Solutions to Midterm Physics 201

1. We can consider this situation as a superposition of a uniformly charged sphere of charge density ρ and radius R, and a second uniformly charged sphere of charge density $-\rho$ and radius $\frac{R}{2}$ at the position of the cavity. At the center of the big sphere, Gauss' law tells us that the field due to the large positively charged sphere will be zero, and thus the only contribution to the field comes from the negatively charged sphere. Calling up the $+y$ direction, we thus get

$$\mathbf{E} = -\frac{Q_-}{4\pi\epsilon_0\left(\frac{R}{2}\right)^2}\mathbf{e}_y \tag{1}$$

$$= \frac{\frac{16}{3}\pi\left(\frac{R}{2}\right)^3\rho}{4\pi\epsilon_0 R^2}\mathbf{e}_y \tag{2}$$

$$= \frac{\rho R}{6\epsilon_0}\mathbf{e}_y \tag{3}$$

2. At the time the particles enter the magnetic field, they have energy

$$E = qV \tag{4}$$

and thus momentum

$$mv = \sqrt{2mE} = \sqrt{2mqV} \tag{5}$$

Once in the magnetic field, the particles travel in circular orbits of radius $r = \frac{d}{2}$. Equating the magnetic force with the total centripetal force thus gives

$$\frac{mv^2}{r} = qvB \tag{6}$$

$$\frac{mv}{r} = qB \tag{7}$$

$$r = \frac{mv}{qB} \tag{8}$$

$$r = \frac{\sqrt{2mqV}}{qB} \tag{9}$$

$$d = \frac{2}{B}\sqrt{\frac{2mV}{q}} \tag{10}$$

3. We know that the potential V_b at the surface of the inner sphere is given (up to a constant) by

$$V_b = \frac{Q}{4\pi\epsilon_0 b} \tag{11}$$

and the potential V_a at the surface of the outer sphere by

$$V_a = \frac{Q}{4\pi\epsilon_0 a} \tag{12}$$

From the definition of capacitance, we find

$$\begin{aligned} C &= \frac{Q}{V} && (13) \\ &= \frac{Q}{V_b - V_a} && (14) \\ &= 4\pi\epsilon_0 \frac{1}{\frac{1}{b} - \frac{1}{a}} && (15) \\ &= 4\pi\epsilon_0 \frac{ab}{a-b} && (16) \end{aligned}$$

The total energy stored in the capacitor is given by

$$\begin{aligned} U &= \frac{\epsilon_0}{2} \int E^2 d^3\mathbf{r} && (17) \\ &= \frac{4\pi\epsilon_0}{2} \int_b^a \frac{Q^2}{(4\pi\epsilon_0)^2 r^4} r^2 dr && (18) \\ &= \frac{Q^2}{8\pi\epsilon_0} \int_b^a \frac{1}{r^2} dr && (19) \\ &= \frac{Q^2}{8\pi\epsilon_0} \frac{a-b}{ab} && (20) \end{aligned}$$

But we also have

$$\begin{aligned} U &= \frac{Q^2}{2C} && (21) \\ &= \frac{Q^2}{8\pi\epsilon_0} \frac{a-b}{ab} && (22) \end{aligned}$$

Thus we see explicitly that the two formulas for energy give the same result.

4. We can consider this wire as a superposition of an infinitely long straight wire with a circular wire. At the center of the circle, Ampere's law tells us that the magnetic field $\mathbf{B}_1$ due to the infinite wire is directed out of the page and has magnitude

$$B_1 = \frac{\mu_0 I}{2\pi a} \tag{23}$$

Similarly, the Biot-Savart law tells us that the field $\mathbf{B}_2$ from the circular wire is directed out of the page, and has magnitude

$$B_2 = \frac{\mu_0 I}{4\pi} \int \frac{dl}{r^2} \tag{24}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a d\theta}{a^2} \tag{25}$$

$$= \frac{\mu_0 I}{2a} \tag{26}$$

The total field is given by the sum $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, and therefore has magnitude

$$B = \frac{\mu_0 I}{2a} \left(1 + \frac{1}{\pi}\right) \tag{27}$$

and is directed out of the page.

5. The magnetic moment of the current loop has magnitude

$$\mu = ILw \tag{28}$$

and is pointed downward. Thus, the torque on the loop due to the magnetic field is directed towards point A and has magnitude

$$\tau_B = ILwB \tag{29}$$

In equilibrium, this must equal $\tau_m = mgw$, the magnitude of the torque due to gravity. Equating and solving for B, we find

$$B = \frac{mg}{IL} \tag{30}$$

6. Amprere's law tells us that at a horizontal distance x from the wire, the magnetic field has magnitude

$$B(x) = \frac{\mu_0 I}{2\pi x} \tag{31}$$

and is directed into the page. The magnetic flux through the loop thus has magnitude

$$\Phi = \int_a^{a+w} \int_0^L B(x) dy dx \tag{32}$$

$$= \frac{\mu_0 IL}{2\pi} \int_a^{a+w} \frac{1}{x} dx \tag{33}$$

$$= \frac{\mu_0 IL}{2\pi} \ln\left(1 + \frac{w}{a}\right) \tag{34}$$

and is directed into the page.

If $I(t) = I_0 \cos \omega t$, then the magnitude of the EMF $\mathcal{E}$ induced in the loop is given by

$$|\mathcal{E}| = \left| \frac{d\Phi}{dt} \right| \tag{35}$$

$$= \frac{\mu_0 L I_0 \omega}{2\pi} \ln\left(1 + \frac{w}{a}\right) |\sin \omega t| \tag{36}$$

which has a maximum value of

$$\mathcal{E}_{max} = \frac{\mu_0 L I_0 \omega}{2\pi} \ln\left(1 + \frac{w}{a}\right) \tag{37}$$

Finally, the mutual inductance M is given by

$$M = \frac{\Phi}{I} = \frac{\mu_0 L}{2\pi} \ln\left(1 + \frac{w}{a}\right) \tag{38}$$

7. Using the Biot-Savart law, we get that the field due to the two straight segments is zero. For the segment of radius a, we find

$$B_a = \frac{\mu_0 I}{4\pi} \int \frac{dl}{r^2} \tag{39}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{\pi} \frac{a d\theta}{a} \tag{40}$$

$$= \frac{\mu_0 I}{4a} \tag{41}$$

directed into the page. An identical integration gives

$$B_b = \frac{\mu_0 I}{4b} \tag{42}$$

directed out of the page. Summing together these contributions, we find that the total field is thus directed into the page and has magnitude

$$B = \frac{\mu_0 I(b-a)}{4ab} \tag{43}$$

With the numbers given,

$$B = \frac{\mu_0 \times 2A}{8} = 3.14 \times 10^{-7} T \tag{44}$$

8. The energy dissipated by the resistor is, by conservation of energy, equal to the difference between the energy E_i stored in the capacitors in the initial configuration and the energy E_f stored in the final configuration. Initially, all charge is on the capacitor C_1, and thus

$$E_i = \frac{1}{2}C_1V_0^2 \tag{45}$$

To find E_f, we use the fact that current ceases to flow when the voltage across both capacitors are equal. By conservation of charge, this occurs via a transfer of charge q from capacitor C_1 to capacitor C_2 Since C_2 is initially uncharged, it will have a final charge of q, while C_1 will have a final charge of

$$C_1V_0 - q \tag{46}$$

Equating the voltages, we find

$$\begin{aligned}\frac{C_1V_0 - q}{C_1} &= \frac{q}{C_2} && (47)\\ V_0 &= q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) && (48)\\ q &= \frac{V_0C_1C_2}{C_1 + C_2} && (49)\end{aligned}$$

Thus, the final voltage across C_2 is given by

$$\begin{aligned}V_{2f} &= \frac{q}{C_2} && (50)\\ &= \frac{V_0C_1}{C_1 + C_2} && (51)\end{aligned}$$

which, as we know, also equals the final voltage across C_1. Thus,

$$\begin{aligned}E_f &= \frac{1}{2}(C_1 + C_2)V_{2f}^2 && (52)\\ &= \frac{1}{2}(C_1 + C_2)\left(\frac{V_0^2C_1^2}{(C_1 + C_2)^2}\right) && (53)\\ &= \frac{1}{2}V_0^2\frac{C_1^2}{C_1 + C_2} && (54)\end{aligned}$$

Thus, the resistor dissipates

$$W = E_i - E_f = \frac{1}{2}C_1V_0^2 - \frac{1}{2}V_0^2\frac{C_1^2}{C_1 + C_2} \tag{55}$$

$$= \frac{1}{2}V_0^2\left(C_1 - \frac{C_1^2}{C_1 + C_2}\right) \tag{56}$$

$$= \frac{1}{2}V_0^2\frac{C_1C_2}{C_1 + C_2} \tag{57}$$

$$= \frac{V_0^2C_1C_2}{2(C_1 + C_2)} \tag{58}$$

as desired.

Note that we can find E_f an alternate way, by realizing that since the voltage on the two capacitors is the same after they discharge, their equivalent capacitance is

$$C_{eq} = C_1 + C_2 \tag{59}$$

and thus by charge conservation

$$E_f = \frac{Q^2}{2C_{eq}} \tag{60}$$

$$= \frac{V_0^2C_1^2}{2(C_1 + C_2)} \tag{61}$$

9. When the switch has been closed for a long time, the current is no longer changing and thus the inductor acts as a short circuit. Thus, no current flows through R_2 and the current leaving the battery is given by

$$I_{bat} = \frac{V_0}{R_1} \tag{62}$$

which is also the current through the inductor.

If the switch is suddenly open, the only complete circuit is the loop containing only the inductor and R_2. Since the sum of the voltage around this loop must be zero, we find

$$-L\frac{dI}{dt} - R_2I = 0 \tag{63}$$

Solving this equation, we find

$$I(t) = I_0e^{-\frac{R_2}{L}t} \tag{64}$$

Since the initial current is $I_0 = I_{bat}$, we get

$$I(t) = \frac{V_0}{R_1} e^{-\frac{R_2}{L}t} \tag{65}$$

The initial voltage across the inductor is then

$$V(0) = -L\frac{dI(0)}{dt} = \frac{R_2}{R_1}V_0 \tag{66}$$

For this to be less that $10V_0$, we must have

$$\frac{R_2}{R_1} < 10 \tag{67}$$