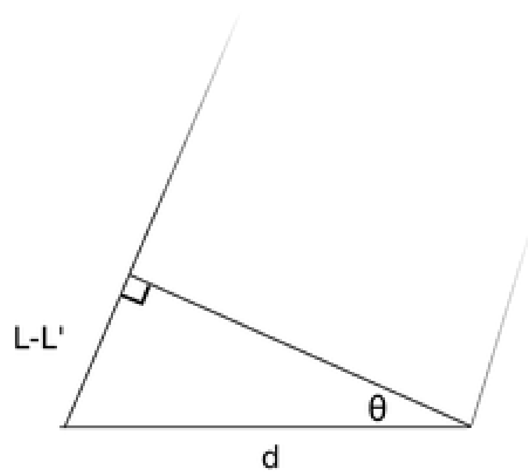
Solutions to PS 10 Physics 201

1. For a screen very far away from the aperture, the rays from each of the N slits emerge at approximately the same angle. Thus, from the diagram below, we see that the path length difference between two adjacent slits is given by

$$\Delta\ell = L - L' = d\sin\theta \tag{1}$$

and thus the phase difference is given by

$$\phi = k\Delta\ell = kd\sin\theta \tag{2}$$



Thus, if we take the phase of the top-most ray to be zero, the phase of the n-th ray is

$$\phi_n = n\phi = nkd\sin\theta \tag{3}$$

Summing up the amplitude of all N rays, we find

$$A = \sum_{n=0}^{N-1} ae^{i\phi_n} \tag{4}$$

$$= a\sum_{n=0}^{N-1} \left(e^{i\phi}\right)^n \tag{5}$$

$$= a\frac{1-e^{iN\phi}}{1-e^{i\phi}} \tag{6}$$

Where in the last line we used the formula for a finite geometric series. The first zero of A occurs when the numerator vanishes (and the denominator is nonzero), which occurs when

$$N\phi = 2\pi \tag{7}$$

$$Nkd\sin\theta = 2\pi \tag{8}$$

$$\frac{2\pi}{\lambda}D\sin\theta = 2\pi \tag{9}$$

$$D\sin\theta = \lambda \tag{10}$$

As desired.

We can write our expression for A as a ratio of sin's by factoring $e^{iN\phi.2}$ out of the numerator, and $e^{i\phi/2}$ out of the denominator. Doing so gives

$$A = ae^{i(N-1)\frac{\phi}{2}}\frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}} \tag{11}$$

2. (a) For a double slit, we have that maxima occur when

$$d\sin\theta = n\lambda \tag{12}$$

and minima when

$$d\sin\theta = (n+\frac{1}{2})\lambda \tag{13}$$

Thus, the first non-central maximum occurs at

$$d\sin\theta = \lambda \tag{14}$$

$$\sin\theta = \frac{\lambda}{d} \tag{15}$$

$$\theta \approx \frac{\lambda}{d} = 0.006rad \tag{16}$$

Similarly, the first minima occurs when

$$d\sin\theta = \frac{\lambda}{2} \tag{17}$$

$$\theta \approx \frac{\lambda}{2d} = 0.003rad \tag{18}$$

(b) Let $L = 2m$ be the distance to the screen, and x the height of the first dark fringe above the center. Then we have

$$\tan\theta \approx \theta = \frac{x}{L} \tag{19}$$

$$x \approx L\theta = 6mm \tag{20}$$

An analogous calculation shows that for the first maximum,

$$x \approx 12mm \tag{21}$$

(c) From the above calculations, we see that in the small angle approximation, maxima and minima are equally spaced (in both angle and position on the screen). Thus, for the second maximum, we must have

$$\theta \approx 0.012rad \tag{22}$$

$$x \approx 24mm \tag{23}$$

and similarly for the second minimum,

$$\theta \approx 0.009rad \tag{24}$$

$$x \approx 18mm \tag{25}$$

(d) Replacing changing our value of λ to $500nm$ is equivalent to multiplying our result for x by $5/6$. Thus, we have for the difference in location of the first maximum,

$$\Delta x = 12(1 - \frac{5}{6})mm = 2mm \tag{26}$$

3. The difference in optical path length due to the presence of the material is given by

$$\Delta\ell = (n-1)t \tag{27}$$

Before the material is placed, the total difference in optical path length is 5λ, while aferwards, it is $\frac{3\lambda}{2}$. Thus,

$$\Delta\ell = \lambda(5 - \frac{3}{2}) \tag{28}$$

$$(n-1)t = \frac{7\lambda}{2} \tag{29}$$

$$t = \frac{7\lambda}{2(n-1)} = 3\mu m \tag{30}$$

4. For a diffraction grating with N lines per meter, we have maxima when

$$\frac{1}{N}\sin\theta = m\lambda \tag{31}$$

Thus, for the first order maxima we have

$$\sin\theta = N\lambda \tag{32}$$

$$\theta \approx 0.157rad \tag{33}$$

similarly for the fourth order maxima we have

$$\sin\theta = 4N\lambda \tag{34}$$

$$\theta = 0.674rad \tag{35}$$

Note that 0.674 radians is a fairly large angle, so we have not used the small angle approximation in the last step.

5. Since both sides of the film are bounded by air, there is no phase shift at the lower boundary. Furthermore, normal incidence implies that all relevant angles are zero. Thus constructive interference occurs when

$$2nt = (m + \frac{1}{2})\lambda \tag{36}$$

We get a minimum t when $m = 0$, and thus

$$t = \frac{\lambda}{4n} = 93.0nm \tag{37}$$

6. The first minimum occurs when

$$d\sin\theta = \lambda \tag{38}$$

which implies

$$\theta = \arcsin\frac{600}{2000} = 0.304rad \tag{39}$$

The angular width is given by $\delta = 2\theta$, and thus

$$\delta = 0.610rad \tag{40}$$

7. For the third order maximum of the grating to be at $\theta = \pi/6$, we must have

$$\frac{1}{N}\sin\frac{\pi}{6} = 3\lambda \tag{41}$$

$$\frac{1}{2N} = 3\lambda \tag{42}$$

$$N = \frac{1}{6\lambda} \tag{43}$$

$$= 3333\frac{1}{cm} \tag{44}$$

8. The final kinetic energy of the electrons is given by

$$E = eV \tag{45}$$

and thus their momentum is

$$p = \sqrt{2mE} = \sqrt{2meV} \tag{46}$$

and therefore they have a wavelength

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}} \tag{47}$$

The first double slit minimum is given by the condition

$$d\sin\theta = \frac{\lambda}{2} \tag{48}$$

and thus the angle of the first minimum is given by (in the small angle approximation)

$$\theta = \frac{\lambda}{2d} = \frac{h}{2d\sqrt{2meV}} \tag{49}$$

For a screen L meters away, we have the height w (note - we use w instead of h to avoid confusion with Planck's constant) of this minimum given by

$$\sin\theta \approx \tan\theta = \frac{h}{2d\sqrt{2meV}} \tag{50}$$

$$\frac{w}{L} = \frac{h}{2d\sqrt{2meV}} \tag{51}$$

$$w = \frac{hL}{2d\sqrt{2meV}} \tag{52}$$

Solving for V, we find

$$V = \frac{1}{2me}\left(\frac{hL}{2wd}\right)^2 \tag{53}$$

Plugging in the numbers given, we find

$$V = 37.6MV \tag{54}$$

9.
$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{E^2 - (mc^2)^2}} \tag{55}$$

Since the rest mass energy of the proton is given by $938MeV$, we see that the ratio $\frac{mc^2}{E}$ is vanishingly small for $E = 3.5TeV$. Thus, we can Taylor expand our formula for λ and keep terms only zeroth order in m, yielding

$$\lambda = \frac{hc}{E} = 3.54 \times 10^{-19}m \tag{56}$$

Note that since we ignored the mass term, this is the same wavelength light would have at that energy.

10. Note that the image of the source lies a distance H below the mirror. Thus, we can treat this as if it were a double slit setup, with $d = 2H$. However, because of the phase shift of π at the mirror, we will get destructive interference at what would normally be the double slit maxima. Thus, we have minima when

$$2H \sin\theta = n\lambda \tag{57}$$

where θ is measured from the mirror. For the height h of the first minimum, this gives

$$\lambda = 2H \sin\theta \approx 2H \tan\theta \tag{58}$$
$$\frac{\lambda}{2H} = \frac{h}{D} \tag{59}$$
$$h = \frac{D\lambda}{2H} \tag{60}$$
$$= 50\mu m \tag{61}$$

11. We have

$$E = \hbar\omega - W \tag{62}$$

where E is the electron kinetic energy. The minimum $\omega = \omega_0$ occurs when the kinetic energy is zero, giving

$$\omega_0 = \frac{W}{\hbar} = 6.08 \times 10^{15}\frac{rad}{s} \tag{63}$$

At $\omega = 2\omega_0$, we have

$$E = 2W - W = W \tag{64}$$

Since $E = mv^2/2$, we find the velocity of the electrons is

$$v = \sqrt{\frac{2W}{m}} = 1.19 \times 10^6\frac{m}{s} \tag{65}$$

12.

$$\lambda_T = \frac{h}{p} \tag{66}$$
$$= \frac{h}{\sqrt{2mE}} \tag{67}$$
$$= \frac{h}{\sqrt{3mkT}} \tag{68}$$

at $T = 300K$, we find

$$\lambda_T = 6.23nm \tag{69}$$

13. (a) From the picture shown, we see that the difference in path length $\Delta\ell$ between the top and bottom rays is given by

$$\Delta\ell = 2d\sin\theta \tag{70}$$

The condition for constructive interference is that $\Delta\ell = m\lambda$, from which we recover the Bragg condition

$$2d\sin\theta = m\lambda \tag{71}$$

(b) Electrons with $E = 54eV$ have a de Broglie wavelength

$$\lambda = \frac{h}{\sqrt{2mE}} = 0.167nm \tag{72}$$

Plugging this into the Bragg formula with $\theta = 65°$, $d = a$ and $m = 1$, we find

$$a = 0.92\mathring{A} \tag{73}$$