Solutions to PS 3 Physics 201

1. $\frac{\partial}{\partial y}(x^2 y) = \frac{\partial}{\partial x}(\frac{x^3}{3}) = x^2$. That is, $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$. Therefore, **F** can be written in the form of **F**$= -\nabla U(x,y)$ with some function $U(x,y)$, which means that **F** is conservative.

   From $-\frac{\partial U}{\partial x} = x^2 y$, $U = \int -x^2 y\, dx = -\frac{1}{3}x^3 y + C(y)$, and then from $-\frac{\partial U}{\partial y} = \frac{x^3}{3} - C'(y) = \frac{x^3}{3}$, we get $C(y)$=const. So finally, $U(x,y) = -\frac{1}{3}x^3 y$+const.

   Using this potential,

$$\int_{(0,0)}^{(2,3)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0)}^{(2,3)} -\nabla U(x,y) \cdot d\mathbf{r} = -U(2,3) + U(0,0) = 8. \tag{1}$$

2.

$$\frac{1.6 \times 10^3 \mathrm{J}}{10\mathrm{V}} = 1.6 \times 10^2 \text{ Coulomb} \tag{2}$$

$$= 1.6 \times 10^2 \text{ Coulomb } \times \frac{6.24 \times 10^{18} \text{ electrons}}{1 \text{ Coulomb}} \tag{3}$$

$$= 1.0 \times 10^{19} \text{ electrons.} \tag{4}$$

3. The potentials at (1,1) and (2,2) are given by

$$V(1,1) = \frac{1}{4\pi\epsilon_0}\frac{(2\ \mu\mathrm{C})}{\sqrt{1^2+1^2}\ \mathrm{m}} + \frac{1}{4\pi\epsilon_0}\frac{(-3\mu\mathrm{C})}{\sqrt{0.8^2+0.5^2}\ \mathrm{m}}, \tag{5}$$

$$V(2,2) = \frac{1}{4\pi\epsilon_0}\frac{(2\ \mu\mathrm{C})}{\sqrt{2^2+2^2}\ \mathrm{m}} + \frac{1}{4\pi\epsilon_0}\frac{(-3\mu\mathrm{C})}{\sqrt{1.8^2+1.5^2}\ \mathrm{m}}. \tag{6}$$

   Therefore,

$$\text{(Work needed)} \tag{7}$$

$$= V(2,2) \times 2\ \mu\mathrm{C} - V(1,1) \times 2\ \mu\mathrm{C} \tag{8}$$

$$= \frac{1}{4 \times 3.14 \times 8.85 \times 10^{-12}\ \mathrm{C^2 J^{-1}\ m^{-1}}} \tag{9}$$

$$\times\left(\left(\frac{4 \times 10^{-12}\ \mathrm{C}^2}{2.83\ \mathrm{m}} - \frac{6 \times 10^{-12}\ \mathrm{C}^2}{2.34\ \mathrm{m}}\right) - \left(\frac{4 \times 10^{-12}\ \mathrm{C}^2}{1.41\ \mathrm{m}} - \frac{6 \times 10^{-12}\ \mathrm{C}^2}{0.94\ \mathrm{m}}\right)\right) \tag{10}$$

$$= 2.16 \times 10^{-2}\ \mathrm{J} \tag{11}$$

4. The potential created by a dipole is given by

$$V(r,\theta) = V(x,y) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} = \frac{p}{4\pi\epsilon_0}\frac{r\cos\theta}{r^3} = \frac{p}{4\pi\epsilon_0}\frac{x}{[x^2+y^2]^{3/2}}. \tag{12}$$

First, in cartesian coordinate,

$$\begin{aligned}
\mathbf{E} = -\nabla V &= -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} && (13)\\
&= -\mathbf{i}\ \frac{p}{4\pi\epsilon_0}\frac{[x^2+y^2]^{3/2} - x\cdot\frac{3}{2}[x^2+y^2]^{1/2}2x}{[x^2+y^2]^3} - \mathbf{j}\ \frac{p}{4\pi\epsilon_0}\frac{-3}{2}\frac{2y}{[x^2+y^2]^{5/2}} && (14)\\
&= \mathbf{i}\ \frac{p}{4\pi\epsilon_0}\frac{(2x^2-y^2)}{[x^2+y^2]^{5/2}} + \mathbf{j}\ \frac{p}{4\pi\epsilon_0}\frac{3xy}{[x^2+y^2]^{5/2}} && (15)
\end{aligned}$$

In polar coordinate, using the fact that $\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}$, we get

$$\begin{aligned}
\mathbf{E} = -\nabla V &= -\mathbf{e}_r\frac{\partial}{\partial r}(\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}) - \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}) && (16)\\
&= \frac{p}{4\pi\epsilon_0}\frac{2\cos\theta}{r^3}\mathbf{e}_r + \frac{p}{4\pi\epsilon_0}\frac{\sin\theta}{r^3}\mathbf{e}_\theta, && (17)
\end{aligned}$$

which can be easily shown to be the same as the result in cartesian coordinate, noting that $\mathbf{e}_r = \mathbf{i}\ (\frac{x}{r}) + \mathbf{j}\ (\frac{y}{r})$ and $\mathbf{e}_\theta = -\mathbf{i}\ (\frac{x}{r}) + \mathbf{j}\ (\frac{x}{r})$.

5. V=0 surface is determined by

$$\frac{q}{\sqrt{(x-a)^2+y^2+z^2}}+\frac{-2q}{\sqrt{x^2+y^2+z^2}}=0 \tag{18}$$

$$\Leftrightarrow \frac{q^2}{(x-a)^2+y^2+z^2}=\frac{4q^2}{x^2+y^2+z^2} \tag{19}$$

$$\Leftrightarrow x^2+y^2+z^2=4\{(x-a)^2+y^2+z^2\} \tag{20}$$

$$\Leftrightarrow (x-\frac{4a}{3})^2+y^2+z^2=(\frac{2a}{3})^2 \tag{21}$$

This gives the surface of a sphere of radius $\frac{2a}{3}$, with the center at $(\frac{4a}{3},0,0)$ (FIG. 1).

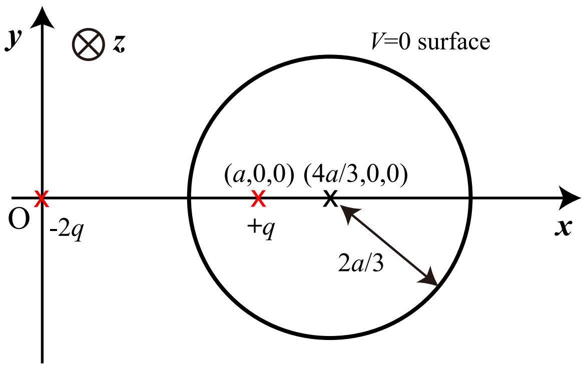

FIG. 1: V=0 surface.

V=const. surface appears if there is a grounded metal surface in the system. The result of this problem can be used to obtain the potential created by a point charge located inside or outside a metallic shell. This is a special case of the general result that when charge Q is put at a distance r from the center of a sphere of radius R, the image equals $-(RQ/r)$ and is located R^2/r from the center towards the external charge. (In our example $R=2a/3$ and $r=4a/3$.) You will be guided towards a proof of this result in PS4.

6. The uniform charge density per area is $\rho = \frac{Q}{\pi R^2}$. The potential is calculated as the sum of the potential created by the charge located at tiny part of the disc, and therefore,

$$V = \int_{\text{disc}} \frac{1}{4\pi\epsilon_0} \frac{\rho dS}{\sqrt{r^2+z^2}} \tag{22}$$

$$= \int_0^R rdr \int_0^{2\pi} d\theta \frac{1}{4\pi\epsilon_0} \frac{\rho}{\sqrt{r^2+z^2}} \tag{23}$$

$$= \frac{1}{2\epsilon_0} \frac{Q}{\pi R^2} [\sqrt{r^2+z^2}]_{r=0}^{r=R} \tag{24}$$

$$= \frac{Q}{2\pi\epsilon_0 R^2} [\sqrt{R^2+z^2} - \sqrt{z^2}] \tag{25}$$

$$= \frac{Q}{2\pi\epsilon_0 R^2} [\sqrt{R^2+z^2} - |z|] \tag{26}$$

In the limit of $|z| \to \infty$,

$$V = \frac{Q}{2\pi\epsilon_0 R^2} \frac{R^2}{\sqrt{R^2+|z|^2}+|z|} \to \frac{Q}{4\pi\epsilon_0 |z|}, \tag{27}$$

which coincides with the potential created by a point charge Q at the origin.

Also, in the limit of $|z| \to 0$,

$$V = \frac{Q}{2\pi\epsilon_0 R^2}[-|z| + \sqrt{R^2+|z|^2}] = \frac{Q}{2\pi\epsilon_0 R^2}[-|z| + R + \frac{|z|^2}{2R} + \cdots] \tag{28}$$

$$\to \frac{Q}{2\pi\epsilon_0 R} - \frac{Q|z|}{2\pi\epsilon_0 R^2}. \tag{29}$$

Next, the electric field in the z direction at $(0,0,z)$ can be calculated by differentiating potential with z. That is, in the region of $z \approx 0$, by differentiating Eq. (29), we get

$$E_z = -\frac{\partial V}{\partial z} \tag{30}$$

$$= \frac{Q}{2\pi\epsilon_0 R^2} \frac{\partial |z|}{\partial z} \tag{31}$$

$$= \frac{Q}{2\pi\epsilon_0 R^2} \frac{z}{|z|}. \tag{32}$$

In other words, in the limit of $z \to \pm 0$,

$$\lim_{z \to \pm 0} E_z = \pm \frac{Q}{2\pi\epsilon_0 R^2}, \tag{33}$$

which coincides with the electric field created by infinitely large sheet with charge density per area $\rho = \frac{Q}{\pi R^2}$.

If V is wrongly given by $\frac{Q}{2\pi\epsilon_0 R^2}[\sqrt{R^2+z^2}-z] \approx \frac{Q}{2\pi\epsilon_0 R} - \frac{Qz}{2\pi\epsilon_0 R^2}$, this leads to

$$\lim_{z\to\pm 0} E_z = -\frac{\partial V}{\partial z} = \frac{Q}{2\pi\epsilon_0 R^2}, \tag{34}$$

which is wrong because this gives the electric field in the same direction on both sides of the disc.

And finally, V calculated above is valid only on the z-axis. Therefore, it cannot be used to calculate the electric field in x and y direction, which requires to use the potential at the point off the axis. To calculate E_x and E_y on the z-axis from V, first we have to calculate V for point (x, y, z) that is not on the axis and then calculate the gradient of V.

7. From the condition given in the problem, we get

$$\begin{cases} 120\text{V} = \frac{Q}{2\pi\epsilon_0 R^2}(\sqrt{1^2+R^2}-1) \\ 100\text{V} = \frac{Q}{2\pi\epsilon_0 R^2}(\sqrt{2^2+R^2}-2) \end{cases} \tag{35}$$

Elliminating Q,

$$\frac{120}{100} = \frac{\sqrt{1+R^2}-1}{\sqrt{4+R^2}-2}, \tag{36}$$

and finally we get $R = 4\sqrt{210}/11 = 5.27$ m. Putting this into the previous equation, we get

$$Q = 120 \text{ V} \times 2\pi\epsilon_0 R^2/((\sqrt{R^2+1}-1)) \tag{37}$$

$$= 120 \text{ J/C} \times 2 \times 3.14 \times 8.85 \times 10^{-12} \text{ C}^2\text{J}^{-1} \times \frac{5.27^2}{\sqrt{5.27^2+1}-1} \tag{38}$$

$$= 4.25 \times 10^{-8} \text{ C}. \tag{39}$$

8. In the same way as Problem 3 of PS2, using Gauss's law and the symmetry of the system, we get

$$\int \mathbf{E}\cdot d\mathbf{S} = 4\pi r^2 E_r = \frac{Q_{\text{enclosed}}}{\epsilon_0}. \tag{40}$$

For $r < R$, this gives us

$$E_r = \frac{Q_{\text{enclosed}}}{4\pi\epsilon_0 r^2} = \frac{Q\frac{r^3}{R^3}}{4\pi\epsilon_0 r^2} = \frac{Qr}{4\pi\epsilon_0 R^3} \tag{41}$$

For $r \geq R$, we have

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}. \tag{42}$$

In both cases, $E_\theta = E_\phi = 0$. To calculate the potential at some point, we have to integrate the work needed to convey test charge from infinite to that point. That is, the potential is given by

$$V(\mathbf{r}) = \int_\infty^{\mathbf{r}} -\mathbf{E} \cdot d\mathbf{r} \tag{43}$$

Therefore, for $r \geq R$, we have

$$V(\mathbf{r}) = \int_\infty^r -\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = \frac{Q}{4\pi\epsilon_0 r}. \tag{44}$$

For $r < R$,

$$V(\mathbf{r}) = \int_\infty^R -\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr + \int_R^r \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r dr \tag{45}$$

$$= \frac{Q}{4\pi\epsilon_0}[\frac{1}{R} + \frac{1}{R^3}\frac{R^2 - r^2}{2}]. \tag{46}$$

9. First we have to calculate the work needed to add a shell of thickness dr on a sphere of radius r. Using the result of the previous problem for $r \geq R$ and replacing Q with $Q\frac{r^3}{R^3}$ and R with r, we get

$$dW = (\text{charge of the shell}) \times (\text{potential at r}) \tag{47}$$

$$= (\frac{Q}{\frac{4}{3}\pi R^3}4\pi r^2 dr)(\frac{Q(r/R)^3}{4\pi\epsilon_0}\frac{1}{r}) \tag{48}$$

$$= \frac{3Q^2}{4\pi\epsilon_0 R^6}r^4 dr. \tag{49}$$

Integrating this with respect to r from 0 to R, we get

$$W = \int_0^R \frac{3Q^2}{4\pi\epsilon_0 R^6}r^4 dr \tag{50}$$

$$= \frac{3Q^2}{4\pi\epsilon_0 R^6}\frac{R^5}{5} \tag{51}$$

$$= \frac{3Q^2}{20\pi\epsilon_0 R}. \tag{52}$$

Meanwhile, the volume integral of electric field energy is given by

$$\begin{aligned}
\int dV\ \frac{\epsilon_0}{2}\mathbf{E^2} &= \frac{\epsilon_0}{2}\int_0^R (\frac{Q}{4\pi\epsilon_0})^2(\frac{r}{R^3})^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty (\frac{Q}{4\pi\epsilon_0})^2\frac{1}{r^4}4\pi r^2 dr && (53)\\
&= \frac{Q^2}{8\pi\epsilon_0}[\int_0^R \frac{r^4}{R^6}dr + \int_R^\infty \frac{1}{r^2}dr] && (54)\\
&= \frac{Q^2}{8\pi\epsilon_0}[\frac{1}{5R} + \frac{1}{R}] && (55)\\
&= \frac{3Q^2}{20\pi\epsilon_0}, && (56)
\end{aligned}$$

which is the same as the previous result.

10. Applying Gauss's law, and using the symmetry of the system, we have (PS2, Problem 4)

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 0\ (r < a) \\ \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e}_r\ (a \le r \le b) \\ 0\ (r > b) \end{cases} \tag{57}$$

The potential can be calculated from this by the relation $V(\mathbf{r}) = \int_\infty^{\mathbf{r}} -\mathbf{E}\cdot d\mathbf{r}$.

For $r > b$, $V(r > b) = 0$.

For $a \le r \le b$,

$$\begin{aligned}
V(\mathbf{r}) &= \int_\infty^{\mathbf{r}} -\mathbf{E}\cdot d\mathbf{r} && (58)\\
&= \int_b^r -\frac{\lambda}{2\pi\epsilon_0 r}dr && (59)\\
&= -\frac{\lambda}{2\pi\epsilon_0}\log(\frac{r}{b}) && (60)\\
&= \frac{\lambda}{2\pi\epsilon_0}\log(\frac{b}{r}), && (61)
\end{aligned}$$

and finally for $r < a$,

$$V(r < a) = \frac{\lambda}{2\pi\epsilon_0}\log(\frac{b}{a}). \tag{62}$$

11. The potential difference is given by

$$V_1 - V_2 = \frac{Q_1}{4\pi\epsilon_0 r_1} - \frac{Q_2}{4\pi\epsilon_0 r_2} \tag{63}$$

$$= \frac{30 \times 10^{-9}\ \mathrm{C}}{4 \times 3.14 \times 8.85 \times 10^{-12}\ \mathrm{C^2 J^{-1}} \times 0.10\ \mathrm{m}} - \frac{-20 \times 10^{-9}\ \mathrm{C}}{4 \times 3.14 \times 8.85 \times 10^{-12}\ \mathrm{C^2 J^{-1}} \times 0.20\ \mathrm{m}} \tag{64}$$

$$= 2.70 \times 10^3 - (-0.90 \times 10^3)\ [\mathrm{V}] \tag{65}$$

$$= 3.60 \times 10^3\ [\mathrm{V}] \tag{66}$$

Next, suppose charge q moves from the sphere 1 to the other when they are connected by a conducting wire. In the end, the potential difference between the two sphere should be 0. This gives us the following condition:

$$\frac{Q_1 - q}{4\pi\epsilon_0 R_1} = \frac{Q_2 + q}{4\pi\epsilon_0 R_2} \tag{67}$$

$$\Leftrightarrow R_2(Q_1 - q) = R_1(Q_2 + q) \tag{68}$$

By solving for q, we get

$$q = \frac{R_2 Q_1 - R_1 Q_2}{R_1 + R_2} \tag{69}$$

$$= \frac{0.20\ \mathrm{m} \times 30\ \mathrm{nC} - 0.10\ \mathrm{m} \times (-20\ \mathrm{nC})}{0.10\ \mathrm{m} + 0.20\ \mathrm{m}} \tag{70}$$

$$= 26.7\ \mathrm{nC} \tag{71}$$

The resulting potential is given by

$$V_1^{\mathrm{final}} = V_2^{\mathrm{final}} = \frac{1}{4\pi\epsilon_0 R_1}(Q_1 - q) \tag{72}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q_1 + Q_2}{R_1 + R_2} \tag{73}$$

$$= \frac{10\ \mathrm{nC}}{4 \times 3.14 \times 8.85 \times 10^{-12}\ \mathrm{C^2 J^{-1}} \times 0.30\ \mathrm{m}} \tag{74}$$

$$= 3.00 \times 10^2\ \mathrm{V} \tag{75}$$

And the charges in the spheres are

$$Q_1^{\mathrm{final}} = Q_1 - q = 3.3\ \mathrm{nC}, \tag{76}$$

and

$$Q_2^{\text{final}} = Q_2 + q = 6.7 \text{ nC}, \tag{77}$$

respectively.

12. Suppose charge +Q and charge -Q are charged on the inner cylinder and on the outer cylinder respectively. Assuming the cylinder is infinitely long, we can use the result of problem 10 by replacing λ with Q/L. Therefore, the potential difference V between the two cylinder is given by

$$V = \frac{Q/L}{2\pi\epsilon_0} \log(b/a). \tag{78}$$

From the relation $Q = CV$,

$$C = Q/V = 2\pi\epsilon_0 L \frac{1}{\log \frac{b}{a}} \tag{79}$$

$$\tag{80}$$

When $b - a = d \ll a$,

$$\begin{aligned} C &= 2\pi\epsilon_0 L \frac{1}{\log \frac{b-a+a}{a}} && (81) \\ &= 2\pi\epsilon_0 L \frac{1}{\log(1 + \frac{d}{a})} && (82) \\ &\approx \epsilon_0 \frac{2\pi a L}{d}, && (83) \end{aligned}$$

which coincides with the capacitance of a parallel plate capacitor with area $2\pi a L$.