Solutions to PS 6 Physics 401a

1. Let $\mathbf{B}$ point along the z axis. Then by circuar symmetry, we have

$$\mathbf{B}(r,t) = B(r,t)\mathbf{e}_z \tag{1}$$

Using Faraday's Law, we can find the electric field at radius r to be

$$\oint \mathbf{E}(\mathbf{r},\mathbf{t}) \cdot d\mathbf{l} = -\frac{d}{dt}\mathbf{B}(r,t) \cdot d\mathbf{A} \tag{2}$$

$$2\pi r E_\phi(r,t) = -\frac{d}{dt}\int_0^{2\pi}\int_0^r B(r,t) r dr d\theta \tag{3}$$

Since the integral on the right hand side is just the integral of the magnetic field over the while circle of radius r, we can replace it by the average value $\bar{B}(t)$ of the field times the area of the circle. Thus

$$2\pi r E_\phi(r,t) = -\frac{d}{dt}(\bar{B}(t)\pi r^2) \tag{4}$$

$$E_\phi(r,t) = -\frac{r}{2}\frac{d\bar{B}(t)}{dt} \tag{5}$$

$$\mathbf{E}(r,t) = -\frac{r}{2}\frac{d\bar{B}(t)}{dt}\mathbf{e}_\phi \tag{6}$$

Using the relativistically invariant Lorentz force law, we se

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = -q\left(\mathbf{E}(r,t) + \mathbf{v} \times \mathbf{B}(r,t)\right) \tag{7}$$

$$= -\frac{qr}{2}\frac{d(\bar{B}(t)}{dt}\mathbf{e}_\phi + q\left(\mathbf{v} \times \mathbf{e}_z\right)B(r,t) \tag{8}$$

To get the tangential component p_T of the momentum, we dot both sides of the Eq. (8) by the unit vector $\mathbf{e}_\phi$, which yields

$$\frac{dp_T}{dt} = -\frac{qr}{2}\frac{d(\bar{B}(t)}{dt} + q\mathbf{e}_\phi \cdot \left(\mathbf{v} \times \mathbf{e}_z\right)B(r,t)) \tag{9}$$

Since we are looking for solutions describing circular orbits, we must have that $\mathbf{v} = v\mathbf{e}_\phi$, and thus the second term in the above equation vanishes. Thus,

$$\frac{dp_T}{dt} = -\frac{qr}{2}\frac{d\bar{B}(t)}{dt} \tag{10}$$

$$p_T(t) = -\frac{qr}{2}\bar{B}(t) \tag{11}$$

and thus the magnitude of the tangential momentum is given by $\frac{qr}{2}\bar{B}(t)$.

To find the available radial force, we now dot both sides of Eq. (8) by the unit vector $\mathbf{e}_r$. This gives

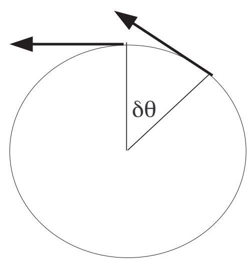
$$F_r = -\frac{qr}{2}\frac{d\bar{B}(t)}{dt}\mathbf{e}_\phi \cdot \mathbf{e}_r + q\left(\mathbf{v} \times \mathbf{e}_z\right) \cdot \mathbf{e}_r B(r,t) \tag{12}$$

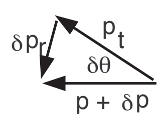
$$= q\mathbf{v} \cdot \left(\mathbf{e}_z \times \mathbf{e}_r\right) B(r,t) \tag{13}$$

$$= -q\mathbf{v} \cdot \mathbf{e}_\phi B(r,t) \tag{14}$$

$$= -qvB_0 \tag{15}$$

Thus the available radial force is qvB_0 pointing towards the center of the circle.





We now want to derive an expression for the rate of change of the radial momentum. Examining the figures above, we see that

$$\sin\delta\theta \approx \delta\theta = \frac{\delta p_r}{p_T} \tag{16}$$

$$\delta p_r = p_T \delta\theta \tag{17}$$

but $\delta\theta$ is the angle traversed by the particle in some infinitesimally small time interval δt, which by definition is equal to

$$\delta\theta = \omega\delta t \tag{18}$$

Thus, we have that

$$\delta p_r = p_T \omega \delta t \tag{19}$$
$$\frac{\delta p_r}{\delta t} = \omega p_T \tag{20}$$
$$\frac{dp_r}{dt} = \omega p_T \tag{21}$$

Combining this with our expressions for p_T and F_r, we find

$$\omega p_T = qvB_0 \tag{22}$$
$$\omega \frac{qr}{2}\bar{B}(t) = qvB_0 \tag{23}$$
$$\left(\frac{v}{r}\right)\frac{qr}{2}\bar{B}(t) = qvB_0 \tag{24}$$
$$\frac{\bar{B}(t)}{2} = B_0 \tag{25}$$
$$\bar{B}(t) = 2B_0 \tag{26}$$

as expected. Note that nowhere in this argument did we need to use the explicit form for the relativistically correct momentum.

2. Let a current I flow through the larger loop. Since $R_1 \ll R_2$, we can assume that the field within the smaller loop is constant, and equal to its value at the center. By the Biot-Savart law, we can compute its magnitude as

$$B = \frac{\mu_0 I}{4\pi}\int \frac{dl}{r^2} \tag{27}$$
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{R_2 d\theta}{R_2^2} \tag{28}$$
$$= \frac{\mu_0 I}{2R_2} \tag{29}$$

Thus, the flux through the inner loop is given by

$$\Phi = BA = \frac{\mu_0 I \pi R_1^2}{2R_2} \tag{30}$$

And therefore the mutual inductance is,

$$M = \frac{\Phi}{I} = \frac{\mu_0 \pi R_1^2}{2R_2} \tag{31}$$

3. Let y be the coordinate of the lower end of the loop, and define $y = 0$ to be the point where it initially enters the magnetic field. Then the magnitude of the EMF through the loop is given by

$$|\mathcal{E}| = \left|\frac{d\Phi}{dt}\right| \tag{32}$$

$$= \frac{d}{dt}(Bwy) \tag{33}$$

$$= Bwv \tag{34}$$

And thus the current through the loop is given by

$$I = \frac{Bwv}{R} \tag{35}$$

By Lenz's law, we know that this current must be flowing counterclockwise through the loop to oppose the increasing flux into the page. Thus, the magnetic force on the loop is given by

$$\mathbf{F}_b = -IwB\mathbf{e}_y \tag{36}$$

$$= -\frac{B^2w^2}{R}v\mathbf{e}_y \tag{37}$$

where we have defined down to be the positive y direction. At terminal velocity, we know that the acceleration on the loop is zero, and thus the magnetic force exactly balances the gravitational force. Thus

$$Mg - \frac{B^2w^2}{R}v_T = 0 \tag{38}$$

$$v_T = \frac{MgR}{B^2w^2} \tag{39}$$

4. With $B(t) = B_0t$, we have that the magnitude of the EMF through the loop is given by

$$|\mathcal{E}| = \left|\frac{d\Phi}{dt}\right| \tag{40}$$

$$= \frac{d}{dt}(\pi A^2 B_0 t) \tag{41}$$

$$= \pi A^2 B_0 \tag{42}$$

The maximum charge on the capacitor is given when the voltage across the capacitor exactly balances the EMF, and thus

$$V_{cap} = |\mathcal{E}| = \pi A^2 B_0 \tag{43}$$

and finally

$$Q = CV_{cap} = \pi A^2 B_0 C \tag{44}$$

The orentiation of the charge can be seen in the figure below, via Lenz's law.

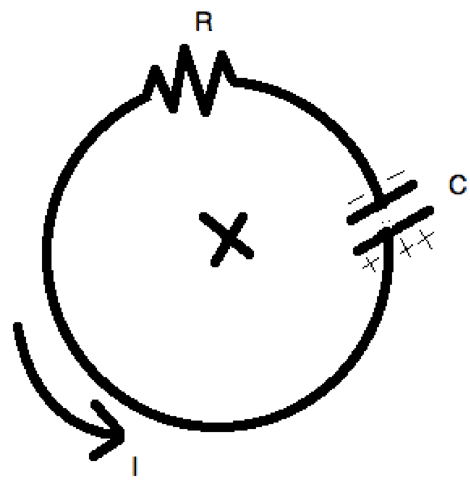


FIG. 1: The X indicates the direction of increasing magnetic flux, and thus by Lenz's law current must flow counterclockwise through the loop, depositing positive charge on the bottom end of the capacitor.

5. (a) Taking the $+x$ direction to be away from the battery, we have that the net EMF in the loop is given by

$$\mathcal{E} = V - \frac{d\Phi}{dt} \tag{45}$$

$$= V - \frac{d}{dt}(Bwx) \tag{46}$$

$$= V - Bwv \tag{47}$$

(b) The magnetic force is

$$\mathbf{F} = IwB\mathbf{e}_x = \frac{V - Bwv}{R} wB\mathbf{e}_x \tag{48}$$

Thus,

$$m\frac{dv}{dt} = \frac{VwB - B^2w^2v}{R} \tag{49}$$

To integrate this equation, we let

$$\alpha = \frac{VwB}{mR} \tag{50}$$

$$\beta = \frac{B^2w^2}{mR} \tag{51}$$

Then we have

$$\frac{dv}{dt} = \alpha - \beta v \tag{52}$$

$$\int \frac{dv}{\alpha - \beta v} = \int dt \tag{53}$$

$$-\frac{1}{\beta}\ln(\alpha - \beta v) = t + D \tag{54}$$

$$\alpha - \beta v = Ce^{-\beta t} \tag{55}$$

$$v(t) = \frac{1}{\beta}\left(\alpha - Ce^{-\beta t}\right) \tag{56}$$

Since $v(0) = 0$, we must have

$$C = \alpha \tag{57}$$

and therefore

$$v(t) = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \tag{58}$$

$$= \frac{V}{Bw}\left(1 - e^{-\frac{B^2w^2}{mR}t}\right) \tag{59}$$

6. Looking at the loop composed of R_1 and R_2, we have by Kirchoff's law that

$$V = (I_1 + I_2)R_1 + I_2R_2 \tag{60}$$

$$I_2 = \frac{V}{R_1 + R_2} - I_1\left(\frac{R_1}{R_1 + R_2}\right) \tag{61}$$

Next, Looking at the loop containing the inductor and R_2 we find

$$0 = -L\frac{dI_1}{dt} + R_2I_2 \tag{62}$$

$$L\frac{dI_1}{dt} = \frac{VR_2}{R_1 + R_2} - I_1\left(\frac{R_1R_2}{R_1 + R_2}\right) \tag{63}$$

$$L\frac{dI_1}{dt} = \frac{VR_2}{R_1 + R_2} - I_1R' \tag{64}$$

Asymptotically, we see that

$$I_1(t \to \infty) = \frac{V}{R_1} \tag{65}$$

Substituting into the equation

$$I_1(t) = \frac{V}{R_1} + \alpha(t) \tag{66}$$

We see that

$$L\frac{d\alpha}{dt} = -R'\alpha(t) \tag{67}$$

$$\frac{d\alpha}{\alpha} = -\frac{R'}{L}dt \tag{68}$$

$$\alpha(t) = Ce^{-\frac{R'}{L}t} \tag{69}$$

Since $I_1(0) = 0$, we must have that

$$C = \frac{V}{R_1} \tag{70}$$

and thus

$$I_1(t) = \frac{V}{R_1}\left(1 - e^{-\frac{R'}{L}t}\right) \tag{71}$$

7. (a)

$$\omega_0 = \sqrt{\frac{1}{LC}} \tag{72}$$

$$C = \frac{1}{\omega_0^2 L} \tag{73}$$

$$= \frac{1}{(2\pi \times 3000Hz)^2 (10mH)} \tag{74}$$

$$= 281nF \tag{75}$$

(b) The total frequency dependent complex impedance is given by

$$Z(\omega) = R + i\left(\omega L - \frac{1}{\omega C}\right) \tag{76}$$

$$= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} e^{i\tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)} \tag{77}$$

Plugging in the numbers for R,L,C, and ω, we find

$$Z(2\pi \times 5000Hz) = (225\Omega)e^{1.11i} \tag{78}$$

(c) We can write the voltage signal in complex form as

$$V(t) = \Re\left[200e^{10000\pi it}\right] V \tag{79}$$

where $\Re\left[\right]$ denotes the real part. Then, by Ohm's Law we have

$$I(t) = \Re\left[\frac{200e^{10000\pi it}}{(225\Omega)e^{1.11i}}\right] \tag{80}$$

$$= 0.89\cos\left(10000\pi t - 1.11\right) A \tag{81}$$

(d) The average power is given by

$$\langle P\rangle = \langle IV\rangle \tag{82}$$

$$= 178\left\langle \cos(10000\pi t)\cos(10000\pi t - 1.11)\right\rangle W \tag{83}$$

$$= \frac{178}{2}\cos(-1.11)W \tag{84}$$

$$= 39.6W \tag{85}$$

(e) The maximum value of the current is given by

$$I_{max} = 0.89A \tag{86}$$

Thus, by Ohm's law, the maximum voltage across the resistor is

$$V^R_{max} = 89V \tag{87}$$

the maximum voltage across the inductor is

$$V^L_{max} = (10000\pi)(0.01)(0.89) = 89\pi V \tag{88}$$

and finally the maximum voltage across the capacitor is given by

$$V^C_{max} = \frac{0.89}{(10000\pi)(281\times 10^{-9})} = 100.7V \tag{89}$$

These numbers add up to greater than $200V$ since the maximum voltage drop across each element do not occur at the same time.

8. At resonance, $Z = R$, and so the current flowing through the circuit is given by

$$I(t) = \frac{V(t)}{R} = 1.1\cos(100\pi t)A \tag{90}$$

We can also solve for L from the resonance condition to find

$$100\pi = \sqrt{\frac{1}{L(20\mu F)}} \tag{91}$$

$$L = 0.51H \tag{92}$$

Using Ohm's law in complex form, we thus find that the voltage V_{LR} across the resistor-inductor segment is given by

$$V_{LR}(t) = \Re\left[1.1(100 + i51\pi)e^{100\pi it}\right] V \tag{93}$$

The maximum value is then given by the amplitude of V_{LR}, which we can see is

$$\begin{aligned} V_{LR}^{max} &= 1.1\sqrt{100^2 + (51\pi)^2}V & (94)\\ &= 207.75V & (95) \end{aligned}$$

9. Since we define the charge on the capacitor as

$$Q = \int I(t)dt \tag{96}$$

We have that the voltage drop across the capacitor is opposite in sign to the direction of the current. Thus, the Kirchoff loop equation gives us

$$-\frac{1}{C}\int I(t)dt - L\frac{dI}{dt} - IR = 0 \tag{97}$$

Substituting in

$$I(t) = I_0 e^{-\alpha t} \tag{98}$$

we find

$$\begin{aligned} 0 &= \frac{1}{\alpha C} + \alpha L - R & (99)\\ &= \alpha^2 L - \alpha R + \frac{1}{C} & (100)\\ \alpha &= \frac{R}{2L} \pm \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L} & (101)\\ &= \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} & (102) \end{aligned}$$

If R is small enough, we get two complex solutions α_+ and α_- which are complex conjugate pairs. The most general solution for $I(t)$ is then

$$I(t) = I_+ e^{-\alpha_+ t} + I_- e^{-\alpha_- t} \tag{103}$$

For this to be real, we must have

$$I_+ = I_-^* \tag{104}$$

where * denotes the complex conjugate operation. Defining

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{105}$$

and letting

$$I_+ = \frac{A}{2} e^{-i\phi} \tag{106}$$

we have

$$\begin{aligned} I(t) &= \frac{A}{2} e^{-i\phi} e^{-\frac{R}{2L}t} e^{i\omega' t} + \frac{A}{2} e^{i\phi} e^{-\frac{R}{2L}t} e^{-i\omega' t} && (107) \\ &= A e^{-\frac{R}{2L}t} \cos\left(\omega' t - \phi\right) && (108) \end{aligned}$$

where A and ϕ are determined by the initial values $I(0)$ and $\frac{dI(0)}{dt}$

10. From the Current divider rule given in the problem, we have that the current flowing through point A is given by

$$I_A = I_{tot} \frac{Z_{R_2} + Z_L}{Z_{R_1} + Z_{R_2} + Z_L + Z_C} \tag{109}$$

and similarly

$$I_B = I_{tot} \frac{Z_{R_1} + Z_C}{Z_{R_1} + Z_{R_2} + Z_L + Z_C} \tag{110}$$

Thus, by Ohm's law, V_A is given by

$$V_A = V - Z_C I_A \tag{111}$$

and similarly

$$V_B = V - Z_{R_2} I_B \tag{112}$$

Combining these, we find

$$\begin{aligned} V_A - V_B &= Z_{R_2} I_B - Z_C I_A && (113) \\ &= \frac{Z_{R_2}\left(Z_{R_1} + Z_C\right) - Z_C\left(Z_{R_2} + Z_L\right)}{Z_{R_1} + Z_{R_2} + Z_L + Z_C} && (114) \\ &= \frac{R_1 R_2 - i\frac{R_2}{\omega C} + \frac{iR_2}{\omega C} - \frac{L}{C}}{Z_{R_1} + Z_{R_2} + Z_L + Z_C} && (115) \\ &= \frac{R_1 R_2 - \frac{L}{C}}{Z_{R_1} + Z_{R_2} + Z_L + Z_C} && (116) \\ & && (117) \end{aligned}$$

Thus, we see that when $\frac{L}{C} = R_1 R_2$,

$$V_A - V_B = 0 \tag{118}$$