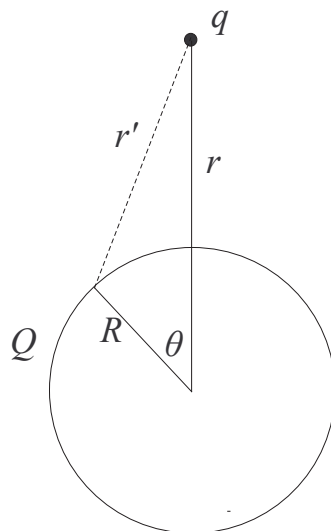## Solutions to PS 4 Physics 201

1. (a) Below is a labelled diagram of the situation. We place the center of the sphere which we are averaging over at the origin.

The potential due to the point charge q at a point on the surface of the sphere is given by

$$V(\mathbf{R}) = \frac{k_e q}{r'} \tag{1}$$

By the law of cosines, we can write

$$r'^2 = r^2 + R^2 - 2rR\cos\theta \tag{2}$$

$$r' = \sqrt{r^2 + R^2 - 2rR\cos\theta} \tag{3}$$

Using this, we can find the average of V over the sphere using the integral given

$$\bar{V} = \frac{1}{4\pi R^2} \int_0^\pi \int_0^{2\pi} \frac{k_e q}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} R^2 \sin\theta \, d\phi \, d\theta \tag{4}$$

$$= \frac{1}{2} \int_0^\pi \frac{k_e q}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} \sin\theta \, d\theta \tag{5}$$

$$= \frac{1}{2} \int_{-1}^1 \frac{k_e q}{\sqrt{r^2 + R^2 - 2rRx}} \, dx \tag{6}$$

$$\tag{7}$$

Next, using the substitution

$$u = r^2 + R^2 - 2rRx \tag{8}$$

$$du = -2rRdx \tag{9}$$

we find

$$\bar{V} = -\frac{1}{4rR} \int_{(R+r)^2}^{(r-R)^2} \frac{k_e q}{\sqrt{u}} du \tag{10}$$

$$= \frac{k_e q}{2rR} [(R+r) - (r-R)] \tag{11}$$

$$= \frac{k_e q}{r} = V(\mathbf{0}) \tag{12}$$

But $V(\mathbf{0})$ is exactly the potential due to the point charge evaluated at the center of the sphere, as desired.

(b) Assuming the sphere is fixed, the energy needed to bring the point charge to a distance r from the sphere is simply

$$U = qV_{sphere}(\mathbf{r}) \tag{13}$$

But since the charge Q is glued uniformly to the sphere, we know that $V_{sphere}(r)$ is simply the same as the potential due to a point charge, and thus

$$U = \frac{k_e qQ}{r} \tag{14}$$

$$= QV(0) \tag{15}$$

On the other hand, we can consider the point charge fixed, in which case the energy is given by

$$U = \int \frac{k_e q dQ}{r'} \tag{16}$$

where r' is the distance between the point charge q and an infinitesimal element dQ of charge on the surface of the sphere. However, because the charge is distributed uniformly, we have

$$dQ = \frac{Q}{4\pi R^2} R^2 \sin\theta d\theta d\phi \tag{17}$$

and thus we find

$$U = \frac{1}{4\pi R^2} \int \frac{k_e Qq}{r'} R^2 \sin\theta d\theta d\phi \tag{18}$$

$$= Q\bar{V} \tag{19}$$

Thus, equating our two expressions for U, we find the desired result

$$V(0) = \bar{V} \tag{20}$$

(c) Assume that $V(\mathbf{r})$ has a maximum at $\mathbf{r} = \mathbf{r}_0$. Then, by the definition of a maximum, the value of V on the surface of a sphere of radius ϵ around $\mathbf{r}_0$ must be strictly less than $V(\mathbf{r}_0)$. But that means that the average $\bar{V}$ of V over this sphere is also strictly less than $V(\mathbf{r}_0)$. But we have just shown that $\bar{V}$ must equal $V(\mathbf{r}_0)$ in a charge free region. Thus, we have a contradiction, and V cannot have a maximum at $\mathbf{r}_0$. The same argument also holds for minima of V, in which case we find $\bar{V} > V(\mathbf{r}_0)$.

(d) Let $\mathbf{r}_0$ be a point of stable equilibrium - i.e. a minimum - of V in a charge-free region. Then for any sufficiently small number ϵ, and any unit vector $\hat{\mathbf{n}}$ we must have that
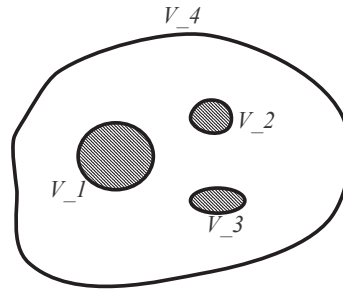
$$V(\mathbf{r}_0 + \epsilon\hat{\mathbf{n}}) > V(\mathbf{r}_0) \tag{21}$$

in particular, this implies that the derivative of V is positive at all points in a sphere of radius ϵ of $\mathbf{r}_0$, which implies that the electric field points towards $\mathbf{r_0}$ for all points on the boundary of that sphere. Thus, the electric flux through that sphere must be strictly less than 0. But by Gauss' law, this implies that there must be negative charge within the sphere, which is a contradiction, and thus no such minimum can exist. For a maximum of V, we find that the electric field must point away from $\mathbf{r_0}$ on the surface of the sphere, leading to a net positive flux and thus the presence of a positive charge, which is still a contradiction

2. Assume $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ satisfy the same boundary conditions at each of the conducting surfaces, as shown below.

Then at each of these surfaces, the function

$$W(\mathbf{r}) = V_1(\mathbf{r}) - V_2(\mathbf{r}) \tag{22}$$

must vanish. Also note that as the superposition of two potential functions, W itself is also a valid electrical potential. Now, assume $W(\mathbf{r})$ is nonzero at some point $\mathbf{r}_0$ in the region between the conductors. Then, since W vanishes on the outer conducting surface, it follows from the continuity of W that it must attain either a maximum or a minimum value in the region free of conductors (at least as large in magnitude as $W(\mathbf{r}_0)$). But this contradicts the result derived in Problem 1 that the potential cannot have a maximum or a minimum in a charge free region. Thus, W must vanish everywhere, and thus

$$V_1(\mathbf{r}) = V_2(\mathbf{r}) \tag{23}$$

at all points.

3. We know that for the circuit described,

$$I(t) = \frac{\mathcal{E}}{R} e^{-\frac{t}{RC}} \tag{24}$$

$$Q(t) = C\mathcal{E} \left(1 - e^{-\frac{t}{RC}}\right) \tag{25}$$

Next, we know that the time-rate change of the work done by the battery - its output power - is given by

$$P_{bat} = I\mathcal{E} = \frac{\mathcal{E}^2}{R} e^{-\frac{t}{RC}} \tag{26}$$

From Ohm's law, we also have that the power dissipated by the resistor is given by

$$P_{res} = I^2 R = \frac{\mathcal{E}^2}{R} e^{-\frac{2t}{RC}} \tag{27}$$

Finally, the rate of change of energy stored in the capacitor is given by

$$P_{cap} = \frac{d}{dt}\left(\frac{Q^2}{2C}\right) \tag{28}$$

$$= \frac{d}{dt}\left[\frac{C\mathcal{E}^2}{2}\left(1 - 2e^{-\frac{t}{RC}} + e^{-\frac{2t}{RC}}\right)\right] \tag{29}$$

$$= \frac{\mathcal{E}^2}{R}e^{-\frac{t}{RC}} - \frac{\mathcal{E}^2}{R}e^{-\frac{2t}{RC}} \tag{30}$$
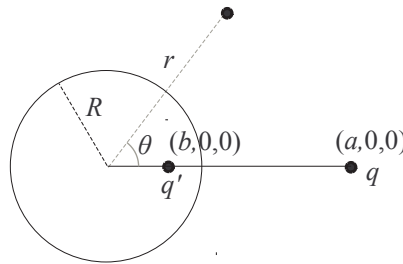
$$= P_{bat} - P_{res} \tag{31}$$

Thus, we have that

$$P_{bat} = P_{cap} + P_{res} \tag{32}$$

which, upon integrating, gives

$$W_{bat}(t) = E_{cap}(t) + E_{res}(t) \tag{33}$$

as desired.

4. (a) We demand that $V = 0$ at the point on the sphere closest to the point charge, and at the point farthest from the point charge.

Examining the figure above, we see that closest to the point charge, we have for the potential

$$V = 0 = \frac{k_e q}{a - R} + \frac{k_e q'}{R - b} \tag{34}$$

$$-q'(a - R) = q(R - b) \tag{35}$$

Farthest from the point charge, we have

$$V = 0 = \frac{k_e q}{a + R} + \frac{k_e q'}{R + b} \tag{36}$$

$$-q'(a + R) = q(R + b) \tag{37}$$

Adding these two equations, we find

$$-2q'a = 2qR \tag{38}$$

$$q' = -q\frac{R}{a} \tag{39}$$

As desired. Substituting this result back into the second equation yields

$$q\frac{R}{a}(a + R) = q(R + b) \tag{40}$$

$$q\frac{R^2}{a} = qb \tag{41}$$

$$b = \frac{R^2}{a} \tag{42}$$

Using the law of cosines, we can show now that these values of q' and b make the potential vanish over the whole sphere. At any point on the sphere we have

$$V = \frac{k_e q}{x} + \frac{k_e q'}{y} \tag{43}$$

$$= \frac{k_e q}{x} - \frac{k_e q R}{ay} \tag{44}$$

From the diagram given, we see that the law of cosines gives

$$x = \sqrt{a^2 + R^2 - 2aR\cos\theta} \tag{45}$$

and

$$y = \sqrt{R^2 + b^2 - 2Rb\cos\theta} \tag{46}$$

Plugging these in to the expression for V, we get

$$V = \frac{k_e q}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{k_e q R}{a\sqrt{R^2 + b^2 - 2Rb\cos\theta}} \tag{47}$$

$$= k_e q \left(\frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{1}{\frac{a}{R}\sqrt{R^2 + \frac{R^4}{a^2} - 2Rb\cos\theta}} \right) \tag{48}$$

$$= k_e q \left(\frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} - \frac{1}{\sqrt{a^2 + R^2 - 2aR\cos\theta}} \right) \tag{49}$$

$$= 0 \tag{50}$$

(b) To find the surface charge density, we start by using the image charge method to find the potential everywhere outside the sphere. Since the sphere is conducting,

we also have that the potential is constant inside the sphere. Differentiating the potential, we can get the electric field everywhere outside the sphere. Finally, since the sphere is a conductor, we know the field inside is zero. Thus, using the relationship between change in electric field across a boundary and the surface charge density on that boundary derived from Gauss' law, we have

$$\sigma(\mathbf{R}) = \epsilon_0 \mathbf{E}(\mathbf{R}) \cdot \mathbf{e}_r \tag{51}$$

where $\mathbf{R}$ is a vector from the center of the sphere to a point on its surface, and $\mathbf{e}_r$ is a unit vector pointing outward from the sphere.

By looking at a sphere surrounding the conductor, Gauss' law tells us that the total induced charge on the sphere needs to be the same as the image charge q'.

(c) The force between the sphere and q is just the force of attraction between the image charge and q. Thus, we have

$$\mathbf{F} = \frac{k_e q q'}{(a-b)^2} \mathbf{e}_r \tag{52}$$

$$= -\frac{k_e q^2 R}{a \left(a - \frac{R^2}{a} \right)^2} \mathbf{e}_r \tag{53}$$

$$= -\frac{k_e q^2 R a}{(a^2 - R^2)^2} \mathbf{e}_r \tag{54}$$

(d) To solve the case of an isolated, neutral sphere, we need to balance out the induced charge of $q' = -\frac{R}{a} q$ on the sphere while still maintaining the sphere as an equipotential surface. This can be done by adding a second image charge of $q'' = \frac{R}{a} q$ at the center of the sphere. Doing this, we find the force on the charge q to be

$$\mathbf{F} = \mathbf{F}_{grounded} + \frac{k_e q q''}{a} \mathbf{e}_r \tag{55}$$

$$= \left(\frac{k_e q^2 R}{a^3} - \frac{k_e q^2 R a}{(a^2 - R^2)^2} \right) \mathbf{e}_r \tag{56}$$

(e) Similar to above, if the sphere is isolated with initial charge Q, we can add a second image charge at the center of the sphere of magnitude $q'' = \frac{R}{a} q + Q$. This

yields a force on the point charge of

$$\mathbf{F} = \mathbf{F}_{grounded} + \frac{k_e q q''}{a}\mathbf{e}_r \tag{57}$$

$$= \left(\frac{k_e q \left(q\frac{R}{a} + Q\right)}{a^2} - \frac{k_e q^2 R a}{(a^2 - R^2)^2}\right)\mathbf{e}_r \tag{58}$$

5. Equating the centripetal force with the magnetic force, we find

$$\frac{mv^2}{r} = qvB \tag{59}$$

$$\frac{v}{r} = \frac{qB}{m} \tag{60}$$

$$\omega = \frac{qB}{m} \tag{61}$$

$$\frac{2\pi}{T} = \frac{qB}{m} \tag{62}$$

where ω is the angular frequency and T is the period. Solving for B we get

$$B = \frac{2\pi m}{qT} \tag{63}$$

letting q be the charge of an electron, m be the mass of an electron, and $T = 1\mu s$, we find

$$B = 3.57 \times 10^{-5} Tesla \tag{64}$$

6. We label the currents flowing through each element as shown in the figure below.

We assume there is a current I flowing into terminal A, and thus a current I flowing out of terminal B. We can see that if we instead were to let the current flow from B to A, that it would face the same resistor paths. This symmetry implies that
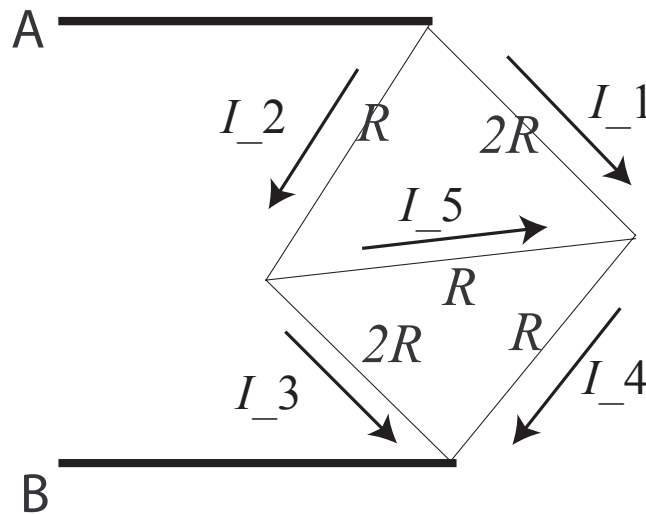
$$I_2 = I_4 \tag{65}$$

$$I_1 = I_3 \tag{66}$$

Also, by current conservation, we need

$$I = I_1 + I_2 \tag{67}$$

$$I_5 = I_2 - I_3 = I_2 - I_1 \tag{68}$$

A

I_2 R $2R$ I_1

I_5

R

$2R$ R

I_3 I_4

B

This means that we can write

$$I_2 = xI \qquad (69)$$

$$I_1 = (1-x)I \qquad (70)$$

$$I_5 = (2x-1)I \qquad (71)$$

where x is some number between zero and one. Since the total voltage around any closed loop is zero, we need

$$RI_2 + RI_5 = 2RI_1 \qquad (72)$$

$$3x - 1 = 2 - 2x \qquad (73)$$

$$x = \frac{3}{5} \qquad (74)$$

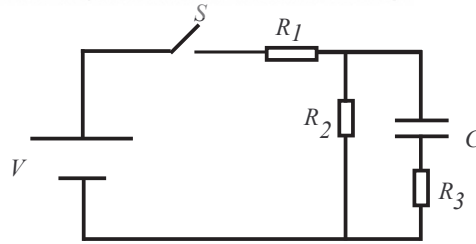Thus, the voltage drop between A and B must be

$$V = 2RI_1 + RI_4 \qquad (75)$$

$$= \frac{4}{5}RI + \frac{3}{5}RI \qquad (76)$$

$$= \frac{7}{5}RI \qquad (77)$$

From which we see that the equivalent resistance is

$$R_{eq} = \frac{7}{5}R \qquad (78)$$

7. (a) For convenience, the diagram for this circuit has been reproduced below.

If the switch S has been closed for a very long time, then we know the capacitor will be fully charged, and thus the current flowing through it must be zero.

(b) Since no current is flowing through C, no current can flow through R_3, and thus the resistors R_1 and R_2 are effectively connected in series. Thus, the current flowing through them is given by

$$I = \frac{V}{R_{eq}} = \frac{V}{R_1 + R_2} \tag{79}$$

(c) We know that the voltage on C must be the same as the voltage drop across the resistor R_2. Thus,

$$V_C = V - IR_1 \tag{80}$$

$$= V - \frac{R_1 V}{R_1 + R_2} \tag{81}$$

$$= \frac{R_2 V}{R_1 + R_2} \tag{82}$$

The charge on the capacitor is then

$$Q = CV_C = \frac{CVR_2}{R_1 + R_2} \tag{83}$$

(d) When the switch S is opened, the only complete circuit has the capacitor C connected in series with resistors R_2 and R_3. Thus, the capacitor with discharge through an equivalent resistance of $R_{eq} = R_2 + R_3$.

(e) Based on the above, we have that the time constant for the decay is given by

$$\tau = R_{eq}C = C(R_2 + R_3) \tag{84}$$

(f) Putting in the numbers, we find that the current flowing through R_1 and R_2 with the switch closed is given by

$$I = \frac{9V}{9k\Omega} = 1mA \tag{85}$$

Similarly, the charge on the capacitor is given by

$$Q = \frac{(100\mu F)(9V)(5k\Omega)}{9k\Omega} = 500\mu C \tag{86}$$

And lastly, the time constant is given by

$$\tau = (100\mu F)(6k\Omega) = 0.6s \tag{87}$$

8. Consider a Gaussian surface between the two conductors that completely encloses the inner conductor. We have by Ohm's law that the current flowing across this surface is
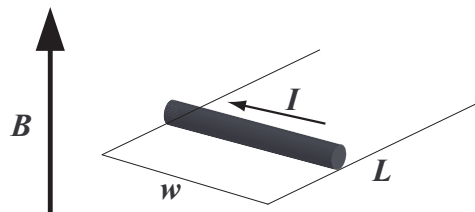
$$I = \int \mathbf{J} \cdot d\mathbf{A} \tag{88}$$

$$= \int \sigma \mathbf{E} \cdot d\mathbf{A} \tag{89}$$

$$= \sigma \frac{Q}{\epsilon_0} \tag{90}$$

$$= \frac{\sigma C V_0}{\epsilon_0} \tag{91}$$

Where Q is the charge on the inner conductor, and we have used $Q = CV_0$ by the definition of capacitance

9. We solve this problem by conservation of energy.

From the figure above, it can be seen that magnitude of the magnetic force on the cylinder is given by

$$F = IwB \tag{92}$$

directed down the rods. Thus, at the time the cylinder exits, the magnetic force has done work

$$W = ILwB \tag{93}$$

This work is equal to the change in kinetic energy of the cylinder. Since the cylinder begins at rest, its initial kinetic energy is 0, and thus

$$ILwB = K_f = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \tag{94}$$

Letting R denote the radius of the cylinder, its moment of inertia is given by

$$I = \frac{1}{2}MR^2 \tag{95}$$

while the rolling without slipping condition gives

$$\omega = \frac{v}{R} \tag{96}$$

Therefore

$$ILwB = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2 \tag{97}$$

$$v = \sqrt{\frac{4BLIw}{3M}} \tag{98}$$