Ανισότητες ΙΙ

Cauchy

Buniakowsky-Cauchy-Schwarz

Βαγγέλης Ψύχας

Τετραγωνικός Μέσος *ΤΜ*

Quadratic Mean

- $\sqrt{\dfrac{\alpha^2+\beta^2}{2}}$

- $\sqrt{\dfrac{\alpha^2+\beta^2+\gamma^2}{3}}$

- $\sqrt{\dfrac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_\nu^2}{\nu}}$

Αριθμητικός Μέσος *AM*

Arithmetic Mean

- $\dfrac{\alpha+\beta}{2}$

- $\dfrac{\alpha+\beta+\gamma}{3}$

- $\dfrac{\alpha_1+\alpha_2+\cdots+\alpha_\nu}{\nu}$

Γεωμετρικός Μέσος *ΓΜ*

Geometric Mean

- $\sqrt{\alpha \cdot \beta}$
- $\sqrt[3]{\alpha \cdot \beta \cdot \gamma}$
- $\sqrt[\nu]{\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_\nu}$

Αρμονικός Μέσος *HM*

Harmonic **M**ean

- $$\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$

- $$\frac{3}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}$$

- $$\frac{\nu}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_\nu}}$$

Ανισότητα Cauchy

- Αριθμητικού-Γεωμετρικού-Αρμονικού Μέσου

- $$\frac{\alpha+\beta+\gamma}{3} \geq \sqrt[3]{\alpha\cdot\beta\cdot\gamma} \geq \frac{3}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}} \qquad \alpha,\beta,\gamma > 0$$

- $\alpha+\beta+\gamma \geq 3\sqrt[3]{\alpha\beta\gamma}$

- $\alpha^3+\beta^3+\gamma^3 \geq 3\alpha\beta\gamma$

- $$\frac{\alpha_1+\alpha_2+\cdots+\alpha_\nu}{\nu} \geq \sqrt[\nu]{\alpha_1\cdot\alpha_2\cdot\cdots\cdot\alpha_\nu} \geq \frac{\nu}{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\cdots+\frac{1}{\alpha_\nu}}$$

$$\alpha_1,\alpha_2,\cdots,\alpha_\nu > 0$$

Ανισότητα B-C-S

Buniakowsky-Cauchy-Schwarz

- $(\alpha^2+\beta^2)(x^2+y^2) \geq (\alpha x+\beta y)^2$
- $\alpha, \beta, x, y \in \mathbb{R}$
- Η ισότητα ισχύει όταν: $\alpha = \lambda x, \beta = \lambda y \quad \lambda \in \mathbb{R}$.

Ανισότητα B-C-S

Εναλλακτική διατύπωση

- $\sqrt{\alpha^2+\beta^2}\sqrt{x^2+y^2} \geq \alpha x + \beta y$
- $\alpha, \beta, x, y \in \mathbb{R}$
- Η ισότητα ισχύει όταν: $\alpha = \lambda x$ και $\beta = \lambda y \quad \lambda \in \mathbb{R}$.

Ανισότητα B-C-S

Απόδειξη Ι

- $$(\alpha^2+\beta^2)(x^2+y^2)\geq(\alpha x+\beta y)^2$$

Από την ταυτότητα του Lagrange έχουμε:

$$(\alpha^2+\beta^2)(x^2+y^2)-(\alpha x+\beta y)^2=\begin{vmatrix}\alpha & \beta\\ x & y\end{vmatrix}^2.$$

Επειδή όμως $\begin{vmatrix}\alpha & \beta\\ x & y\end{vmatrix}^2=(\alpha y-\beta x)^2\geq 0$, έχουμε:

$$(\alpha^2+\beta^2)(x^2+y^2)-(\alpha x+\beta y)^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow(\alpha^2+\beta^2)(x^2+y^2)\geq(\alpha x+\beta y)^2.$$

Ανισότητα B-C-S

Απόδειξη ΙΙ

- $(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2)\ge(\alpha_1\beta_1+\alpha_2\beta_2)^2$

Θα αποδείξουμε ότι: $(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2)\ge(\alpha_1\beta_1+\alpha_2\beta_2)^2$

$\cdots \Leftrightarrow \alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2+\alpha_2^2\beta_2^2\ge\alpha_1^2\beta_1^2+2\alpha_1\beta_1\alpha_2\beta_2+\alpha_2^2\beta_2^2 \Leftrightarrow$

$\Leftrightarrow \alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2\ge 2\alpha_1\beta_1\alpha_2\beta_2 \Leftrightarrow (\alpha_1\beta_2-\alpha_2\beta_1)^2\ge 0$ πού ισχύει.

Ανισότητα B-C-S

Απόδειξη ΙΙΙ

Θεωρούμε τα διανύσματα $\vec{\alpha} = (\alpha_1, \alpha_2)$ και $\vec{\beta} = (\beta_1, \beta_2)$.

Τότε: $|\vec{\alpha}| = \sqrt{\alpha_1^2 + \alpha_2^2}$, $|\vec{\beta}| = \sqrt{\beta_1^2 + \beta_2^2}$

$$\vec{\alpha} \cdot \vec{\beta} = |\vec{\alpha}| \cdot |\vec{\beta}| \sigma\upsilon\nu\left(\widehat{\vec{\alpha}, \vec{\beta}}\right) = \alpha_1\beta_1 + \alpha_2\beta_2.$$

Από τη γνωστή ανισότητα

$|\vec{\alpha} \cdot \vec{\beta}| = \left| |\vec{\alpha}| \cdot |\vec{\beta}| \sigma\upsilon\nu\left(\widehat{\vec{\alpha}, \vec{\beta}}\right) \right| \leq |\vec{\alpha}| \cdot |\vec{\beta}|$ έχουμε

$$|\alpha_1\beta_1 + \alpha_2\beta_2| \leq \sqrt{\alpha_1^2 + \alpha_2^2} \cdot \sqrt{\beta_1^2 + \beta_2^2} \Rightarrow$$

$$\Rightarrow \left(\alpha_1^2 + \alpha_2^2\right)\left(\beta_1^2 + \beta_2^2\right) \geq (\alpha_1\beta_1 + \alpha_2\beta_2)^2.$$

Ανισότητα B-C-S

Buniakowsky-Cauchy-Schwarz

Γενίκευση I

- $(\alpha^2+\beta^2+\gamma^2)(x^2+y^2+z^2) \geq (\alpha x+\beta y+\gamma z)^2$

- $\alpha, \beta, \gamma, x, y, z \in \mathbb{R}$

- Η ισότητα ισχύει όταν: $\alpha = \lambda x, \beta = \lambda y$ $\kappa\alpha\iota$ $\gamma = \lambda z \quad \lambda \in \mathbb{R}$.

Ανισότητα B-C-S

Γενίκευση Ι

Απόδειξη Ι

Από την ταυτότητα του Lagrange έχουμε:

$$(\alpha^2+\beta^2+\gamma^2)(x^2+y^2+z^2)-(\alpha x+\beta y+\gamma z)^2=$$

$$=\begin{vmatrix}\alpha & \beta\\ x & y\end{vmatrix}^2+\begin{vmatrix}\alpha & \gamma\\ x & z\end{vmatrix}^2+\begin{vmatrix}\beta & \gamma\\ x & z\end{vmatrix}^2.$$

Επειδή όμως $\begin{vmatrix}\alpha & \beta\\ x & y\end{vmatrix}^2+\begin{vmatrix}\alpha & \gamma\\ x & z\end{vmatrix}^2+\begin{vmatrix}\beta & \gamma\\ x & z\end{vmatrix}^2 \geq 0$**, έχουμε:**

$$(\alpha^2+\beta^2+\gamma^2)(x^2+y^2+z^2)\geq(\alpha x+\beta y+\gamma z)^2.$$

Ανισότητα B-C-S

Γενίκευση I

Απόδειξη II

Θα αποδείξουμε ότι:

$$\left(\alpha_1^2+\alpha_2^2+\alpha_3^2\right)\left(\beta_1^2+\beta_2^2+\beta_3^2\right)\geq\left(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3\right)^2$$

$$\cdots\Leftrightarrow \alpha_1^2\beta_1^2+\alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_2^2\beta_1^2+\alpha_2^2\beta_2^2+\alpha_2^2\beta_3^2+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2\geq$$

$$\geq \alpha_1^2\beta_1^2+\alpha_2^2\beta_2^2+\alpha_3^2\beta_3^2+2(\alpha_1\beta_1\alpha_2\beta_2+\alpha_1\beta_1\alpha_3\beta_3+\alpha_2\beta_2\alpha_3\beta_3)\Leftrightarrow$$

$$\Leftrightarrow \alpha_1^2\beta_2^2+\alpha_1^2\beta_3^2+\alpha_2^2\beta_1^2+\alpha_2^2\beta_3^2+\alpha_3^2\beta_1^2+\alpha_3^2\beta_2^2\geq$$

$$\geq 2(\alpha_1\beta_1\alpha_2\beta_2+\alpha_1\beta_1\alpha_3\beta_3+\alpha_2\beta_2\alpha_3\beta_3)\Leftrightarrow$$

$\Leftrightarrow(\alpha_1\beta_2-\alpha_2\beta_1)^2+(\alpha_1\beta_3-\alpha_3\beta_1)^2+(\alpha_2\beta_3-\alpha_3\beta_2)^2\geq 0$ πού ισχύει.

Ανισότητα B-C-S

Απόδειξη ΙΙΙ

Γενίκευση Ι

Θεωρούμε τα διανύσματα $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ και $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$.

Τότε: $|\vec{\alpha}| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, $\quad |\vec{\beta}| = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$

$$\vec{\alpha} \cdot \vec{\beta} = |\vec{\alpha}| \cdot |\vec{\beta}| \sigma\upsilon\nu\left(\widehat{\vec{\alpha}, \vec{\beta}}\right) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3.$$

Από τη γνωστή ανισότητα

$|\vec{\alpha} \cdot \vec{\beta}| = \left| |\vec{\alpha}| \cdot |\vec{\beta}| \sigma\upsilon\nu\left(\widehat{\vec{\alpha}, \vec{\beta}}\right) \right| \leq |\vec{\alpha}| \cdot |\vec{\beta}|$ έχουμε

$$|\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3| \leq \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} \cdot \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \Rightarrow$$

$$\Rightarrow (\alpha_1^2 + \alpha_2^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2 + \beta_3^2) \geq (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)^2.$$

Ανισότητα B-C-S

Buniakowsky-Cauchy-Schwarz

Γενίκευση ΙΙ

- $(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_\nu^2)(\beta_1^2 + \beta_2^2 + \cdots + \beta_\nu^2) \geq$

$$\geq (\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_\nu\beta_\nu)^2$$

- Η ισότητα ισχύει όταν: $\alpha_i = \lambda\beta_i \quad i = 1, 2, \cdots \nu \quad \lambda \in \mathbb{R}$.

Ανισότητα B-C-S

Γενίκευση ΙΙ

Απόδειξη

$$\left(\alpha_1^2+\alpha_2^2+\cdots+\alpha_\nu^2\right)\left(\beta_1^2+\beta_2^2+\cdots+\beta_\nu^2\right)-(\alpha_1\beta_1+\alpha_2\beta_2+\cdots+\alpha_\nu\beta_\nu)^2=$$

$$=\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}^2+\cdots+\begin{vmatrix}\alpha_1 & \alpha_\nu\\ \beta_1 & \beta_\nu\end{vmatrix}^2+\begin{vmatrix}\alpha_2 & \alpha_3\\ \beta_2 & \beta_3\end{vmatrix}^2+\cdots+\begin{vmatrix}\alpha_2 & \alpha_\nu\\ \beta_2 & \beta_\nu\end{vmatrix}^2+\cdots\cdots+\begin{vmatrix}\alpha_{\nu-1} & \alpha_\nu\\ \beta_{\nu-1} & \beta_\nu\end{vmatrix}^2=$$

$$=\sum_{1\le i<j\le\nu}\begin{vmatrix}\alpha_i & \alpha_j\\ \beta_i & \beta_j\end{vmatrix}^2\ge 0.$$

Ανισότητα B-C-S

Γενίκευση ΙΙΙ

$$\left(f(x)-\lambda g(x)\right)^2 \geq 0 \Leftrightarrow f^2(x)-2\lambda f(x)g(x)+\lambda^2 g^2(x) \geq 0.$$

$$\int_a^b f^2(x)dx - 2\lambda \int_a^b f(x)g(x)dx + \lambda^2 \int_a^b g^2(x)dx \geq 0.$$

Πρέπει $\Delta = 4\left(\int_a^b f(x)g(x)dx\right)^2 - 4\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx \leq 0.$

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$$