Faraday and Lenz: A loop in two frames R.Shankar February 17, 2010 .

I would like to clarify in some detail how it all works out for the loop in two frames.

The equation we need is

$$\mathcal{E} = \oint (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} + \oint_{\partial S} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} \qquad (1)$$

In the Lab frame, (part A of Fig. 1) $\mathbf{B}$ is time-independent and going into the page and the loop is moving to the right with velocity $\mathbf{v}$ as in the upper half of the figure. It feels an EMF $\mathcal{E} = vBw$, which is the $\mathbf{v} \times \mathbf{B}$ force integrated over the width w of the leading edge, which comes from the right most term in Eq. 1.

In the loop frame, (part B of Fig. 1), the magnet or solenoid producing $\mathbf{B}$ is moving to the left at speed v. The EMF is from the first term in the RHS of Eq. 1. There must be an electric field $\mathbf{E}'$ in this frame obeying

$$\oint_{\partial S} \mathbf{E}' \cdot d\mathbf{l} = -\int \int_S \frac{\partial \mathbf{B}'}{\partial t} \cdot d\mathbf{A}. \qquad (2)$$

What does such a field look like? We can get an answer as follows.

First consider small velocities when a nonrelativistic treatment is allowed. Since in the lab frame a unit charge in the leading edge of the loop feels an upward (Lorentz) force vB, it must feel the same upward force the loop rest frame. (We are using the fact that since m and a are unaffected by relative velocity in Newtonian mechanics, $F = F'$.)

Further this force has to be electric in origin since the loop is at rest. Thus we expect to see a field $E' = vB$ pointing up as shown in the lower figure. This field must exist wherever $\mathbf{B}$ does since the leading edge of the loop could be anywhere in the shaded region. Thus $\mathbf{E}'$ is nonzero and upwards wherever $\mathbf{B} \neq 0$, i.e., in the entire shaded semi-infinite region.

Notice that the EMF is same as in the Lab frame:$\mathcal{E}' = wE' = Bvw = \mathcal{E}$. This is required in the nonrelativistic theory where F and the width w are invariant.

How about Faraday's equation 2? In the lab frame it is $0 = 0$ since there is no $\mathbf{E}$ and no $\frac{\partial \mathbf{B}}{\partial t}$. In the loop frame, consider a Faraday contour (dotted line) that straddles the two regions in the lower figure. The anticlockwise line integral of $\mathbf{E}'$ receives a contribution wE' from the right edge and nothing from the other three. The flux change inside the contour in time dt is $wBvdt$ since an extra piece of the shaded region of height w and width vdt carrying a nonzero B enters the Faraday contour. (In the nonrelativistic limit $B = B'$.) If we took a contour that is entirely in one region or the other we will get $0 = 0$.

In a relativistic treatment, corrections of order v^2/c^2 will affect the results. For example the EMF will not be same in both frames. But then neither are forces the same in frames if we include corrections to Newtonian mechanics of order b^2/c^2.

An interesting twist is the following. Suppose I add to the lower figure *a uniform downward* electric field of strength E' *in all of space*. The the result is an $\mathbf{E}'$ that is nonzero and pointing down to the left of the shaded region and zero in the shaded region, as in part C of the figure. This new set of fields obeys all the Maxwell equations as the old one *since the constant $\mathbf{E}'$ field added has no surface integral (over any closed surface) or line*

integral around any closed loop (check this out). How can there be two solutions both obeying Maxwell equations sustained by the same currents (in the solenoid)? Did I not say the equations had unique solutions with given surface and line integrals? The answer is that two solutions can still differ by fields ($\mathbf{E}$ or $\mathbf{B}$) which are constant, since these do not affect the line or surface integrals. Only by demanding that all fields vanish at infinity do we banish such constant fields and get a unique solution. The constant field I added (which does not vanish anywhere, especially at infinity) violates this condition. In any event, solution (C) is not acceptable to the loop problem since it is not the Lorentz transformed version of the Lab solution (A), only (B) is.
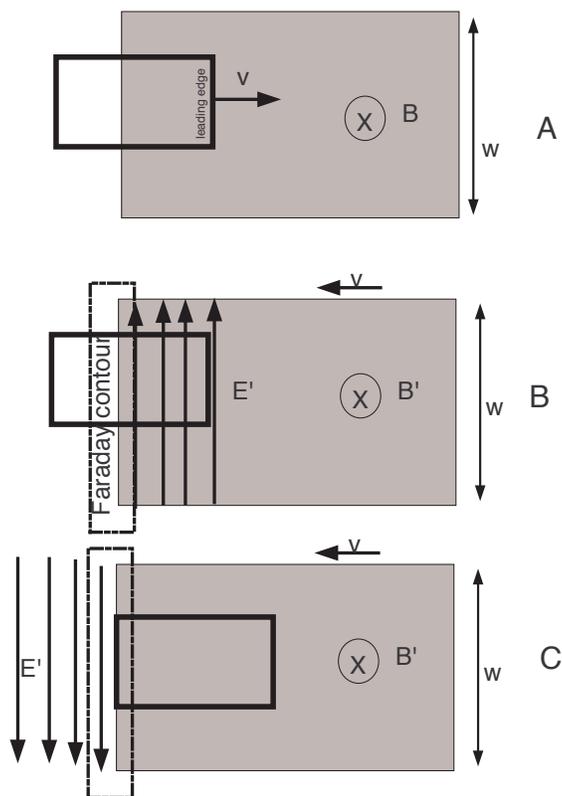
Figure 1: The lab frame (A) the loop moves to the right at speed v. The **B** field is nonzero in the shaded region, assumed to be semi-infinite. In the loop frame (B) the loop is fixed and the source of the **B** field (a solenoid) moves to the left at speed v. In this frame there is an upward **E**$'$ field as shown (and ti extends oveer the half-palne with the nonzero **B** field. Figure (C) corresponds to adding a constant **E**$'$ field in all of space to (B). This does not change any line or surface integrals. It is however not obtained from (A) by a Lorentz transform, only (B) is.