Solutions to PS 5 Physics 201

1. Force acting small part $d\mathbf{l}$ of the wire carrying current is given by

$$\mathbf{F} = I\ d\mathbf{l} \times\ \mathbf{B}. \tag{1}$$

By integrating this over the entire loop, the total force is obtained:

$$\mathbf{F}_{\text{tot}} = \oint I\ d\mathbf{l} \times\ \mathbf{B} = I\ (\oint d\mathbf{l}) \times \mathbf{B} = 0. \tag{2}$$

Suppose two wires which draw two different paths C_1 and C_2 from A to B are carrying the same amount of current I. Then, from the above result, it follows that for forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the two wire

$$\mathbf{F}_1 - \mathbf{F}_2 = \int_{C_1} I\ d\mathbf{l} \times\ \mathbf{B} - \int_{C_2} I\ d\mathbf{l} \times\ \mathbf{B} \tag{3}$$

$$= I\ (\int_{C_1} d\mathbf{l} - \int_{C_2} d\mathbf{l}) \times \mathbf{B} \tag{4}$$

$$= I\ \oint d\mathbf{l} \times\ \mathbf{B} \tag{5}$$

$$= 0. \tag{6}$$

Therefore, $\mathbf{F}_1 = \mathbf{F}_2$, which also holds when C_1 is arbitrary and C_2 is a straight line.

2. From the balance between centrifugal force and Lorentz force,

$$m\frac{v^2}{R} = qvB. \tag{7}$$

Therefore, we get for the kinetic energy of the particle,

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\frac{qB}{mR})^2 = \frac{q^2B^2R^2}{2m}. \tag{8}$$

In case of protons circulating with $K = 4$MeV in a magnetic field of 4T,

$$R = \sqrt{\frac{2mK}{q^2B^2}} = \frac{\sqrt{2 \times 1.67 \times 10^{-21}\ \text{kg} \times 4\ \text{MeV}}}{1.6 \times 10^{-19}\ \text{C} \times 4\ \text{T}} = 72.3\ \text{m}. \tag{9}$$

For protons, $mc^2 = 938$ MeV, and therefore when $K = 4$ MeV, we have

$$\gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}} = \frac{942}{938}, \tag{10}$$

and

$$\beta \equiv \frac{v}{c} = \sqrt{1 - (\frac{938}{942})^2} = 0.092. \tag{11}$$

Therefore, when we expand E as a series of β, that is,

$$E = \frac{mc^2}{\sqrt{1-\beta^2}} = mc^2(1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \cdots), \tag{12}$$

the third and successive terms are at least $\beta^2 = 0.00085$ times smaller than the second term. Therefore, we have $K \approx mc^2 \frac{1}{2}\beta^2 = \frac{1}{2}mv^2$.

3. Note that the forces acting on the two parts parallel to x-axis cancel each other. Therefore, the net force comes from the other two parts directing $\pm y$ direction. The difference between the magnetic field at the left side and the right side is 0.2 T/m $\times$ 0.2 m $=$ 0.04 T. Then, the total force is

$$|\mathbf{F}|_{\text{tot}} = I\ B_{\text{right}}\ l - I\ B_{\text{left}}\ l \tag{13}$$

$$= I\ (B_{\text{right}} -\ B_{\text{left}})\ l \tag{14}$$

$$= 3\ \text{A} \times 0.04\ \text{T} \times 0.2\ \text{m} \tag{15}$$

$$= 2.4 \times 10^{-2}\ \text{N}, \tag{16}$$

in the positive x direction.

4. In the limit of $(\delta/R) \to 0$, when seen from the tiny element of one loop, the other loop current looks as if it is straight (Fig 1).

Therefore, we can apply the formula for straight infinite current as an approximation. That is, the magnetic field created by the upper current at the lower loop is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\delta}\mathbf{e}_y, \tag{17}$$

where the coordinate is taken as shown in Fig. 1. Therefore the force acting on this small element is

$$d\mathbf{F} = Idl\mathbf{e}_x \times \mathbf{B} = Idl\mathbf{e}_x \times \frac{\mu_0 I}{2\pi\delta}\mathbf{e}_y = \frac{\mu_0 I^2 dl}{2\pi\delta}\mathbf{e}_z. \tag{18}$$

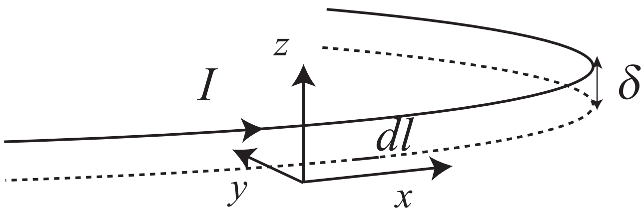

FIG. 1: Two closely spaced loop currents.

Integrating over the lower loop, we get

$$\mathbf{F} = \frac{\mu_0 I^2 R}{\delta}\mathbf{e}_z. \tag{19}$$

As is clear from the result above, the force between the two loops is attractive.

5. By the superposition principle, we can consider the system as an imaginary superposition of two parts. That is, one is the current in the direction of the actual current, distributed over the entire cylinder of radius a (denoted by A) and the other is the current in the opposite direction flowing over the removed region (denoted by cylinder B) (Fig. 2). For this superposition to reproduce the given current distribution, we need to assign both currents the same magnitude of current density (current per area) $j = I/(\frac{3}{4}\pi a^2)$, which is the same as the actual current density.

Using Ampere's law, we get for **B** at the centers of the two cylinders as field created by each other's current. That is

$$\mathbf{B}_A = \frac{\mu_0 I_{\text{enclosed}}}{2\pi\frac{a}{2}}\mathbf{e}_x = \frac{\mu_0 \frac{I}{\frac{3}{4}\pi a^2}\frac{\pi a^2}{4}}{\pi a} = \frac{2\mu_0 I}{3\pi a}\mathbf{e}_x, \tag{20}$$

and

$$\mathbf{B}_B = \frac{\mu_0 I_{\text{enclosed}}}{2\pi\frac{a}{2}}\mathbf{e}_x = \frac{\mu_0 \frac{I}{\frac{3}{4}\pi a^2}\frac{\pi a^2}{4}}{\pi a} = \frac{2\mu_0 I}{3\pi a}\mathbf{e}_x. \tag{21}$$

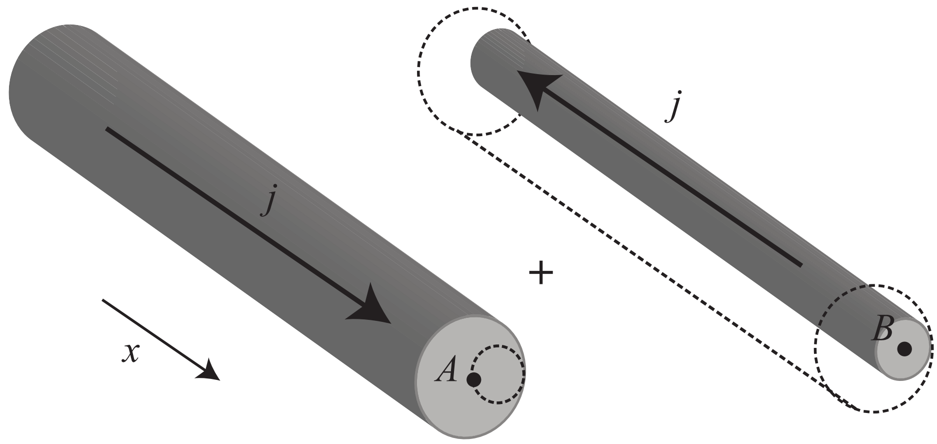

FIG. 2: Decomposition of the system into two imaginary parts.

6. In the same way as problem 5, we can consider the system as an imaginary superposition of currents over three cylinders whose centers are shown by dots in Fig. 3. The large cylinder marked by 1 carries a current coming out of the paper, and for the other two cylinders, the currents are in the opposite direction.

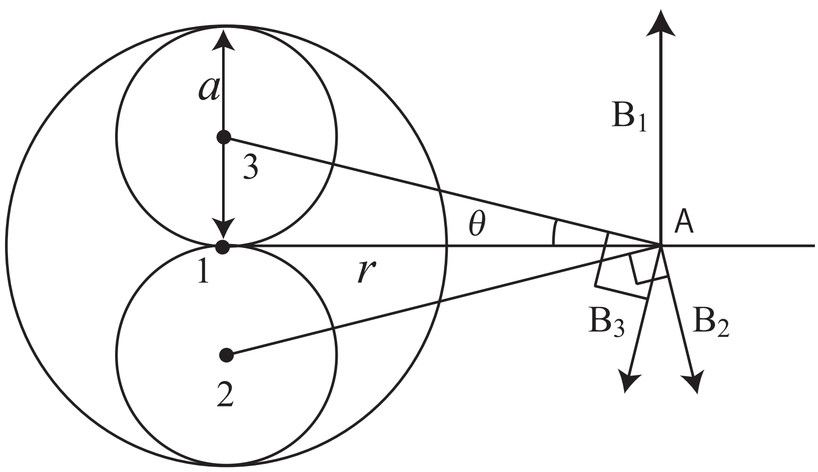

FIG. 3: The current configuration.

Using Ampere's law, and summing fields by these three parts, we get the magnetic field at the point A shown in Fig. 3, which is in positive-y direction:

$$\mathbf{B} = \mathbf{B}_{\text{tot}} = [|\mathbf{B}_1| - |\mathbf{B}_2|\cos\theta - |\mathbf{B}_3|\cos\theta]\mathbf{e}_y = [|\mathbf{B}_1| - 2|\mathbf{B}_2|\cos\theta\ ]\mathbf{e}_y \tag{22}$$

$$= [\frac{\mu_0 I_{\text{enclosed},1}}{2\pi r} - 2\frac{\mu_0 I_{\text{enclosed},2}}{2\pi\sqrt{r^2+(a/2)^2}}\cos\theta\ ]\mathbf{e}_y \tag{23}$$

$$= [\frac{\mu_0 \frac{I}{\frac{1}{2}\pi a^2}\pi a^2}{2\pi r} - 2\frac{\mu_0 \frac{I}{\frac{1}{2}\pi a^2}\frac{\pi a^2}{4}}{2\pi\sqrt{r^2+(a/2)^2}}\frac{r}{\sqrt{r^2+(a/2)^2}}]\mathbf{e}_y \tag{24}$$

$$= [\frac{\mu_0 I}{\pi r} - \frac{\mu_0 I r}{2\pi\{r^2+a^2/4\}}]\mathbf{e}_y. \tag{25}$$

7. Using Biot-Savart's law, we can calculate magnetic field created by one side of the square.

$$\mathbf{B}_{\text{one side}} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{\mu_0}{4\pi}\frac{dx(I,0,0)\times(-x,a/2,0)}{[(a/2)^2+x^2]^{3/2}} \tag{26}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{\mu_0 I}{4\pi}\frac{dx(0,0,a/2)}{[(a/2)^2+x^2]^{3/2}} \tag{27}$$

$$= \mathbf{e}_z\frac{\mu_0 I a}{8\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{dx}{[(a/2)^2+x^2]^{3/2}} \tag{28}$$

$$= \mathbf{e}_z\frac{\mu_0 I}{2\pi a}\int_{-1}^{1}\frac{ds}{[1+s^2]^{3/2}} \tag{29}$$

$$= \mathbf{e}_z\frac{\mu_0 I}{2\pi a}[\frac{s}{\sqrt{1+s^2}}]_{-1}^{1} \tag{30}$$

$$= \mathbf{e}_z\frac{\mu_0 I}{2\pi a}\sqrt{2} \tag{31}$$

$$= \mathbf{e}_z\frac{\mu_0 I}{\sqrt{2}\pi a} \tag{32}$$

$$\tag{33}$$

Noting that 4 sides of the square give the same contribution, we finally get

$$\mathbf{B} = 4\mathbf{B}_{\text{one side}} = \mathbf{e}_z\frac{2\sqrt{2}\mu_0 I}{\pi a} \tag{34}$$

8. First, we can think of the system of a superposition of $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ in Fig. 4.

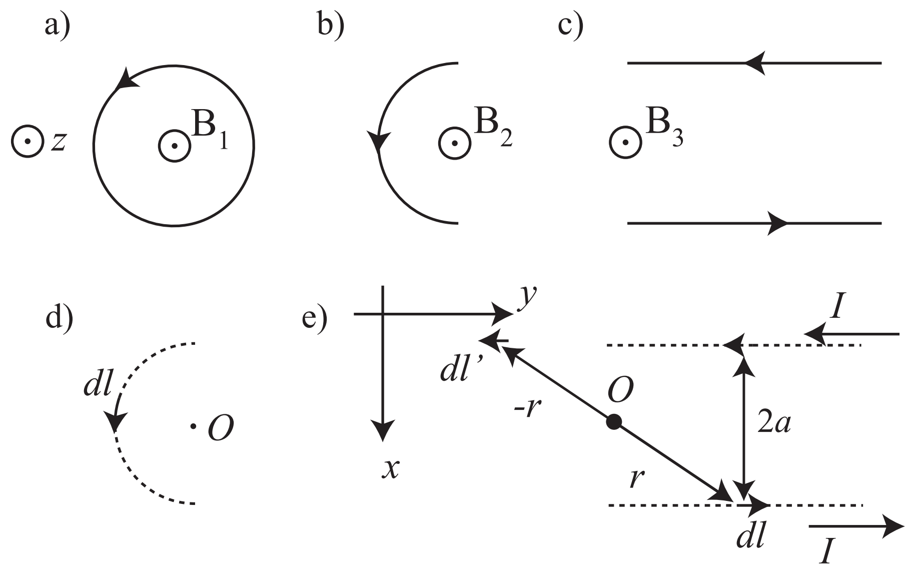

FIG. 4: Current elements for calculation.

To evaluate $\mathbf{B}_1$ and $\mathbf{B}_2$, we note that the magnetic field at the center O created by the small element shown in Fig. 4(d) is given by

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l} \times (\mathbf{0} - \mathbf{l})}{|\mathbf{l}|} = \frac{\mu_0 I}{4\pi a} |d\mathbf{l}|\ \mathbf{e}_z, \tag{35}$$

which is always pointing the positive-z direction and whose magnitude is proportional to the arc length. So we easily get

$$\mathbf{B}_1 + \mathbf{B}_2 = \frac{\mu_0 I}{4\pi a} 3\pi a\ \mathbf{e}_z = \frac{3\mu_0 I}{4}\ \mathbf{e}_z. \tag{36}$$

Next, let's consider $\mathbf{B}_3$. For this purpose, it is good to first show that a small element dl (at $\mathbf{r} = (a, l, 0)$) and an imaginary current element dl' shown in the Fig. 4 (e) give the same contribution to $\mathbf{B}$.

The field element created by the small element dl can be calculated using Biot-Savart's law:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l} \times (\mathbf{0} - \mathbf{r})}{|\mathbf{r}|} \tag{37}$$

$$= \frac{\mu_0 I}{4\pi} \frac{(0, dl, 0) \times (-a, -l, 0)}{\sqrt{a^2 + l^2}} \tag{38}$$

$$= \frac{\mu_0 I}{4\pi} \frac{a\ dl}{\sqrt{a^2 + l^2}}\ \mathbf{e}_z. \tag{39}$$

And in the same way, dl' gives,

$$d\mathbf{B}' = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l}' \times (\mathbf{0} - (-\mathbf{r}))}{|\mathbf{r}|} = \frac{\mu_0 I}{4\pi} \frac{(0, -dl, 0) \times (a, l, 0)}{\sqrt{a^2 + l^2}} = \frac{\mu_0 I}{4\pi} \frac{a\ dl}{\sqrt{a^2 + l^2}}\ \mathbf{e}_z, \tag{40}$$

which is equivalent to that by dl. Thus, it follows that the field created by the two semi-infinite currents are equivalent to that created by a single infinite current. Therefore,

$$\mathbf{B}_3 = \frac{\mu_0 I}{2\pi a} \mathbf{e}_z. \tag{41}$$

Finally, we get the total field,

$$\mathbf{B} = (\frac{3\mu_0 I}{4} + \frac{\mu_0 I}{2\pi a}) \mathbf{e}_z. \tag{42}$$

9. (i) As in usual image charge problems, for $y = 0$ plane to be an equipotential surface, we need to imagine a pair of charges q and q' located at $(0, Y, Z)$ and $(0, -Y, Z)$. For $x = 0$ plane, a pair of charges at $(0, Y, Z)$ and $(0, Y, -Z)$ are needed. Therefore, we should put image charge $-q$ at $(0, -a, a)$ and $(0, a, -a)$, and q at $(0, -a, -a)$. Then, the attraction between the charge and the plates is the same as that between the charge and the image charges. Therefore, we get

$$\mathbf{F} = +\frac{-q^2}{4\pi\epsilon_0 (2a)^2} \mathbf{e}_y + \frac{-q^2}{4\pi\epsilon_0 (2a)^2} \mathbf{e}_z + \frac{q^2}{4\pi\epsilon_0 (2\sqrt{2}a)^2} \frac{1}{\sqrt{2}} \{\mathbf{e}_y + \mathbf{e}_z\} \tag{43}$$

$$= -\frac{(4 - \sqrt{2})q^2}{64\pi\epsilon_0 a^2} \mathbf{e}_y - \frac{(4 - \sqrt{2})q^2}{64\pi\epsilon_0 a^2} \mathbf{e}_z \tag{44}$$

(ii)When the charge is at infinity, the potential energy is 0. Let's take a path from infinity to (a,a) on the line $y = z$, $x = 0$, parametrized by $\{(0, -s, -s), (s : -\infty \to$

$-a)\}$ Therefore,

$$W_{\text{needed}} = \int_{\infty}^{(a,a)} -\mathbf{F}\cdot d\mathbf{l} \tag{45}$$

$$= \int_{s=-\infty}^{-a} \{\frac{(4-\sqrt{2})q^2}{64\pi\epsilon_0 s^2}\mathbf{e}_y + \frac{(4-\sqrt{2})q^2}{64\pi\epsilon_0 s^2}\mathbf{e}_z\}\cdot\{(-ds)(\mathbf{e}_y+\mathbf{e}_z)\} \tag{46}$$

$$= -\{\frac{(4-\sqrt{2})q^2}{32\pi\epsilon_0}\int_{s=-\infty}^{-a}\frac{1}{s^2}ds \tag{47}$$

$$= -\frac{(4-\sqrt{2})q^2}{32\pi\epsilon_0 a}. \tag{48}$$

$$\tag{49}$$

(iii) Let's consider $x = 0$ plane. If we think independently 4 pairs of charge, that is, $(\pm a, a, a), (\pm a, -a, a), (\pm a, a, -a)$, and $(\pm a, -a, -a)$, each of these pairs gives equipotential surface $x = 0$, as is in an usual image charge problem. Therefore, from the superposition principle, all the eight charges combined also give an equi-potential surface $x = 0$.

10. Using Ampere's law, magnetic field created by the strip at the wire is given by

$$B_{\text{by strip}} = \int_0^w dl\frac{\mu_0\frac{I}{w}}{2\pi(a+l)} \tag{50}$$

$$= \frac{\mu_0 I}{2\pi w}[\ln(a+l)]_0^w \tag{51}$$

$$= \frac{\mu_0 I}{2\pi w}\ln\frac{a+w}{a}. \tag{52}$$

Therefore, the force per unit length between the wire and the strip is given by

$$\mathbf{F} = IB_{\text{by strip}} = \frac{\mu_0 I^2}{2\pi w}\ln\frac{a+w}{a}. \tag{53}$$

In the limit of $w \to 0$, this gives

$$\mathbf{F} = \frac{\mu_0 I^2}{2\pi w}\ln(1+\frac{w}{a}) = \frac{\mu_0 I^2}{2\pi w}\{\frac{w}{a} - \frac{1}{2}(\frac{w}{a})^2 + \cdots\} \to \frac{\mu_0 I^2}{2\pi a}, \tag{54}$$

which reproduces the force between the two straight currents.

11. From the result of Problem 1, the force acting on the arc PQ is equivalent to a force acting on an imaginary straight current flowing from P to Q (Fig. 5). Therefore,

$$\mathbf{F} = -IB\ |PQ|\ \mathbf{e}_y. \tag{55}$$

$$= -2IB\sqrt{R^2-a^2}\ \mathbf{e}_y. \tag{56}$$

$$\tag{57}$$

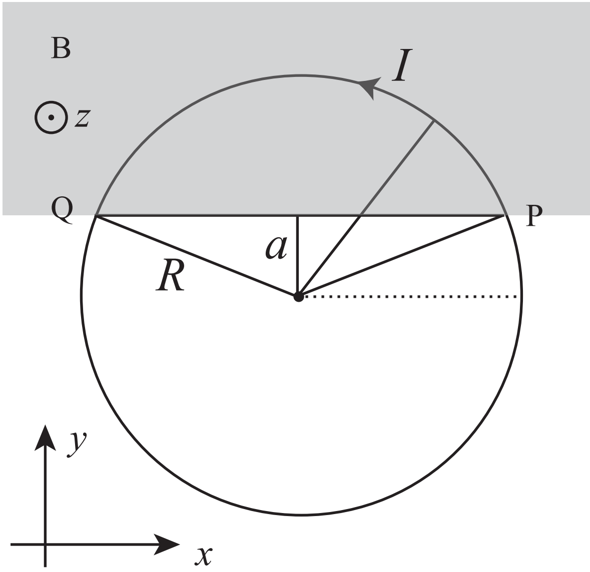

FIG. 5: Arc-shape current and equivalent straight current flowing from P to Q.

12. From the balance between the forces, we get

$$k\delta + IBL - mg = 0. \tag{58}$$

Therefore,

$$k = \frac{-IBL + mg}{\delta} = \frac{-2\text{ A} \times 0.2\text{ T} \times 2\text{ m} + 0.12\text{ kg} \times 9.8\text{ m/s}^2}{0.012\text{ m}} = 31.3\text{ N/m} \tag{59}$$

13. (i)By applying Ampere's law to the red loop shown in Fig. 6, we can calculate the magnetic field at distance r from the center axis:

$$\mathrm{B}(r) = \frac{\mu_0 I_{\text{enclosed}}}{2\pi r} = \frac{\mu_0 N I}{2\pi r} \tag{60}$$

Integrating this over the hatched square, we get

$$\Phi = \int_{\text{Square}} \mathbf{B}\ d\mathbf{S} = a \int_{R-\frac{a}{2}}^{R+\frac{a}{2}} B(r) dr \tag{61}$$

$$= \frac{\mu_0 N I a}{2\pi} \int_{R-\frac{a}{2}}^{R+\frac{a}{2}} \frac{1}{r} dr \tag{62}$$

$$= \frac{\mu_0 N I a}{2\pi} \ln[\frac{R+\frac{a}{2}}{R-\frac{a}{2}}] \tag{63}$$

$$\tag{64}$$

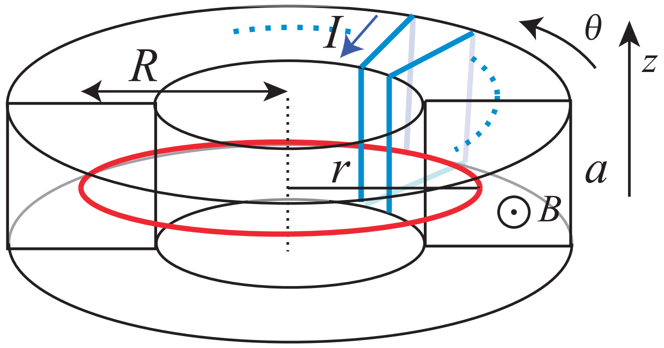

FIG. 6: Toroidal solenoid.

(ii)Using the result above,

$$E_{\text{stored}} = \int dV \frac{B^2}{2\mu_0} \tag{65}$$

$$= \int dzdr(rd\theta)\frac{\mu_0 N^2 I^2}{8\pi r^2} \tag{66}$$

$$= \frac{\mu_0 N^2 I^2}{8\pi} a2\pi \int_{R-\frac{a}{2}}^{R+\frac{a}{2}} dr\frac{1}{r} \tag{67}$$

$$= \frac{\mu_0 N^2 I^2 a}{4} \ln[\frac{R+\frac{a}{2}}{R-\frac{a}{2}}]. \tag{68}$$

(iii) Because the energy stored in the solenoid is given by $\frac{1}{2}LI^2$, from the result of part (ii), we get

$$L = 2E/I^2 = \frac{\mu_0 N^2 a}{2} \ln[\frac{R+\frac{a}{2}}{R-\frac{a}{2}}]. \tag{69}$$

(iv)

$$\mathcal{E} = L\frac{dI}{dt} = -LI_0\omega \sin\omega t = \frac{-\mu_0 N^2 I_0 \omega a}{2} \ln[\frac{R+\frac{a}{2}}{R-\frac{a}{2}}] \sin\omega t \tag{70}$$

14. (i) Just before the switch is closed, the circuit has been carrying a current $I_0 = V/R_2$. Just after the switch is closed, we can think that the flowing current remains the same. That is,

$$I = V/R_2. \tag{71}$$

(Although this is not required, we can also calculate V_L using the relation $V = I(R_1//R_2) + L\frac{dI}{dt}$. That is, $V_L = L\frac{dI}{dt} = V - I(R_1//R_2) = V - \frac{V}{R_2}\frac{R_1R_2}{R_1+R_2} = \frac{VR_2}{R_1+R_2}$.)

(ii) In this limit ($t \to \infty$), the system is stable. That is, $\frac{dI}{dt} = 0$. Then, from $V = I(R_1//R_2) + L\frac{dI}{dt}$, we get

$$I = \frac{(R_1 + R_2)V}{R_1R_2}. \tag{72}$$

(iii) We can think in the same way as part (i). Just before the switch is opened, $I = \frac{(R_1+R_2)V}{R_1R_2}$. And just after the switch is closed, the current remains the same. Therefore,

$$V_{R_2} = IR_2 = \frac{(R_1 + R_2)V}{R_1}. \tag{73}$$