Solutions to PS 7 Physics 201

1. The impedance of the circuit is given by

$$Z(\omega) = R + \frac{1}{i\omega C} + i\omega L \tag{1}$$

$$= R + i(\omega L - \frac{1}{\omega C}). \tag{2}$$

Noting the relation between the amplitudes, $|I| = |V|/|Z|$, we have

$$\frac{|I(\omega)|}{|I_{\max}|} = \frac{|I(\omega)|}{|I(\omega_0)|} = \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}. \tag{3}$$

When $\omega = \omega_0 \pm \delta = \omega_0 \pm R/2L$, provided $\delta/\omega_0 << 1$, we have

$$\frac{|I(\omega_0 \pm \delta)|}{|I_{\max}|} = \frac{R}{\sqrt{R^2 + \{(\omega_0 \pm \delta)L - \frac{1}{(\omega_0 \pm \delta)C}\}^2}} \tag{4}$$

$$= \frac{R}{\sqrt{R^2 + \{(\omega_0 \pm \delta)L - \frac{1}{\omega_0 C}(1 \mp \frac{\delta}{\omega_0})\}^2}} \tag{5}$$

$$= \frac{R}{\sqrt{R^2 + (\pm \delta L \pm \frac{1}{\omega_0 C}\frac{\delta}{\omega_0})^2}} \tag{6}$$

$$= \frac{R}{\sqrt{R^2 + (\frac{R}{2} + \frac{R}{2})^2}} \tag{7}$$

$$= \frac{1}{\sqrt{2}}. \tag{8}$$

2. From the relation $1/Z_{//} = \sum 1/Z_i$, we get

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{\frac{1}{i\omega C}} + \frac{1}{i\omega L} \tag{9}$$

$$= \frac{1}{R} + i\omega C - \frac{i}{\omega L}, \tag{10}$$

and therefore,

$$Z = \frac{1}{\frac{1}{R} + i\omega C - \frac{i}{\omega L}} = \frac{R\omega L}{\omega L + i(\omega^2 CL - 1)R}. \tag{11}$$

3. The impedance of the circuit element shown in the figure satisfies the relation

$$\frac{1}{Z} = \frac{1}{\frac{1}{i\omega C}} + \frac{1}{R + i\omega L} \tag{12}$$
$$= i\omega C + \frac{1}{R + i\omega L} \tag{13}$$
$$= i\omega C + \frac{R - i\omega L}{R^2 + \omega^2 L^2} \tag{14}$$
$$= \frac{R + i\{(R^2 + \omega^2 L^2)\omega C - \omega L\}}{R^2 + \omega^2 L^2}. \tag{15}$$
$$\tag{16}$$

Noting that Im$[Z]$ =0 (Z is real.) $\Leftrightarrow$ Im$[1/Z]$ =0, we have

$$\mathrm{Im}[Z] = 0 \Leftrightarrow (R^2 + \omega^2 L^2)\omega C - \omega L = 0 \tag{17}$$
$$\Leftrightarrow (R^2 + \omega^2 L^2)C - L = 0, \text{or } \omega = 0 \tag{18}$$
$$\Leftrightarrow \omega = 0, \sqrt{\frac{L - CR^2}{CL^2}}. \tag{19}$$

Of course, $\sqrt{\frac{L-CR^2}{CL^2}}$ is real only if $L > CR^2$. Otherwise, the impedance is real only for $\omega = 0$ (Note that $Z = \infty$ for $\omega = 0$).

4. As seen in problem 1, the impedance is given by

$$Z(\omega) = R + \frac{1}{i\omega C} + i\omega L \tag{20}$$
$$= R + i(\omega L - \frac{1}{\omega C}). \tag{21}$$

Clearly, R_1 =100 Ω gives the minimum impedance, and R_2 =200 Ω gives the maximum impedance. Next, we have to consider the imaginary part of the impedance. For $\omega = 2000$, we get

$$\omega L_1 - \frac{1}{\omega C_1} = 2000 \text{ s}^{-1} \times 1 \text{ mH} - \frac{1}{2000 \text{ s}^{-1} \times 1\ \mu\text{F}} = -498\Omega\ , \tag{22}$$

$$\omega L_1 - \frac{1}{\omega C_2} = 2000 \text{ s}^{-1} \times 1 \text{ mH} - \frac{1}{2000 \text{ s}^{-1} \times 100\ \mu\text{F}} = -3\ \Omega, \tag{23}$$

$$\omega L_2 - \frac{1}{\omega C_1} = 2000 \text{ s}^{-1} \times 2 \text{ mH} - \frac{1}{2000 \text{ s}^{-1} \times 1\ \mu\text{F}} = -496\Omega\ , \tag{24}$$

and

$$\omega L_2 - \frac{1}{\omega C_2} = 2000 \text{ s}^{-1} \times 2 \text{ mH} - \frac{1}{2000 \text{ s}^{-1} \times 100 \ \mu\text{F}} = -1 \ \Omega. \tag{25}$$

Therefore, (R_1, C_2, L_2) gives the minimum impedance $|Z_{\min}| = \sqrt{100^2 + 1^2} \approx 100 \ \Omega$, and (R_2, C_1, L_1) gives the maximum impedance $|Z_{\max}| = \sqrt{200^2 + 498^2} \approx 537 \ \Omega$.

5. The impedance Z_2 at $\omega = 500$ is given by

$$Z_2(\omega = 500) = 15 \ \Omega + \frac{1}{i \times 500 \text{ s}^{-1} \times 2 \ \mu\text{F}} = (15 - 1000i) \ \Omega \approx 1000.1 \ e^{-1.556i} \ \Omega, \tag{26}$$

and the total impedance is

$$Z_{\text{tot}}(\omega = 500) = (25 - 1000i) \ \Omega \approx 1000.3 \ e^{-1.545i} \ \Omega. \tag{27}$$

Using these, we can calculate the power loss across Z_2. However, we have to note that $P_2 = I_2 V_2 = \text{Re}[\tilde{I}_2]\text{Re}[\tilde{V}_2] \neq Re[\tilde{I}_2\tilde{V}_2]$, where $\tilde{A}$ is the imaginary expression of A. (Operations such as derivative or integration commute with an operation of taking Re[], that is, the order of operations does not matter. Actually, this fact makes use of complex number convenient for this kind of problems. However, multiplication does not commute with Re[]. Also note that complex numbers are "imaginary" tool to make calculation easier and that physical quantities we can observe in experiments are always real.) Therefore,

$$P_2 \equiv I_2 V_2 = I(IZ_2) = (\frac{V}{Z_{\text{tot}}})\frac{VZ_2}{Z_{\text{tot}}} \tag{28}$$

$$= \text{Re}[\frac{30e^{i500t}[\text{V}]}{1000.3 \ e^{-1.545i} \ \Omega}] \ \text{Re}[\frac{30e^{i500t}[\text{V}](1000.1 \ e^{-1.556i} \ \Omega)}{1000.3 \ e^{-1.545i} \ \Omega}] \tag{29}$$

$$= 0.900 \ \text{Re}[e^{i(500t+1.545)}] \ \text{Re}[e^{i(500t-0.011)}] \ [\text{W}] \tag{30}$$

$$= 0.900 \ \cos(500t + 1.545) \ \cos(500t - 0.011) \ [\text{W}] \tag{31}$$

$$= 0.450 \ \{\cos(1000t + 1.534) + \cos 1.556\} \ [\text{W}] \tag{32}$$

$$\tag{33}$$

Also from this, we can easily calculate

$$(\text{Time average of power loss}) = 0.450 \ \cos 1.556 = 6.66 \text{ mW}. \tag{34}$$

6. The electric field between the plates is

$$\mathbf{E}(r) = \begin{cases} \frac{V(t)}{d} & (r < a) \\ 0 & (r > a), \end{cases} \tag{35}$$

where $d = 2$ cm is the separation between the plates and $a = 4$ cm is a radius of the plates. Noting that the capacitance has rotation symmetry about the central axis, we have from Maxwell equation,

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2\pi r B_\theta(r) = \epsilon_0\mu_0 \int d\mathbf{S}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2}\int d\mathbf{S}\frac{\partial \mathbf{E}}{\partial t}. \tag{36}$$

Therefore,

$$\mathbf{B}(r) = \begin{cases} \frac{r}{2c^2 d}\frac{dV(t)}{dt}\mathbf{e}_\theta & (r < a) \\ \frac{a^2}{2c^2 rd}\frac{dV(t)}{dt}\mathbf{e}_\theta & (r > a), \end{cases} \tag{37}$$

where $\frac{dV(t)}{dt} = (-200\pi \times 200 \sin 200\pi t)$V/s, whose amplitude is 40000π V/s. B reaches its maximum at $r = a$. With actual numbers plugged in,

$$|B_{\max}| = \frac{a}{2c^2 d}|\frac{dV(t)}{dt}| \tag{38}$$

$$= \frac{2\text{cm}}{2 \times (3 \times 10^8 \text{m/s})^2 \times 4\text{cm}} \times 40000\pi\text{V/s} \tag{39}$$

$$= 1.11 \times 10^{-13} \text{ T}. \tag{40}$$