Open Yale courses

Physics 201
Final Exam
Solutions

I.   i) The condition at equilibrium is

$$\sum F = \frac{Qq}{4\pi\epsilon_0}\frac{1}{y^2} - mg = 0 \implies y_0 = \sqrt{\frac{Qq}{4\pi\epsilon_0 mg}}.$$

ii) To find the spring constant due to the electric force, we write $y = y_0 + (y - y_0) = y_0 + \Delta y$ where Δy is the displacement from equilibrium and note that if Δy is small compared to y_0

$$\frac{1}{y^2} = \frac{1}{(y_0 + \Delta y)^2} = \frac{1}{y_0^2}\frac{1}{(1 + \Delta y/y_0)^2} \approx \frac{1}{y_0^2}\left(1 - \frac{2}{y_0}\Delta y\right) = \frac{1}{y_0^2} - \frac{2}{y_0^3}\Delta y.$$

The constant term cancels the mg from gravity by part (i), so the force on the charge is

$$F = -\frac{2Qq}{4\pi\epsilon_0 y_0}\frac{1}{y_0^2}\Delta y = -\frac{2Qq}{4\pi\epsilon_0 y_0}\frac{4\pi\epsilon_0 mg}{Qq}\Delta y = -\frac{2mg}{y_0}\Delta y,$$

which means our spring constant k is $2mg/y_0$. Then

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2g}{y_0}}.$$

II. i) Since the electric field in the interior of a conductor is zero, a Gaussian surface drawn in between the two surfaces must enclose zero charge. So the charge on the inner surface is $-q$. Meanwhile, the sphere was initially uncharged, so the total charge of the shell must be zero. To balance the charge $-q$ on the inner surface, the charge on the outer surface must be q, because no charge resides in the interior of the shell.

ii) Outside the sphere where $r > b$ the whole thing looks like a point charge q, so the potential is simply

$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{r}, \qquad r > b.$$

The whole conducting shell is at a constant potential, so

$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{b}, \qquad a < r < b.$$

Finally, inside the inner surface of the sphere the potential from the two surfaces adds to the potential due to the point charge to give

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{1}{b} - \frac{1}{a}\right), \qquad 0 < r < a.$$

III. i) By Ampère's law $I_{\mathrm{enc}} = 0$ implies $B = 0$.

ii) Using a circular loop of radius r in Ampère's law gives

$$2\pi r B = \mu_0 \frac{I}{\pi(b^2 - a^2)}\pi(r^2 - a^2) = \mu_0 I\frac{r^2 - a^2}{b^2 - a^2} \implies B = \frac{\mu_0 I}{2\pi r}\frac{r^2 - a^2}{b^2 - a^2}.$$

The direction of the magnetic field is always clockwise.

iii) We can use Ampère's law again with $I_{\mathrm{enc}} = I$, or just note that the magnetic field is continuous at the boundary where $b = r$ so that

$$B = \frac{\mu_0 I}{2\pi r}.$$

The direction is again clockwise.

IV. a) As the emf increases so does the current, and with increasing current the upward force on the loop
due to the magnetic field will increase until it counteracts the fall.

b) The change in flux is $d\Phi = Bwv\,dt$, so the induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{Bwv}{R}.$$

It's clear that this current flows counterclockwise in the loop, which means the force due to the
magnetic field points upward (the sides do not contribute to this upward force.) The terminal speed
is reached when the total force is zero, i.e.,

$$IwB = \frac{B^2 w^2 v}{R} = mg \implies v_{\text{terminal}} = \frac{mgR}{B^2 w^2}.$$

c) The loop current flows counterclockwise to counteract the increasing flux, by Lenz's law.

V. We will treat the inductor like a solenoid, i.e., the magnetic field points straight upwards. Then if we draw
a rectangular Amperian loop through the length l of the inductor and assume that it has N turns,

$$Bl = \mu_0 NI \implies B = \frac{\mu_0 NI}{l}.$$

The total flux is then

$$\Phi = \frac{\mu_0 NI}{l} \cdot N\pi \left(\frac{d}{2}\right)^2 = \frac{\pi \mu_0 d^2 N^2}{4l} I,$$

and we can read of the inductance as

$$L = \frac{\Phi}{I} = \frac{\pi \mu_0 d^2 N^2}{4l} \implies N = \sqrt{\frac{4lL}{\pi \mu_0 d^2}} = \sqrt{\frac{4(0.15\text{ m})(5.8 \times 10^{-3}\text{ H})}{\pi(4\pi \cdot 10^{-7}\text{ Wb/A} \cdot \text{m})(0.022\text{ m})^2}} = 1350.$$

VI. i) At resonance the total impedance of the circuit is $Z = R$, so the maximum current is simply $I = V/R$.
The capacitor then feels the voltage

$$V_C = \frac{I}{\omega_0 C} = \frac{V}{\omega_0 RC} \leq V_{\text{max}} \implies R \geq \frac{1}{\omega_0 C} \frac{V}{V_{\text{max}}}.$$

But the resonance frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}},$$

so that the value of the resistance must satisfy

$$R \geq \frac{V}{V_{\text{max}}} \sqrt{\frac{L}{C}} = \frac{32\text{ V}}{400\text{ V}} \sqrt{\frac{1.5\text{ H}}{250 \times 10^{-6}\text{ F}}} = 6.2\ \Omega.$$

ii) The total impedance at frequency $\omega = 2\omega_0 = 2/\sqrt{LC}$ is

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right) = R + i\frac{3}{2}\sqrt{\frac{L}{C}} \implies |Z| = \sqrt{R^2 + \frac{9L}{4C}},$$

so that

$$I_0 = \frac{V}{|Z|} = \frac{V}{\sqrt{R^2 + \frac{9L}{4C}}} = \frac{32\text{ V}}{\sqrt{(6.2\ \Omega)^2 + \frac{9(1.5\text{ H})}{4(250 \times 10^{-6}\text{ F})}}} = 0.28\text{ A}.$$

VII. We have

$$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}, \qquad o + i = d,$$

so that

$$\frac{1}{o} + \frac{1}{d - o} = \frac{1}{f} \implies o^2 - do + fd = 0.$$

Solving for o gives

$$o = \frac{d \pm \sqrt{d^2 - 4fd}}{2} = \frac{d}{2}\left(1 \pm \sqrt{1 - \frac{4f}{d}}\right) = \frac{70\text{ cm}}{2}\left(1 \pm \sqrt{1 - \frac{4(17\text{ cm})}{70\text{ cm}}}\right) = 41\text{ cm},\ 29\text{ cm}.$$

2

VIII. This is the same idea behind diffraction from a double slit. The distance from P to the two speakers is

$$\sqrt{D^2 + \left(\frac{d}{2} \pm x\right)^2} = D\sqrt{1 + \left(\frac{x \pm d/2}{D}\right)^2} \approx D\left[1 + \frac{1}{2}\left(\frac{x \pm d/2}{D}\right)^2\right] = D + \frac{(x \pm d/2)^2}{2D},$$

and the difference is dx/D, which we want to equal to $\lambda/2$ for destructive interference (and hence, silence). But $\lambda = c/f$ where c is the speed of sound, so

$$\frac{dx}{D} = \frac{c}{2f} \implies x = \frac{cD}{2fd} = \frac{(300 \text{ m/s})(10 \text{ m})}{2(3000 \text{ Hz})(1 \text{ m})} = 0.5 \text{ m}.$$

IX.   i)

$$\int_0^{1/2} N^2 \sin^2 \pi x \, dx = \frac{N^2}{2}\int_0^{1/2}(1 - \cos 2\pi x)\, dx = \frac{N^2}{4} - \frac{N^2}{4\pi}\sin 2\pi x\Big|_0^{1/2} = \frac{N^2}{4} = 1 \implies N = 2.$$

ii) Note that $\psi(x)$ is *not* a state of definite energy, and moreover that it is difficult to obtain the answer by inspection.

$$A_2 = \int_0^{1/2} \sqrt{2}\sin 2\pi x \cdot 2\sin \pi x\, dx = \sqrt{2}\int_0^{1/2} 2\sin 2\pi x \sin \pi x\, dx = \sqrt{2}\int_0^{1/2}(\cos \pi x - \cos 3\pi x)\, dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\sin \pi x - \frac{1}{3}\sin 3\pi x\right]_0^{1/2} = \frac{4\sqrt{2}}{3\pi},$$

where we have used the identity
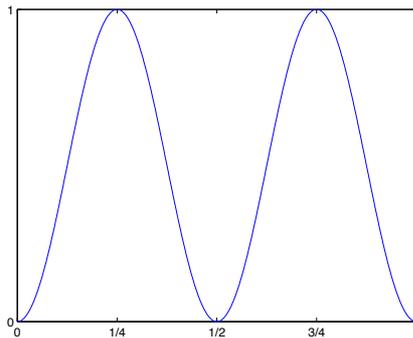
$$2\sin x \sin y = \cos(x - y) - \cos(x + y).$$

Therefore

$$P(E_2) = |A_2|^2 = \frac{32}{9\pi^2}.$$

iii)
$$\psi(x) = \sqrt{2}\sin 2\pi x.$$

The function $\sin^2 2\pi x$ is symmetric about $x = 1/2$, and each part in turn is symmetric about $x = 1/4$ and $x = 3/4$:

Therefore

$$P\left(\frac{1}{4} < x < \frac{3}{4}\right) = \frac{1}{2},$$

by inspecting the wavefunction.

iv) To absorb light from the $n = 2$ state, it must make a transition to a higher state. The next state is the $n = 3$ state, and the difference in energy is

$$E = \hbar\omega = E_3 - E_2 = \frac{9\pi^2\hbar^2}{2m} - \frac{4\pi^2\hbar^2}{2m} = \frac{5\pi^2\hbar^2}{2m} \implies \omega = \frac{5\pi^2\hbar}{2m},$$

which implies

$$f = \frac{\omega}{2\pi} = \frac{5\pi\hbar}{4m}.$$

To emit light, there is only one possible transition, to the ground state $n = 1$. Then

$$E = \hbar\omega = E_2 - E_1 = \frac{4\pi^2\hbar^2}{2m} - \frac{\pi^2\hbar^2}{2m} = \frac{3\pi^2\hbar^2}{2m} \implies \omega = \frac{3\pi^2\hbar}{2m},$$

or

$$f = \frac{3\pi\hbar}{4m}.$$

X.   i) By now we are experts at reading such a wavefunction. The possible momenta are

$$p = \pm\frac{4\pi\hbar}{L},$$

each with equal probability $1/2$.

ii) Both states correspond to the energy

$$E = \frac{p^2}{2m} = \frac{8\pi^2\hbar^2}{mL^2},$$

and this energy occurs with certainty, i.e., probability 1.

iii) Since this is a state of definite energy, we find

$$\psi(x, t) = 3\cos\frac{4\pi x}{L}\, e^{-iEt/\hbar},$$

with E as given above.