

Physics 200a PSIV SOLUTIONS

1. (i) Using the dot product show that if $\mathbf{A} + \mathbf{B}$ is perpendicular to $\mathbf{A} - \mathbf{B}$, then $A = B$. (ii) Use the dot product to relate vector components (A_x, A_y) in the standard frame to (A'_x, A'_y) measured in a frame rotated counterclockwise by an angle θ . Use the fact that components of a vector are just dot products with the corresponding basis vectors: $A_x = \mathbf{A} \cdot \mathbf{i}$ etc.

(i)

$$(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{B} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} \quad (1)$$

$$= A^2 - B^2 \quad (2)$$

$$\text{since } \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad (3)$$

So, in order for the dot product to be 0, A^2 must equal B^2 and $A = B$.

(ii)

$$A'_x = \mathbf{A} \cdot \mathbf{i}' = \mathbf{A} \cdot (\mathbf{i} \cos \theta + \mathbf{j} \sin \theta) = A_x \cos \theta + A_y \sin \theta \quad (4)$$

and so on.

2. If $F(x) = Ax^{3/2}$ find the work done by it in moving a body from $x = 0$ to $x = A$.

Work done

$$W = \int_0^A Ax^{3/2} dx = \left. \frac{Ax^{5/2}}{5/2} \right|_0^A = \frac{2}{5} A^{7/2}. \quad (5)$$

3. You have to choose between a 225-W fridge that costs \$ 1150 and a 425-W model that costs \$ 850. The first runs 11% of the time and the second runs 20% of the time. If electricity costs 9.5 cents/kWh when will you break even? (Use hours as the unit of time at first, go to days to report final answer.)

Let us run these machines for t hours. Equating the cost of cheap machine

$$C_c = \$850 + (.425 \text{ t kWh})(.2)(.095\$/kWh) \quad (6)$$

to running the fancy one

$$C_f = \$1150 + (.225 \text{ t kWh})(.11)(.095\$/kWh) \quad (7)$$

we get t in hours. We convert to days and find 2180 days or about 6 years.

4. See the masses in Figure (1) which start out at rest. (i) Find the velocity of the 14 kg mass just before it hits the ground. (ii) Find the maximum height reached by the 8 kg (and don't worry about hitting the pulley). (iii) Find the fraction of mechanical energy left when the system finally comes to rest.

Initially we have the 14 kg with a PE of $mgh = 14 * 9.8 * 5$. Just before it hits the ground, it has a speed v , the other mass has a height $5m$ and the same speed. Thus

$$14 * 9.8 * 5 = 8 * 9.8 * 5 + \frac{1}{2} v^2 (14 + 8) \quad (8)$$

which yields $v = 5.17 m/s$. (ii) The 8kg mass keeps going upwards till it comes to rest. The extra height it gets (above the 5m) is $h = v^2/2g = 1.36m$.

(iii) At the point we reach in part (ii) we have no energy of any kind for the 14kg mass. The energy of the 8kg mass is best measured when it is at rest at highest point of 6.36m. Initially the only energy was the potential energy of the 14kg mass at height 5m. Thus

$$\frac{E_{final}}{E_{initial}} = \frac{8 * 6.36 * 9.8}{14 * 5 * 9.8} = .727 \quad (9)$$

so that roughly 27% of energy is lost.

The system will come entirely to rest after the 8 kg mass bounces around a little bit more and then comes to rest 5 m above the ground. The ratio of these energies is

$$\frac{E_{final}}{E_{initial}} = \frac{8 * 5 * 9.8}{14 * 5 * 9.8} = \frac{8}{14} = 0.57$$

so 43% of the initial energy is lost.

(Note: this question was not worded well, so we will accept either answer.)

5. A loop-the-loop car of mass m comes down from a height h . (i) What is the minimum h (in terms of the loop radius R) that will ensure a safe trip? (ii) What will be the speed of the car at the lowest part of the circle? (iii) What will be the force exerted by the track on the car at this point? (iv) If the track after the loop is horizontal and has a coefficient of friction μ_k how far will the car travel before stopping? (v) If instead of using gravity, a spring of force constant k gives it the initial speed by how much should it be compressed? Get the actual numbers for part (v) if $m = 840\text{ kg}$, $R = 6.2\text{ m}$, $k = 31\text{ kN/m}$.

(i) At the top we have seen that we need $v^2 = Rg$ and a total energy of $E = mg*2R + \frac{1}{2}m*Rg = \frac{5}{2}mgR$. The ride must then begin at a height $\frac{5}{2}R$.

(ii) At the lowest point we have all the energy in kinetic form: $\frac{1}{2}mv^2 = \frac{5}{2}mgR$ so that $v^2 = 5gR$.

(iii) At the lowest point, the car has an acceleration $v^2/R = 5g$ towards the center of the loop, which is straight up. This is produced by a track force T acting up and mg acting down. Using $F = ma$ we get $T - mg = m(5g)$ or $T = 6mg$.

(iv) It will lose all its energy to friction:

$$\frac{5}{2}mgR = \mu_k(mg)d \tag{10}$$

which gives $d = \frac{5R}{2\mu_k}$.

(v) We equate the total energy to spring energy: $\frac{5}{2}mgR = \frac{1}{2}kA^2$ to get $A = \sqrt{\frac{5mgR}{k}} = 2.87\text{ m}$.

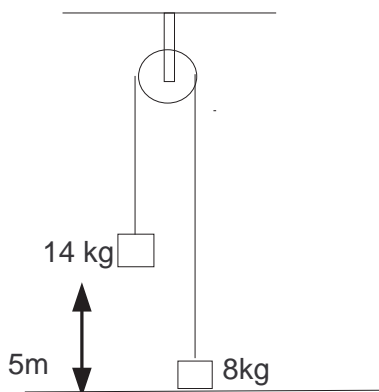


FIG. 1: The pulley is massless. Initially and finally all masses are at rest.

6. A block of mass m comes down an inclined plane at angle at $\pi/6$ from a height h after being given a slight tap to get it going. If the coefficient of friction is μ_k what is the velocity when it reaches ground? Find this using Newton's Law as well as energy principles.

If h is the height, the hypotenuse is $h/(\sin(\pi/6))$. This is the distance over which friction does its dirty work. The energy principle says

$$E_2 - E_1 = \frac{1}{2}mv^2 - mgh = \text{work by friction} = -\mu_k mg \cos(\pi/6) * \frac{h}{\sin(\pi/6)} \quad (11)$$

which gives

$$v^2 = 2g(h - \mu_k h \cot(\pi/6)). \quad (12)$$

On the other hand the downhill force on the block is $F = mg \sin(\pi/6) - \mu_k mg \cos(\pi/6)$. (The first term is the component of gravity acting downhill and the second is the frictional force uphill.) If we take the corresponding acceleration and apply to it $v^2 = 0^2 + 2ad$ where $d = h/\sin(\pi/6)$, we get the same result.

7. A mass m falls from a height h on to a spring of force constant k . Show that the maximum spring compression is $(mg/k)(1 + \sqrt{1 + 2kh/mg})$.

Let $x = 0$ be where the top of the spring is before anything lands on it. If the spring compresses by x , before the comes temporarily to rest, the mass falls a total height $h + x$ and the spring has energy $\frac{1}{2}kx^2$. Solving

$$mg(h + x) = \frac{1}{2}kx^2$$

we get $x = (mg/k)(1 \pm \sqrt{1 + 2kh/mg})$ The positive root is what we want since x is positive. The other root gives a negative x and stands for the fact that if the mass remains attached to the spring, then it will bounce upwards and once again come to rest above the origin.

8. Find the work done by a force $\mathbf{F} = ix^2y^3 + jy^3x^2$ on going from from $(0,0)$ to $(1,1)$ along three paths: (i) first along x and then along y (ii) at 45° (iii) along the curve $y = x^2$. (iv) Show that this is a conservative force and find the potential energy $U(x,y)$. Find the work done earlier in terms of difference in U . (v) Repeat parts (i) and (ii) for $\mathbf{F} = ixy^3 + jxy$

(i) When we go along x , $d\mathbf{r} = i dx$, $y = 0$, and

$$\int_{x=0}^{x=1} \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{x=1} F_x dx = \int_{x=0}^{x=1} x^2 0^3 dx = 0 \quad (13)$$

In the vertical part $d\mathbf{r} = j dy$ and $\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_y dy = \int_0^1 1^3 y^2 dy = 1/3$. Thus the work done in the full trip is $1/3J$.

(ii) The equation for the 45° line is $y = x$ so that $dy = 1 \cdot dx$ and we can replace any y by x so that

$$W = \int_{(0,0)}^{(1,1)} (F_x dx + F_y dy) = \int_{x=0}^{x=1} (F_x dx + F_y \left(\frac{dy}{dx}\right) dx) = \int_0^1 (x^2 * x^3 + x^3 * x^2) dx = \left. \frac{x^6}{3} \right|_0^1 = 1/3. \quad (14)$$

(iii) Now we must note that we may set $y = x^2$ on this curve and also use $dy = 2x dx$ to get

$$W = \int_0^1 (x^2(x^2)^3 + x^3(x^2)^2 2x) dx = 1/3. \quad (15)$$

This is a conservative force since

$$\frac{\partial F_x}{\partial y} = 3x^2 y^2 = \frac{\partial F_y}{\partial x} \quad (16)$$

By inspection of $F_x = x^2 y^3 = -\frac{\partial U}{\partial x}$ it follows $U(x, y) = -\frac{x^3 y^3}{3} + g(y)$ where g is any function of y . If we now look at $F_y = x^3 y^2 = -\frac{\partial U}{\partial y}$ we see that $g = \text{constant}$ and $U(x, y) = -\frac{x^3 y^3}{3}$ where the constant has been dropped. The work done is $U(0, 0) - U(1, 1) = \frac{1}{3}$.

(v) For a repeat of part (i) we now have $W = \int_0^1 1 \cdot y dy = \frac{1}{2}$. For a repeat of part (ii), on the line $y = x$ we have $W = \int_0^1 (x * x^3 + x * x) dx = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$. This is not a conservative force and one can check that $\frac{\partial F_x}{\partial y} \neq \frac{\partial F_y}{\partial x}$.

9. Show that the acceleration due to gravity at a height h above the earth can be approximated (for small h) by $g_h = (1 - 2h/R_E)g_0$ where g_0 is just g and R_E is the radius of the earth. What is g_h on top of Mount Everest? Hint: Use the very useful result $(1 + x)^n \simeq 1 + nx$ for $x \ll 1$.

Begin with the definition $g(r) = GM/r^2$ (Force of gravity divided by mass of object) and proceed as follows:

$$g(r) = \frac{GM}{(R_E + h)^2} = \frac{GM}{R_E^2(1 + h/R_E)^2} = \frac{GM}{R_E^2}(1 + h/R_E)^{-2} = g_0(1 - 2h/R_E)$$

where we have used $(1 + x)^n = 1 + nx + \dots$ with $x = h/R_E$ and $n = -2$.

For Mount Everest, $h = 8.85 \text{ km}$. $g(8.85 \text{ km}) = 9.77 \text{ m/s}^2$.

10. Find the period of the circular orbit of our sun around the center of our Galaxy (take as point mass $4 \cdot 10^{41} \text{ kg}$) at a radius $3 \cdot 10^4$ light years.

Start with $T = 2\pi\sqrt{R^3/GM}$ and plug in the numbers to get $T = 6 \cdot 10^{15} \text{ s} = 200 \text{ Million years}$.

11. A meteorite 80,000 km from the earth is moving towards the earth at 2000 m/s. Ignoring air friction what will be its velocity on impact?

Conservation of energy says

$$-\frac{GMm}{R} + \frac{1}{2}m(v_0)^2 = -\frac{GMm}{R_E} + \frac{1}{2}mv^2$$

where M is the mass of the earth and R_E is its radius, the initial separation is $R = 80,000 * 10^3 \text{ m}$, and initial velocity is $v_0 = 2000 \text{ m/s}$ from which we get $v = 1.09 * 10^4 \text{ m/s}$.

12. Calculate the radius of the geosynchronous orbit about the earth.

From $T = 2\pi\sqrt{\frac{R^3}{GM}}$ we get upon setting $T = 24 \text{ hrs}$ $R = 42,200 \text{ km}$.

13. A rocket is launched at escape velocity from the surface of the earth (radius R_E). What is its velocity when it is at a distance r from the center of the earth in terms of G and M_E ?

At escape velocity the total energy is zero. When it is a distance r from the earth we get $0 = -\frac{GM_E m}{r} + \frac{1}{2}mv^2$ so that $v = \sqrt{2GM_E/r}$. (If you set $r = R_E$ here you get the escape velocity.)

14. A satellite is in an elliptical orbit around the earth with altitudes ranging from 230 to 890 km. At the high point it is moving at 7.23 km/s. What is its speed at the low point?

The law of conservation of energy says that at any two points on the orbit labelled 1 and 2,

$$-\frac{GMm}{r_1} + \frac{1}{2}mv_1^2 = -\frac{GMm}{r_2} + \frac{1}{2}mv_2^2. \quad (17)$$

Cancelling m and plugging in $r_1 = R_E + 890 \text{ km} = 6.37 * 10^6 \text{ m} + 8.9 * 10^5 \text{ m}$ and $v_1 = 7.23 * 10^3 \text{ m/s}$ and similarly for r_2 , we find $v_2 = 7.95 \text{ km/s}$.