Solutions to PS 13 Physics 201

1. By plugging in the assumed form to the equation, we get

$$\frac{d^2\psi(x)}{dx^2}F(t) = \frac{1}{v^2}\frac{d^2F(t)}{dt^2}\psi(x). \tag{1}$$

Dividing by $F(t)\psi(x)$,

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = \frac{1}{v^2}\frac{1}{F(t)}\frac{d^2F(t)}{dt^2}. \tag{2}$$

The left hand side of this equation is a function of only x, while the right hand side is a function of only t. The only possibility is that both of these are just constant. Then, we can assume this constant is $-\beta^2$ with some β. (Here, β is generally a complex number and adding a - sign gives the same result. But this convention will make the calculation simpler by use of sin and cos.) This assumption leads to the following two equations:

$$\frac{d^2\psi(x)}{dx^2} = -\beta^2\psi(x), \tag{3}$$

and

$$\frac{d^2F(t)}{dt^2} = -\beta^2v^2F(t). \tag{4}$$

The solution to the eq. (3) and (4) is given by

$$\psi(x) = \tilde{A}\cos\beta x + \tilde{B}\sin\beta x, \tag{5}$$

and

$$F(t) = \tilde{C}\cos\beta vt + \tilde{D}\sin\beta vt. \tag{6}$$

However, we have to impose the boundary condition $\psi(0) = \psi(L) = 0$. This leads to $\tilde{A} = 0$ and $\beta L = 2\pi m$ with some integer m. Then, by defining new coefficients $A' = \tilde{B}\tilde{C}$ and $B' = \tilde{B}\tilde{D}$, we finally get

$$\psi(x,t) = \sin\frac{2\pi m}{L}x\ (A'\cos\frac{2\pi mv}{L}t + B'\sin\frac{2\pi mv}{L}t). \tag{7}$$

Because the string is at rest at $t = 0$, that is, $\frac{d\psi(x,0)}{dt} = 0$, we have $B' = 0$. Also, from the condition that $\psi(x,0) = A\sin\frac{2\pi n}{L}x$, we get $A' = A$ and $m = n$. Therefore,

$$\psi(x,t) = A\sin\frac{2\pi n}{L}x\cos\frac{2\pi nv}{L}t. \tag{8}$$

2. (i)The normalized momentum eigenstate is given by

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{i\frac{2\pi n}{L}x}. \tag{9}$$

Then,

$$\begin{aligned}\hat{H}\psi_n(x) &= -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + V(x)\psi_n(x) &(10)\\ &= -\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} &(11)\\ &= -\frac{\hbar^2}{2m}(i\frac{2\pi n}{L})^2\psi_n(x) &(12)\\ &= \frac{2\pi^2\hbar^2 n^2}{mL^2}\psi_n(x). &(13)\end{aligned}$$

Thus, $\psi_n(x)$ satisfies $\hat{H}\psi_n(x) = E\psi_n(x)$ with $E = \frac{2\pi^2\hbar^2 n^2}{mL^2} \equiv E_n$.

(ii)Let's define normalized wavefunction $\bar{\psi}(x,t) = C\psi(x,t)$. From normalization condition, we get

$$\begin{aligned}1 &= |C|^2\int_0^L |\psi(x,0)|^2 dx &(14)\\ &= |C|^2\int_0^L (9|\psi_2(x)|^2 + 12\psi_2^*(x)\psi_3(x) + 12\psi_2(x)\psi_3^*(x) + 16|\psi_3(x)|^2 dx &(15)\\ &= |C|^2(9+16) = 25|C|^2, &(16)\end{aligned}$$

where the orthonomality of the states was used. We can simply take C=1/5. Finally we have

$$\bar{\psi}(x,0) = \frac{3}{5}\psi_2(x) + \frac{4}{5}\psi_3(x) \tag{17}$$

(Note that we have only to impose normalization condition at $t = 0$, because the conservation of probability holds from the time-dependent Schrödinger equation.)

Noting that time evolutions of $\psi_0(x)$ and $\psi_1(x)$ under the time-dependent Schrödinger equation are given by

$$\psi_2(x,t) = \psi_2(x)e^{-i\frac{E_2}{\hbar}t} = \psi_2(x)e^{-i\frac{8\pi^2\hbar}{mL^2}t} \tag{18}$$

and

$$\psi_3(x,t) = \psi_3(x)e^{-i\frac{E_3}{\hbar}t} = \psi_3(x)e^{-i\frac{18\pi^2\hbar}{mL^2}t}, \tag{19}$$

we get

$$\bar{\psi}(x,t) = \frac{3}{5}\psi_2(x)e^{-i\frac{8\pi^2\hbar}{mL^2}t} + \frac{4}{5}\psi_3(x)e^{-i\frac{18\pi^2\hbar}{mL^2}t} \tag{20}$$

$$= \frac{3}{5}\frac{1}{\sqrt{L}}e^{i\frac{4\pi}{L}x}e^{-i\frac{8\pi^2\hbar}{mL^2}t} + \frac{4}{5}\frac{1}{\sqrt{L}}e^{i\frac{6\pi}{L}x}e^{-i\frac{18\pi^2\hbar}{mL^2}t}. \tag{21}$$

From this, we have

$$P(x,t) = |\bar{\psi}(x,t)|^2 \tag{22}$$

$$= \frac{1}{L}\{\frac{9}{25} + \frac{12}{25}(e^{i\frac{4\pi}{L}x}e^{-i\frac{8\pi^2\hbar}{mL^2}t})^*(e^{i\frac{6\pi}{L}x}e^{-i\frac{18\pi^2\hbar}{mL^2}t}) + \frac{12}{25}(e^{i\frac{4\pi}{L}x}e^{-i\frac{8\pi^2\hbar}{mL^2}t})(e^{i\frac{6\pi}{L}x}e^{-i\frac{18\pi^2\hbar}{mL^2}t})^* + \frac{16}{25}\} \tag{23}$$

$$= \frac{1}{L}\{1 + \frac{24}{25}\cos(\frac{2\pi}{L}x - \frac{10\pi^2\hbar}{mL^2}t)\}. \tag{24}$$

3. (i)It is useful to define "characteristic length scale" $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. Then, we have

$$\psi_0(x) = [\frac{1}{\pi x_0^2}]^{\frac{1}{4}} e^{-\frac{x^2}{2x_0^2}}. \tag{25}$$

(Note that x/x_0 is a dimensionless quantity.) Fig.1 is a plot of this function.

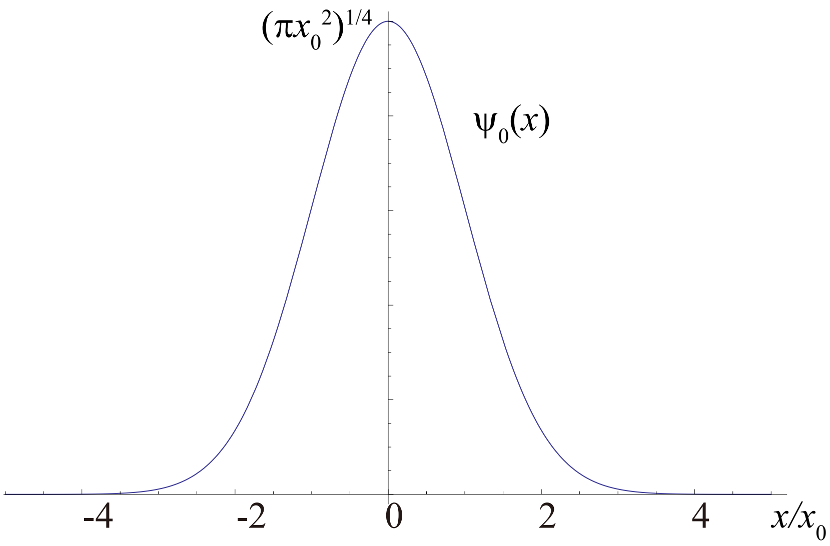

FIG. 1:

(ii) From normalization condition, we have

$$\int_{-\infty}^{\infty} |\psi_1(x)|^2 dx = |A|^2 \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx = 1 \tag{26}$$

Using the formula $\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$ with $\alpha = m\omega/\hbar$, we get

$$|A|^2 \frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}} = \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} = 1 \tag{27}$$

and therefore,

$$A = [\frac{4m^3\omega^3}{\pi\hbar^3}]^{\frac{1}{4}}. \tag{28}$$

Again using x_0,

$$\psi_1(x) = [\frac{4m^3\omega^3}{\pi\hbar^3}]^{\frac{1}{4}} x e^{-\frac{m\omega x^2}{2\hbar}} = [\frac{4}{\pi x_0^6}]^{\frac{1}{4}} x e^{-\frac{x^2}{2x_0^2}} \tag{29}$$

Fig.2 is a plot of this function.

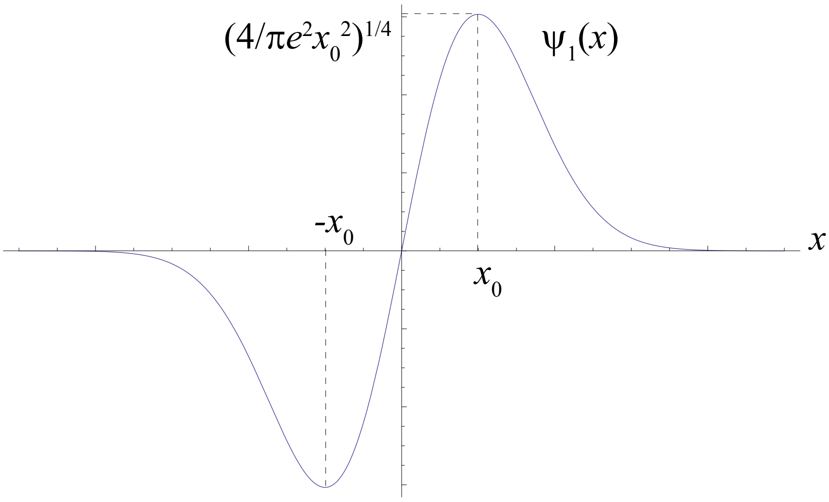

FIG. 2:

(iii)

$$\frac{d^2\psi_1(x)}{dx^2} = A\frac{d^2}{dx^2}(xe^{-\frac{m\omega x^2}{2\hbar}}) \tag{30}$$

$$= A\frac{d}{dx}\{\left(1 - x\frac{m\omega x}{\hbar}\right)e^{-\frac{m\omega x^2}{2\hbar}}\} \tag{31}$$

$$= A\{-\frac{2m\omega x}{\hbar}e^{-\frac{m\omega x^2}{2\hbar}} + \left(1 - \frac{m\omega x^2}{\hbar}\right)(-\frac{m\omega x}{\hbar})e^{-\frac{m\omega x^2}{2\hbar}}\} \tag{32}$$

$$= A(\frac{m^2\omega^2x^2}{\hbar^2} - \frac{3m\omega x}{\hbar})e^{-\frac{m\omega x^2}{2\hbar}} \tag{33}$$

Using this, we find

$$\hat{H}\psi_1(x) = -\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + \frac{1}{2}m\omega x^2\psi_1(x) \tag{34}$$

$$= A\left(\frac{3\hbar\omega}{2} - \frac{1}{2}m\omega^2x^3 + \frac{1}{2}m\omega^2x^3\right)e^{-\frac{m\omega x^2}{2\hbar}} \tag{35}$$

$$= \frac{3\hbar\omega}{2}\psi_1(x), \tag{36}$$

and thus$\hat{H}\psi_1(x) = E\psi_1(x)$ with

$$E = \frac{3\hbar\omega}{2} \equiv E_1. \tag{37}$$

(iv)

$$\int_{\infty}^{\infty} \psi_0^*(x)\psi_1(x)dx = \int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx \tag{38}$$

$$= [\frac{1}{\pi x_0^8}] \int_{-\infty}^{\infty} x e^{-\frac{m\omega x^2}{\hbar}} dx, \tag{39}$$

but this integral is simply 0 because $xe^{-\frac{m\omega x^2}{\hbar}}$ is an odd function of x.

(v) Let's define normalized wavefunction $\bar{\psi}(x,t) = C\psi(x,t)$. Using the above result, we have from normalization condition,

$$1 = |C|^2 \int_{-\infty}^{\infty} |\psi(x,0)|^2 dx \tag{40}$$

$$= |C|^2 \int_{-\infty}^{\infty} (9|\psi_0(x,0)|^2 + 12\psi_0^*(x)\psi_1(x) + 12\psi_0(x)\psi_1^*(x) + 16|\psi_1(x)|^2 dx \tag{41}$$

$$= |C|^2 \int_{-\infty}^{\infty} (9|\psi_0(x,0)|^2 + 24\psi_0(x)\psi_1(x) + 16|\psi_1(x)|^2)dx \tag{42}$$

$$= |C|^2 \int_{-\infty}^{\infty} (9|\psi_0(x,0)|^2 + 16|\psi_1(x)|^2)dx \tag{43}$$

$$= 25|C|^2. \tag{44}$$

We can simply take $C = 1/5$. Then, we have $\bar{\psi}(x,0) = \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x)$. (Note that again we have only to impose normalization condition at $t = 0$.)

Noting that time evolutions of $\psi_0(x)$ and $\psi_1(x)$ under the time-dependent Schrödinger equation are given by

$$\psi_0(x,t) = \psi_0(x)e^{-i\frac{E_0}{\hbar}t} = \psi_0(x)e^{-i\frac{\omega}{2}t} \tag{45}$$

and

$$\psi_1(x,t) = \psi_1(x)e^{-i\frac{E_1}{\hbar}t} = \psi_1(x)e^{-i\frac{3\omega}{2}t}, \tag{46}$$

we get

$$\bar{\psi}(x,t) = \frac{3}{5}\psi_0(x)e^{-i\frac{\omega}{2}t} + \frac{4}{5}\psi_1(x)e^{-i\frac{3\omega}{2}t} \tag{47}$$

$$= \frac{3}{5}[\frac{m\omega}{\pi\hbar}]^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} e^{-i\frac{\omega}{2}t} + \frac{4}{5}[\frac{4m^3\omega^3}{\pi\hbar^3}]^{\frac{1}{4}} x e^{-\frac{m\omega x^2}{2\hbar}} e^{-i\frac{3\omega}{2}t}. \tag{48}$$

Therefore,

$$P(x,t) = |\bar{\psi}(x,t)|^2 \tag{49}$$

$$= \frac{9}{25}[\frac{m\omega}{\pi\hbar}]^{\frac{1}{2}} e^{-\frac{m\omega x^2}{\hbar}} + \frac{24}{25}\sqrt{\frac{2}{\pi}}\frac{m\omega}{\hbar} x e^{-\frac{m\omega x^2}{\hbar}} \cos\omega t + \frac{16}{25}[\frac{4m^3\omega^3}{\pi\hbar^3}]^{\frac{1}{2}} x^2 e^{-\frac{m\omega x^2}{\hbar}} \tag{50}$$

$$= (\frac{9}{25}\sqrt{\frac{m\omega}{\pi\hbar}} + \frac{24}{25}\sqrt{\frac{2}{\pi}}\frac{m\omega}{\hbar} x \cos\omega t + \frac{32}{25}\sqrt{\frac{m^3\omega^3}{\pi\hbar^3}} x^2) e^{-\frac{m\omega x^2}{\hbar}}. \tag{51}$$

Or,

$$P(x,t) = \frac{1}{\sqrt{\pi x_0}}(\frac{9}{25} + \frac{24}{25}\sqrt{2}\frac{x}{x_0}\cos\omega t + \frac{32}{25}\frac{x^2}{x_0^2}) e^{-\frac{x^2}{x_0^2}}. \tag{52}$$