(b+c)}+\frac{b^2}{b(c+a)}+\frac{c^2}{c(a+b)}$$

$$\frac{a^2}{a(b+c)}+\frac{b^2}{b(c+a)}+\frac{c^2}{c(a+b)}\overset{(I)}{\geq}\frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)}$$

$$\frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)}\overset{(II)}{\geq}\frac{3(ab+bc+ca)}{2(ab+bc+ca)}=\frac{3}{2}$$

♢ Αν $a, b, c > 0$, να αποδείξετε ότι:

$$\frac{a}{a+2c}+\frac{b}{b+2a}+\frac{c}{c+2b}\geq 1.$$

♢ $\dfrac{a}{a+2c}=\dfrac{a^2}{a^2+2ac}$ $\quad(x_1=a, y_1=a^2+2ac)$

♢ $\dfrac{b}{b+2a}=\dfrac{b^2}{b^2+2ab}$ $\quad(x_2=b, y_2=b^2+2ab)$

♢ $\dfrac{c}{c+2b}=\dfrac{c^2}{c^2+2bc}$ $\quad(x_3=c, y_3=c^2+2bc)$

♢ $$\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+\frac{x_3^2}{y_3}\geq\frac{(x_1+x_2+x_3)^2}{y_1+y_2+y_3}$$

Εφαρμογή B-C-S

Με τη βοήθεια των αντικαταστάσεων:

◊ $(x_1 = a, y_1 = a^2 + 2ac)$

◊ $(x_2 = b, y_2 = b^2 + 2ab)$

◊ $(x_3 = c, y_3 = c^2 + 2bc)$

στην ανισότητα

◊ $$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} \geq \frac{(x_1 + x_2 + x_3)^2}{y_1 + y_2 + y_3}$$

έχουμε:

◊ $$\frac{a^2}{a^2 + 2ac} + \frac{b^2}{b^2 + 2ab} + \frac{c^2}{c^2 + 2bc} \geq \frac{(a + b + c)^2}{a^2 + 2ac + b^2 + 2ab + c^2 + 2bc} = 1.$$

$$\frac{a}{a + 2c} + \frac{b}{b + 2a} + \frac{c}{c + 2b} \geq 1.$$

♢ Αν $a, b, c > 0$, να αποδείξετε ότι:

$$\frac{a}{a+2b+3c}+\frac{b}{b+2c+3a}+\frac{c}{c+2a+3b}\geq\frac{1}{2}.$$

♢ $\dfrac{a}{a+2b+3c}=\dfrac{a^2}{a^2+2ab+3ac}$ $\quad(x_1=a, y_1=a^2+2ab+3ac)$

♢ $\dfrac{b}{b+2c+3a}=\dfrac{b^2}{b^2+2bc+3ab}$ $\quad(x_2=b, y_2=b^2+2bc+3ab)$

♢ $\dfrac{c}{c+2a+3b}=\dfrac{b^2}{c^2+2ac+3bc}$ $\quad(x_3=c, y_3=c^2+2ac+3bc)$

♢ $\dfrac{x_1^2}{y_1}+\dfrac{x_2^2}{y_2}+\dfrac{x_3^2}{y_3}\geq\dfrac{(x_1+x_2+x_3)^2}{y_1+y_2+y_3}$

Με τη βοήθεια των αντικαταστάσεων:

- $\left(x_1 = a, y_1 = a^2 + 2ab + 3ac\right)$
- $\left(x_2 = b, y_2 = b^2 + 2bc + 3ab\right)$
- $\left(x_3 = c, y_3 = c^2 + 2ac + 3bc\right)$

στην ανισότητα

- $$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} \geq \frac{(x_1 + x_2 + x_3)^2}{y_1 + y_2 + y_3}$$

έχουμε:

- $$\frac{a^2}{a^2 + 2ab + 3ac} + \frac{b^2}{b^2 + 2bc + 3ab} + \frac{c^2}{c^2 + 2ac + 3bc} \geq \frac{(a + b + c)^2}{a^2 + b^2 + c^2 + 5ab + 5bc + 5ac}.$$

Αρκεί να αποδείξουμε ότι:

Αρκεί να αποδείξουμε ότι :

$$\frac{(a+b+c)^2}{a^2+b^2+c^2+5ab+5bc+5ac} \geq \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ac.$$

♦ Αν $a > 0$, να αποδείξετε ότι:

$$\frac{(2a+1)(2a+3)(2a+5)(2a+7)}{a(a+1)(a+2)(a+3)} \geq 16\sqrt{\frac{a+4}{a}}.$$

$x + y \geq 2\sqrt{xy}$

$x, y \geq 0$

- $\frac{2a+1}{a} = 1 + \frac{a+1}{a} \geq 2\sqrt{\frac{a+1}{a}}$
- $\frac{2a+3}{a+1} = 1 + \frac{a+2}{a+1} \geq 2\sqrt{\frac{a+2}{a+1}}$
- $\frac{2a+5}{a+2} = 1 + \frac{a+3}{a+2} \geq 2\sqrt{\frac{a+3}{a+2}}$
- $\frac{2a+7}{a+3} = 1 + \frac{a+4}{a+3} \geq 2\sqrt{\frac{a+4}{a+3}}$

Πολλαπλασιάζουμε κατά μέλη τις παραπάνω ανισότητες.

◆ Αν $a, b, c > 0$, να αποδείξετε ότι:

$$\left(a+\frac{1}{b}\right)+\left(b+\frac{1}{c}\right)+\left(c+\frac{1}{a}\right)\geq 8.$$

$x + y \geq 2\sqrt{xy}$

$x, y \geq 0$

$$\left.\begin{array}{l} a+\dfrac{1}{b}\geq 2\sqrt{a\dfrac{1}{b}}=2\sqrt{\dfrac{a}{b}} \\ b+\dfrac{1}{c}\geq 2\sqrt{b\dfrac{1}{c}}=2\sqrt{\dfrac{b}{c}} \\ c+\dfrac{1}{a}\geq 2\sqrt{c\dfrac{1}{a}}=2\sqrt{\dfrac{c}{a}} \end{array}\right\} \left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)\geq 2\sqrt{\frac{a}{b}}\cdot 2\sqrt{\frac{b}{c}}\cdot 2\sqrt{\frac{c}{a}}=8$$

♦ Αν $a, b, c > 0$, να αποδείξετε ότι:

$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab}\geq 1.$$

● $x^2 + y^2 \geq 2xy$

$$b^2+c^2\geq 2bc \Leftrightarrow a^2+b^2+c^2\geq a^2+2bc \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{a^2+2bc} \geq \frac{1}{a^2+b^2+c^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{a^2}{a^2+2bc} \geq \frac{a^2}{a^2+b^2+c^2}$$

$$\cdots \Leftrightarrow \frac{b^2}{b^2+2ca} \geq \frac{b^2}{a^2+b^2+c^2}$$

$$\cdots \Leftrightarrow \frac{c^2}{c^2+2ab} \geq \frac{c^2}{a^2+b^2+c^2}$$

$$\overset{(+)}{\Longrightarrow} \frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab}\geq 1.$$