Problem Set 4 Physics 201b February 3, 2010. Due Feb 10

1. (i) Consider the electrostatic potential $V(\mathbf{r})$ in a region free of charge. (There can of course be charges outside this region producing the V. Show that $\bar{V}$, *its average value over a sphere of any radius R equals $V(0)$, its value at the center* . (Note that this sphere is a mathematical surface and lies within the charge free region.)

   (i) One way is to use brute force and show by explicitly averaging over the sphere the potential due to single charge q located a distance $r > R$ from the center and then arguing by the superposition principle for the general case of many charges.

   Hint: Note that the average of any function over the sphere is

   $$\bar{F} = \frac{1}{4\pi R^2} \int F(\theta, \phi) R^2 \sin\theta d\theta d\phi,$$

   use the law of cosines to find distances and choose the point charge to lie on the z-axis. (I have the left the R^2 which cancels just so you know where the formula comes from.)

   (ii) Now try the clever proof which goes as follows. Consider the potential energy U of a real nonconducting sphere of radius R with charge Q glued *uniformly* over its surface and a charge q at a distance r from the center of the sphere as in Fig. **??**. First find U assuming the sphere is fixed and the point charge is brought in from infinity to a distance r from its center. Next find the same U by assuming the point charge is fixed and the sphere is brought from infinity to the same final location. Show that the latter involves the average of $V(r')$ (Figure **??**) over the sphere. Equating the U for the two equivalent procedures will give the desired result.

   (iii) Deduce from the above that V *cannot have a minimum or maximum in a charge free region.* (Consider a tiny sphere around the supposed maximum or minimum.)

   (iv) Show (iii) another way. Suppose $\mathbf{r}_0$ is a point of stable equilibrium for a positive test charge. Which way should E point as you approach $\mathbf{r}_0$ from various directions? If you use Gauss' Law what does that imply? This rules out a minimum, what about a maximum?

2. So far we were used to finding the potential $V(\mathbf{r})$ due to a collection of charges. Given $V(\infty) = 0$, the answer was unique (just do the sum or integral over $(q/(4\pi\varepsilon_0 r))$. But consider a different specification of the problem depicted in Figure **??**. It shows a closed region that contains many conductors (shaded blobs) held at various potentials V_1, V_2.. , say by connecting them to ground via a batteries of prescribed voltages. The outermost boundary could even be the sphere at infinity, and it too is at some potential (V_4). So we do not know exactly where the charges are, only that in the end they bring each conductor to the advertised potential. Question: do these data determine the potential $V(\mathbf{r})$ in the empty region? All we know is that it is required to agree with the assigned values on the boundary, i.e., on the conductors' surfaces and the surrounding surface. I will help you show that *there is a unique $V(\mathbf{r})$ in the empty space obeying these "boundary conditions".*

   Suppose there are two solutions $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ obeying these boundary conditions. Consider their difference $W(\mathbf{r}) = V_1(\mathbf{r}) - V_2(\mathbf{r})$ and write down the boundary conditions it obeys. Show that these imply W must vanish everywhere in the empty region or else there will have a maximum or minimum somewhere, which is disallowed by the pervious result.) Thus $V_1(\mathbf{r}) = V_2(\mathbf{r})$ everywhere. This is an example of a *uniqueness theorem.*

There are generalizations which show uniqueness persists even if there are charges in the space between conductors. I just want you to get the flavor, so you understand why the method of images works.

3. Consider the circuit with a battery of EMF $\mathcal{E}$, connected to a resistor R and an uncharged capacitor C in series at $t = 0$. Using the solution for $Q(t)$ and $I(t)$ given in class show that at any time $t > 0$, the work done by the battery up to that point equals the stored energy in the capacitor and energy dissipated in the resistor.

4. Consider a point charge q a distance a along the x-axis from a sphere of radius R as in Fig. **??**. First assume the sphere is grounded. You want to show that q and an image charge $q' = -(R/a)q$ at a distance $b = R^2/a$ will make the sphere an equipotential $V = 0$.

   (i) Show that $b = R^2/a$ and $q' = -(R/a)q$ are required if we demand $V = 0$ at two special points on the sphere: one closest to q and one farthest from q. This does not mean V will vanish over the entire sphere $r = R$, but it just does! Show this using the law of cosines to find distances and the above value of q' and b.

   (ii) Say in words how you will find the surface charge density. What will it integrate to over the sphere and why?

   (iii) Find the force of attraction between the sphere and q.

   (iv) Repeat part (iii) for the case of an isolated and neutral sphere.

   (v) Repeat part (iii) for the case of an isolated neutral sphere with charge Q to begin with.

5. Electrons enter a region of uniform B in the plane perpendicular to $\mathbf{B}$ and finish an orbit in $1\mu s$. What is B in Tesla?

6. Find the effective resistance between points A and B in Fig **??**. Hint: Apply a voltage between A and B, assume a current I flows and find V/I.

7. In Fig **??** assume S has been closed for along time. (i) What is the current flowing through C?(ii) What is the current flowing through R_1 and R_2? (iii) What is the charge on C? (iv) If S is now opened, describe what will happen. (iv) What will be the time constant for the charge decay? (v) Now put in numbers for all parts using $V = 9V, R_1 = 4k\Omega, R_2 = 5k\Omega, R_3 = 1k\Omega, C = 100\mu F$.

8. Imagine a capacitor made of two hollow conductors, one inside the other, with a capacitance C. (You may imagine they are concentric spheres but they need not be concentric or spherical for this problem.) The space between them is filled with a substance of conductivity σ. If I connect a battery with voltage V_0 between the two,

show that the current will be $I = \frac{V_0 \sigma C}{\varepsilon_0}$. Hint: Start with $\mathbf{J} = \sigma \mathbf{E}$ at the surface of the inner conductor.

9. The cylindrical rod of length w shown in Fig. **??** carries a current I as shown and is bathed in a field $\mathbf{B}$ perpendicular to the plane in which is a $\sqcup$ shaped rail. The rod rolls without slipping on the rails, its length perpendicular to the two parallel rails and equal to the space between them. It starts at rest and rolls off after going a distance L. Show that its exit velocity is

$$v = \sqrt{\frac{4BLIw}{3M}}.$$

Notice that the radius of the cylinder is not given. This problem requires you to go back and look up rolling without slipping and torques.

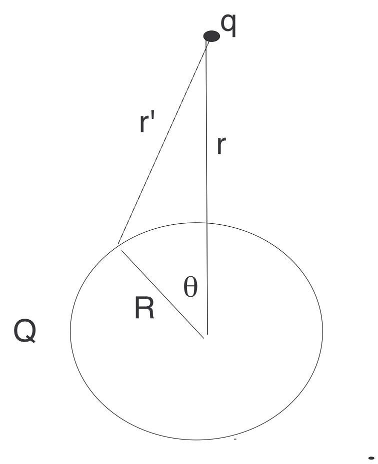

Figure 1: A point charge q and a nonconducting sphere with Q uniformly glued over its surface. r' is the distance from a generic point on the sphere to q.

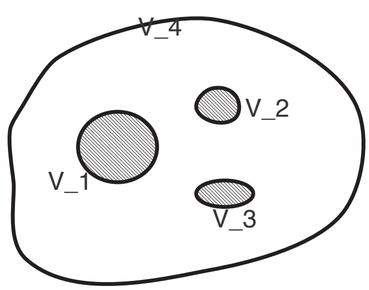

Figure 2: A closed region (big outer curve) with some conductors (shaded blobs) at fixed potentials $V_1, ..V_4$

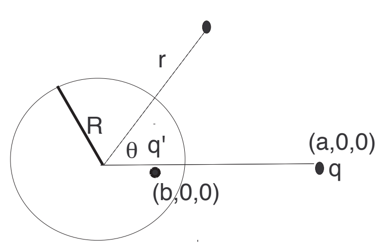

Figure 3: A point charge q at a distance a from the conducting sphere of radius R. The image is q' and located at $(b, 0, 0)$.

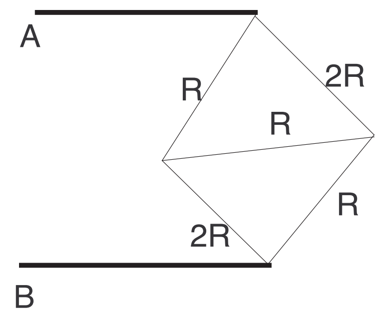

Figure 4: The leads (thick lines) have no resistance, the thin ones have values R and $2R$ as shown.

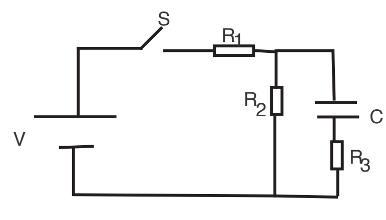

Figure 5: A two loop circuit with a switch S.

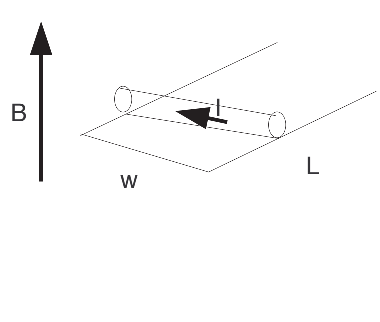

Figure 6: The rod rolls without slipping.