Ενότητα 31

**8.** $f(x) = x+1 - \dfrac{2e^x}{1+e^x}$

Νδο $y = x-1$ ασύμπτωτη $+\infty$

$$\boxed{\text{Αρκεί νδο } \lim_{x\to+\infty} f(x) - (x-1) = 0}$$

$$\lim_{x\to+\infty}\left(x+1 - \frac{2e^x}{1+e^x} - x + 1\right) = \lim_{x\to+\infty}\left(2 - \frac{2e^x}{1+e^x}\right) = 0$$

$$\rightarrow \lim_{x\to+\infty} \frac{2e^x}{1+e^x} = \lim_{x\to+\infty} \frac{2e^x}{e^x} = 2$$

**10.** ε $f(x) = x + \dfrac{\ln x}{x}, \; x>0$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\left(x + \frac{\ln x}{x}\right) = \lim_{x\to0^+}\left(x + \ln x \,\frac{1}{x}\right) =$$

$$= 0 + (-\infty)\cdot(+\infty) = -\infty$$

$\boxed{\text{Εις } x=0 \text{ κατακόρυφη.}}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x + \cancel{\frac{\ln x}{x}}^{\,0}\right) = +\infty$$

Δεν έχει οριζόντια στο $+\infty$.

$$\rightarrow \lim_{x\to+\infty}\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{1}{x} = 0$$

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{x+\frac{\ln x}{x}}{x} = \lim_{x\to+\infty}\frac{x^2+\ln x}{x^2}$$

$$= \lim_{x\to+\infty}\frac{2x+\frac{1}{x}}{2x} = \lim_{x\to+\infty}\frac{2x^2+1}{2x^2} = \underline{1}$$

$$\lim_{x\to+\infty}\left(f(x)-x\right) = \lim_{x\to+\infty} x + \frac{\ln x}{x} - x =$$

$$= \lim_{x\to+\infty}\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{1}{x} = 0$$

$$y = 1\cdot x + 0$$

$+\infty$ πλάγια

$$\underline{y = x}$$

**15.** $$\lim_{x\to-\infty} \frac{xf(x)+3x^2}{\sqrt{x^2+1}} = 2$$

Βρείτε ασύμπτωτη στο $-\infty$

$$g(x) = \frac{xf(x)+3x^2}{\sqrt{x^2+1}} \quad \text{άρα} \quad \lim_{x\to-\infty} g(x) = 2.$$

$$g(x)\sqrt{x^2+1} = xf(x)+3x^2$$

$$xf(x) = g(x)\sqrt{x^2+1} - 3x^2$$

$$f(x) = \frac{g(x)\sqrt{x^2+1} - 3x^2}{x}$$

$$\boxed{f(x) = \frac{g(x)\sqrt{x^2+1}}{x} - 3x}$$

$$\text{Άρα} \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\frac{g(x)\sqrt{x^2+1}}{x} - 3x}{x} =$$

$$= \lim_{x\to+\infty} \frac{g(x)\sqrt{x^2+1} - 3x^2}{x^2} =$$

$$= \lim_{x \to +\infty} \frac{g(x)\sqrt{x^2+1}}{x^2} - \frac{3\cancel{x^2}}{\cancel{x^2}} =$$

$$= \lim_{x \to +\infty} \frac{g(x)\cancel{x}\sqrt{1+\frac{1}{x^2}}}{x^{\cancel{2}}} - 3 = \frac{2 \cdot 1}{+\infty} - 3 = -3$$

$$\lim_{x \to +\infty} f(x) - (-3)x = \lim_{x \to +\infty} \frac{g(x)\sqrt{x^2+1}}{x} \cancel{-3x} \cancel{+3x}$$

$$= \lim_{x \to +\infty} \frac{g(x) \cdot \cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = 2$$

$$y = -3x + 2$$

4. (γ) $f(x) = \ln\left(x + \sqrt{x^2+1}\right),\ x \in R$

Εφαρμογή

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$$f'(x) = \frac{1}{x+\sqrt{x^2+1}} \left(x+\sqrt{x^2+1}\right)'$$

$$f'(x) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right)$$

$$f'(x) = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$f'(x) = \frac{1}{\cancel{x+\sqrt{x^2+1}}} \cdot \frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}$$

$$f'(x) = \frac{1}{\sqrt{x^2+1}} > 0$$

$f \nearrow$

$$f''(x) = \frac{(1)'\sqrt{x^2+1} - 1\cdot\left(\sqrt{x^2+1}\right)'}{\sqrt{x^2+1}^{\,2}}$$

$$f''(x) = \frac{-\frac{2x}{2\sqrt{x^2+1}}}{x^2+1} = -\frac{2x}{(x^2+1)\sqrt{x^2+1}}$$

| $x$ | | $0$ | |
|---|---|---|---|
| $f''$ | $+$ | | $-$ |
| $f$ | $\cup$ | | $\cap$ |

Αφου $D_f = R$ δεν εχει κατακορυφες.

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \ln\left(x+\sqrt{x^2+1}\right) = -\infty$$

δεν εχει
οριζοντια

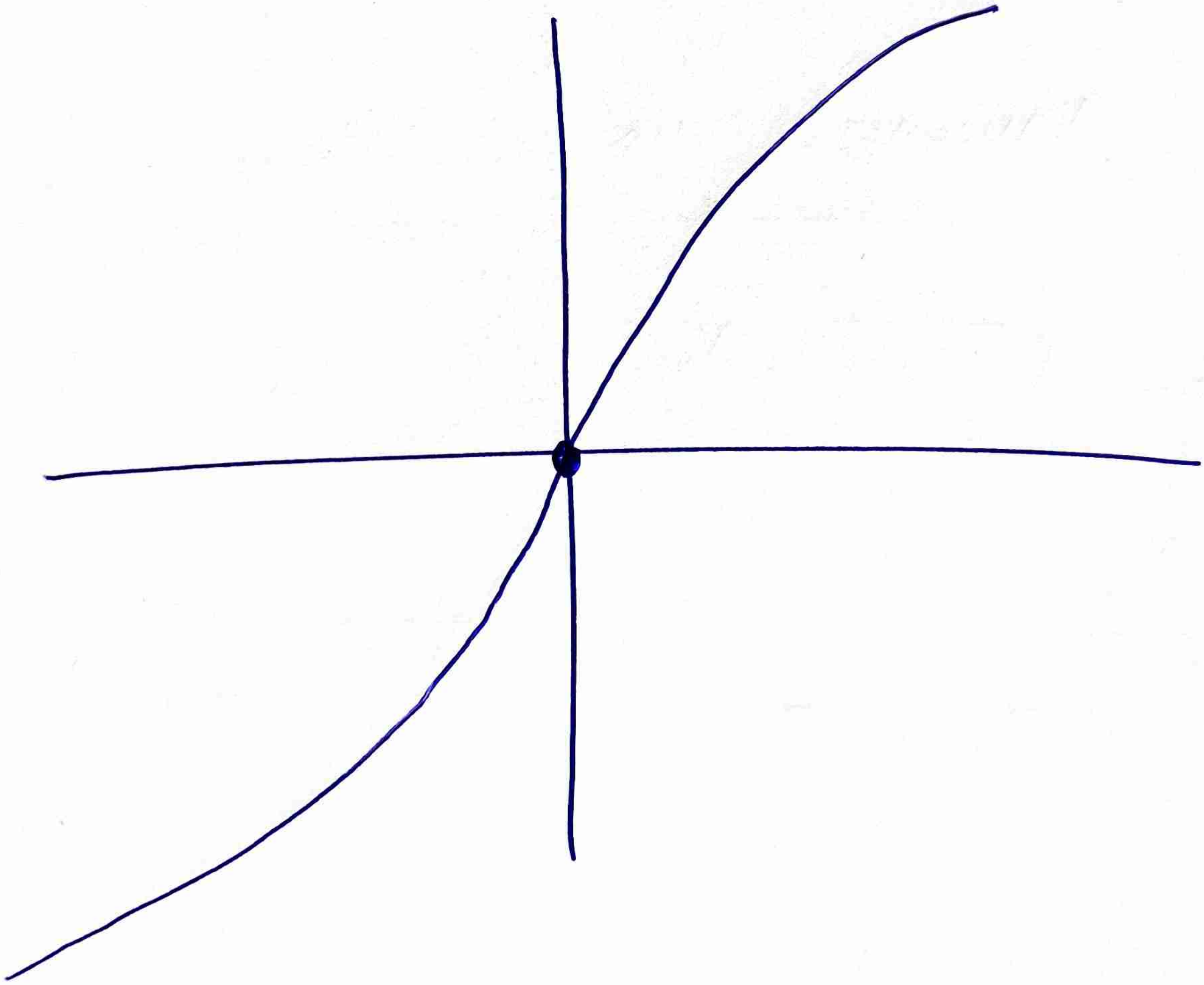
$$\to \lim_{x\to -\infty}\left(\sqrt{x^2+1}+x\right) = \lim_{x\to +\infty}\frac{1}{\sqrt{x^2+1}-x} = 0$$

$\lim\limits_{x\to+\infty} f(x) = +\infty$ δεν εχει οριζοντια στο $+\infty$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\ln\left(x+\sqrt{x^2+1}\right)}{x} =$$

$$= \lim_{x\to-\infty}\frac{1}{\sqrt{x^2+1}} = 0$$ δεν έχει πλάγια στο $-\infty$.

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}} = 0$$ δεν έχει πλάγια στο $+\infty$,

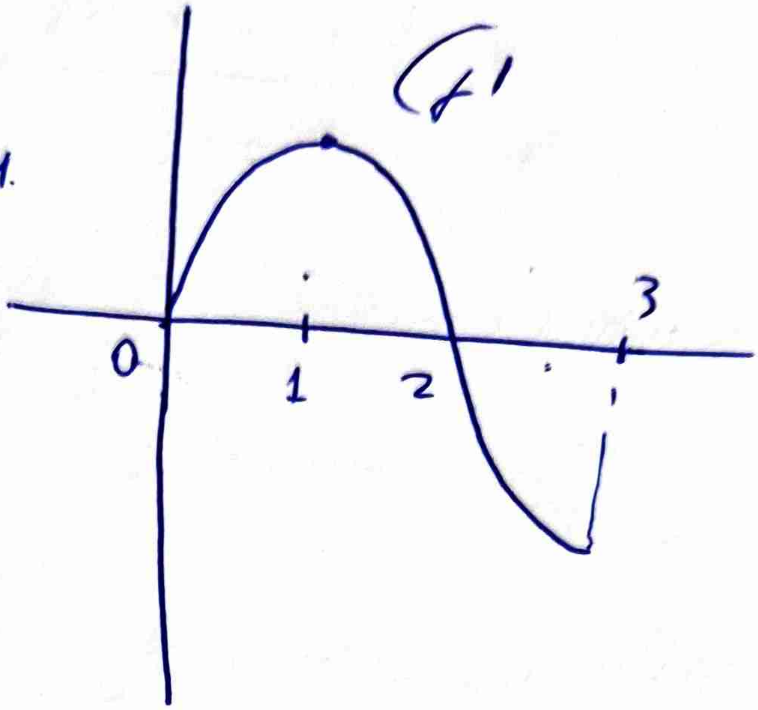


**5.** $f:[0,3]\to R$

$f(2)=2f(1)=2f(3)=4f(0)=4.$

$\boxed{f(2)=4}$ $\boxed{f(3)=2}$

$\boxed{f(1)=2}$ $\boxed{f(0)=1}$

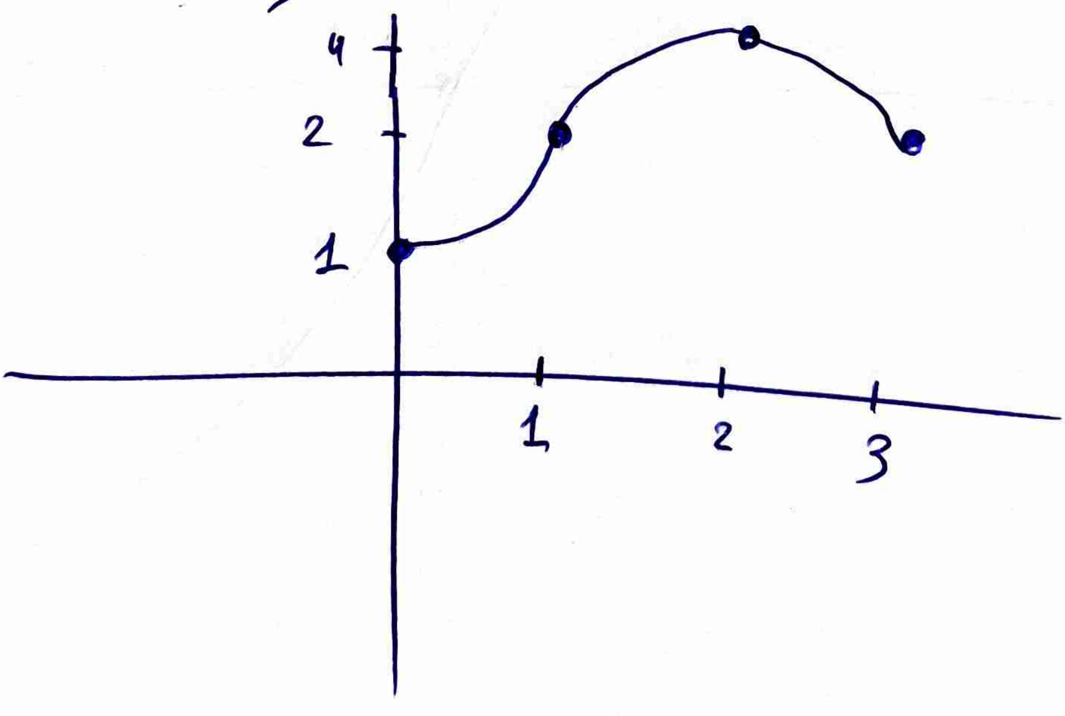



| $x$ | 0 | 2 | 3 |
|---|---|---|---|
| $f'$ | + | − | |
| $f$ | ↗ | ↘ | |

| $x$ | | 1 |
|---|---|---|
| $f'$ | ↗ | ↘ |
| $f$ | ∪ | ∩ |

Κ.Σ: 2

Π.Θ.Α: 2, 0, 3.



**14.** $f: \mathbb{R} \to \mathbb{R}$ δύο φορές παρ/μη

**Ενότητα 30**

$f''(x) \neq 0 \quad \forall x \in \mathbb{R}$.

$f(1) - f'(1) < f(0)$

Νδο $f$ κυρτή.

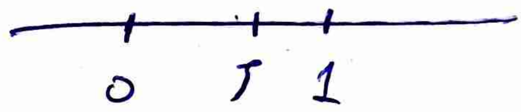
$f$ συν+παρ

$f'$ συν+παρ

$f''$ συνεχή.

Αφού $f''(x) \neq 0$ και συνεχής

$f''(x) > 0$ ή $f''(x) < 0$




$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0}$$

$f'(\xi) = f(1) - f(0)$

$f(1) - f(0) < f'(1)$

$f'(\xi) < f'(1)$

$f'(1) - f'(\xi) < 0$

+

$$f''(\rho) = \frac{f'(1) - f'(\xi)}{1 - \xi} > 0$$

+

$f''(\rho) > 0$

$f''(x) > 0$ $f$ κυρτή

**19.** $f(x) = e^x - \frac{x^3}{6} - 1$

(a) $f'(x) = e^x - \frac{1}{6} 3x^2$

$f''(x) = e^x - \frac{1}{6} 6x$

$\boxed{f''(x) = e^x - x > 0}$ f κυρτή.

• $e^x \geq x+1$

$e^x - x \geq 1$

$e^x - x > 0$

(β) $y - f(0) = f'(0)(x - 0)$

$y - 0 = x$

$y = x$

(γ) Αν $a \in (0,1)$ $\dfrac{f'(a)-1}{x} + \dfrac{f(a^2)-a^2}{x-2} + \dfrac{f^2(a)-a^2}{x-1} = 0$

$$\underbrace{(x-2)(x-1)\left(f'(a)-1\right) + x(x-1)\left(f(a^2)-a^2\right) + x(x-2)\left(f^2(a)-a^2\right)}_{g(x)} = 0$$

$g(x)$.

Αφού $f$ κυρτή

$f' \uparrow$

$0 < a < 1$

$f' \uparrow$

$f'(0) < f'(a) < f'(1)$

$\underline{\underline{1 < f'(a)}}$

$g(0) = 2\left(f'(a)-1\right) > 0$

$g(1) = -\left(f^2(a)-a^2\right) < 0$ (+)

$g(2) = 2\left(f(a^2)-a^2\right) \geq 0$ (+)

Αφού $f$ κυρτή τότε $f(x) \geq x \quad \forall x \in R$.

Έστω $(0,1)$ $f^2(x) \geq x^2$

$f^2(a) \geq a^2$

$f^2(a) - a^2 > 0$.

$f(x) \geq x$

$f(a^2) > a^2$

$g(0)g(1) < 0$ Bolzano $\exists\, \xi_1 \in (0,1)$
τ.ω $g(\xi_1) = 0$

$g(1)g(2) < 0$
Bolzano $\exists\, \xi_2 \in (1,2)$
τ.ω $g(\xi_2) = 0$.

Η $g(x)$ ειναι πολυωνυμο 2ου
βαθμου αρα εχει το πολυ 2
ριζες.
Βρηκα ηδη δυο αρα εχει
ακριβως δυο.

8 Αν $g(x) > f(x)$

$$\lim_{x\to+\infty} g(x) > \lim_{x\to+\infty} f(x)$$

$$\lim_{x\to+\infty} g(x) > \lim_{x\to+\infty} \left(e^x - \frac{x^3}{6} - 1\right)$$

$$\lim_{x\to+\infty} g(x) > +\infty$$

$$\lim_{x\to+\infty} g(x) = +\infty$$

$$\to \lim_{x\to+\infty} e^x - \frac{x^3}{6} - 1$$

$$= \lim_{x\to+\infty} e^x\left(1 - \frac{1}{6}\frac{x^3}{e^x} - \frac{1}{e^x}\right) = +\infty$$

$$\to \lim_{x\to+\infty} \frac{x^3}{e^x} = \lim_{x\to+\infty} \frac{3x^2}{e^x} = \lim_{x\to+\infty} \frac{6x}{e^x}$$

$$= \lim_{x\to+\infty} \frac{6}{e^x} = 0$$

**18.** $f(0)=0$

$g(x)=(x-1)f(x)$ κυρτή $\Rightarrow g' \nearrow$

Νδο $f(x)>0$

$$g'(\xi)=\frac{g(1)-g(0)}{1-0}=\frac{0-0}{1}=0$$

$g'(\xi)=0$

| $x$ | $0$ | | $\xi$ | | $1$ |
|---|---|---|---|---|---|
| $g'$ | | $-$ | $0$ | $+$ | |
| $g$ | | $\searrow$ | | $\nearrow$ | |

$$\frac{x>0}{g(x)<g(0)}$$

$(x-1)f(x)<0$

$\ominus$

$f(x)>0$

$$\frac{x<1}{g(x)<g(1)}$$

$(x-1)f(x)<0$

$\ominus$

$f(x)>0$ ✓

**17.** $f(1) = f(2) = 0$ $\qquad$ $f$ κοίλη.

(α) Ν.δ.ο $\exists!\ \xi \in (1,2)$ τ.ω $f'(\xi) = 0$

Αφού $\underset{0}{\underset{\|}{f(1)}} = f(2)$ Rolle $\exists\ \xi \in (1,2)$ τ.ω $f'(\xi) = 0$.

Αφού $f$ κοίλη η $f'(x) \downarrow$ το $\xi$ μοναδικό.

(β) Ν.δ.ο $f(x) > 0 \quad \forall x \in (1,2)$

| $x$ | $1$ | $\xi$ | $2$ |
|---|---|---|---|
| $f'$ | $\searrow$ $+$ | $0$ | $\searrow$ $-$ |
| $f$ | $+$ $\nearrow$ | | $\searrow$ $+$ |

$x < \xi \Rightarrow f'(x) > f'(\xi)$
$f'(x) > 0$

$x > \xi \Rightarrow f'(x) < f'(\xi)$
$f'(x) < 0$

$\xi > x > 1$

$f \uparrow$

$f(x) > f(1)$

$f(x) > 0$

$\xi < x < 2$

$f \downarrow$

$f(x) > f(2)$

$f(x) > 0$

**16.** $f:[-1,+\infty)\to\mathbb{R}$ κυρτή $\Longrightarrow f'\uparrow$

Η $\varepsilon: y=2$ εφάπτεται της $C_f$ στο $x_0=-1$

$$\begin{cases} f'(-1)=0 \\ f(-1)=0\cdot(-1)+2 \end{cases}$$

$y=0\cdot x+2$

$$\boxed{f'(-1)=0 \qquad f(-1)=2}$$

$$g(x)=\frac{f(x)-2}{x+1}, \quad x>-1.$$

$$g'(x)=\frac{f'(x)(x+1)-(f(x)-2)}{(x+1)^2}=\frac{f'(x)(x+1)-f(x)+2}{(x+1)^2}$$

$$\boxed{\text{Αρκεί νδο } f'(x)(x+1)-f(x)+2>0}$$

$$f'(\xi)=\frac{f(x)-f(-1)}{x-(-1)}=\frac{f(x)-2}{x+1}$$

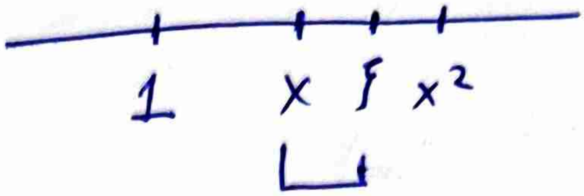
$-1 \quad \xi \quad x$

$$\xi<x \Rightarrow f'(\xi)<f'(x) \Rightarrow \frac{f(x)-2}{x+1}<f'(x)$$

$$f(x)-2<f'(x)(x+1) \Rightarrow f'(x)(x+1)-f(x)+2>0$$

**15.** $f$ κυρτή $\Rightarrow f' \uparrow$

(α) ν.δ.ο $f(x^2) - f(x) > (x^2 - x) f'(x) \quad \forall x > 1$



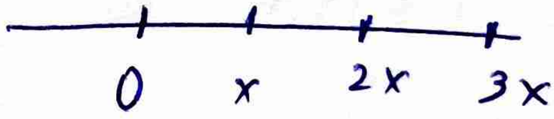

$$f'(\xi) = \frac{f(x^2) - f(x)}{x^2 - x}$$

$$x < \xi \Rightarrow f'(x) < f'(\xi)$$

$$f'(x) < \frac{f(x^2) - f(x)}{x^2 - x}$$

$$f'(x)(x^2 - x) < f(x^2) - f(x) \quad \checkmark$$

(β) ν.δ.ο $f(x) + f(3x) > 2 f(2x) \quad \forall x > 0.$



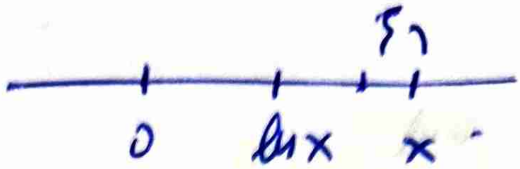

$$f'(\xi_1) = \frac{f(2x) - f(x)}{2x - x} = \frac{f(2x) - f(x)}{x}$$

$$f'(\xi_2) = \frac{f(3x) - f(2x)}{3x - 2x} = \frac{f(3x) - f(2x)}{x}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2) \Rightarrow \frac{f(2x) - f(x)}{x} < \frac{f(3x) - f(2x)}{x}$$

$$2 f(2x) < f(3x) + f(x)$$

(γ) Ν.δ.ο. $f(\ln x) - f(x) > (\ln x - x)\, f'(x) \quad \forall x>0.$




- $\ln x \le x-1$

$\ln x - x \le -1$

$\ln x - x < 0$

$\ln x < x$

$$f'(\xi) = \frac{f(x) - f(\ln x)}{x - \ln x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x)$$

$$\frac{f(x) - f(\ln x)}{x - \ln x} < f'(x)$$

$$f(x) - f(\ln x) < (x - \ln x) f'(x)$$

**26.** Αν $\alpha \in (-1,1)$.

νδο $f(x) = -\frac{x^4}{12} - \frac{\alpha x^3}{3} - \frac{x^2}{2} + 3x - 1$ κοίλη

$f'(x) = -\frac{4}{12}x^3 - \frac{3\alpha}{3}x^2 - \frac{1}{2}2x + 3$

$f''(x) = -\frac{12}{12}x^2 - 2\alpha x - 1$

$$\boxed{f''(x) = -x^2 - 2\alpha x - 1}$$

$\Delta = (-2\alpha)^2 - 4(-1)(-1)$

$\Delta = 4\alpha^2 - 4 = 4(\alpha^2 - 1)$

| $\alpha$ | | $-1$ | | $1$ | |
|---|---|---|---|---|---|
| $\Delta$ | + | | − | | + |

$\Delta < 0$

$\Rightarrow f''(x) < 0$ $f$ κοίλη.

25. $f''(x) \neq 4f'(x) - 4f(x)$

$g(x) = e^{-2x} f(x)$

Έστω ότι έχει σημείο καμπής στο $x_0$

$\Rightarrow g''(x_0) = 0$

$g'(x) = -2e^{-2x} f(x) + e^{-2x} f'(x)$

$g''(x) = 4e^{-2x} f(x) - 2e^{-2x} f'(x) - 2e^{-2x} f'(x) + e^{-2x} f''(x)$

$g''(x) = 4e^{-2x} f(x) - 4e^{-2x} f'(x) + e^{-2x} f''(x)$

$g''(x) = e^{-2x} \left( 4f(x) - 4f'(x) + f''(x) \right)$

$g''(x_0) = 0 \Leftrightarrow 4f(x_0) - 4f'(x_0) + f''(x_0) = 0$

$f''(x_0) = 4f'(x_0) - 4f(x_0)$

Άτοπο

Άρα δεν έχει σημείο καμπής.

22. $f(x) = ax^3 + \beta x^2 + 6x$

Ακρότατο στο $-1$ : Fermat $f'(-1)=0$

Καμπή στο $\frac{1}{2}$ : $f''(\frac{1}{2}) = 0$

κτλ.

**20.** $f(x) = \lambda^2 x^3 + 3\lambda x^2 + 2x - 1$

Αφου παρουσιαζει καμπη στο 1.

$f''(1) = 0.$

$f'(x) = 3\lambda^2 x^2 + 6\lambda x + 2$

$f''(x) = 6\lambda^2 x + 6\lambda$

$f''(1) = 6\lambda^2 + 6\lambda = 0$

$6\lambda(\lambda + 1) = 0$

$\lambda = 0$ $\lambda = -1$

**17.** $f(x) = e^{x-2} - \dfrac{x^3}{6}$ , $x \in \mathbb{R}$

$f'(x) = e^{x-2} - \frac{1}{6} 3x^2$

$f''(x) = e^{x-2} - x$

$f'''(x) = e^{x-2} - 1$

$\longrightarrow f'''(x) = 0$

$e^{x-2} - 1 = 0$

$e^{x-2} = 1$

$x - 2 = 0$

$x = 2$

- $e^x \geq x + 1$

$e^{x-2} \geq x - 2 + 1$

$e^{x-2} - x \geq -1$ ;

| $x$ | $-\infty$ | $2$ | $+\infty$ |
|---|---|---|---|
| $f'''$ | $-$ | $0$ | $+$ |
| $f''$ | $+\infty \searrow$ | $-1$ | $\nearrow +\infty$ |
| $f$ | | | |

$\Sigma T_{f''}$

$f''(2) = e^{2-2} - 2 = 1 - 2 = -1$

$\lim_{x \to -\infty} f''(x) = +\infty$

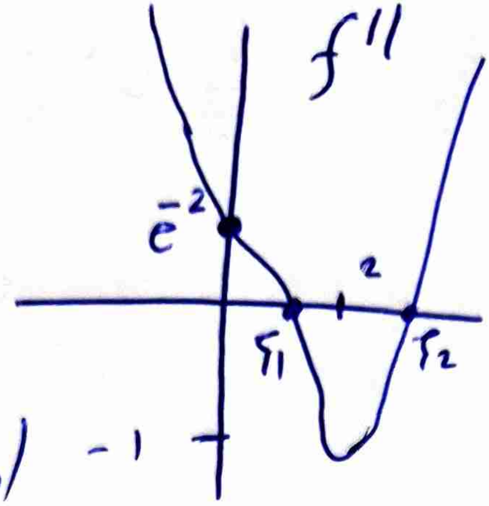
$\lim_{x \to +\infty} f''(x) = \lim_{x \to +\infty} e^{x-2} - x = \lim_{x \to +\infty} x\left(\dfrac{e^{x-2}}{x} - 1\right)$

$= +\infty$

$\rightarrow \lim_{x \to +\infty} \dfrac{e^{x-2}}{x} = \lim_{x \to +\infty} \dfrac{e^{x-2}}{1} = +\infty$

$x < 2$

- $f''$ συνεχής
- $f''$ ↓
- $\Sigma T_{f''} = (-1, +\infty)$




$x \geq 2$

- $f''$ συνεχής
- $f''$ ↗
- $\Sigma T_{f''} = [-1, +\infty)$

Το $0 \in \Sigma T_{f''}$ άρα $\exists!\ \xi_1$ τ.ω $f''(\xi_1) = 0$

Το $0 \in \Sigma T_{f''}$ άρα $\exists!\ \xi_2$ τ.ω $f''(\xi_2) = 0$

| $x$ | | $\xi_1$ | | $2$ | | $\xi_2$ | |
|---|---|---|---|---|---|---|---|
| $f'''$ | $-$ | | $-$ | $0$ | $+$ | | $+$ |
| $f''$ | $+$ ↘ | $0$ | ↘ $-$ | | $-$ ↗ | $0$ | ↗ $+$ |
| $f$ | $\cup$ | | $\cap$ | | $\cap$ | | $\cup$ |

**15.** $f(x) = e^x - \frac{x^2}{2} - x + \ln x \quad , x > 0$

$f'(x) = e^x - \frac{1}{2} 2x - 1 + \frac{1}{x}$

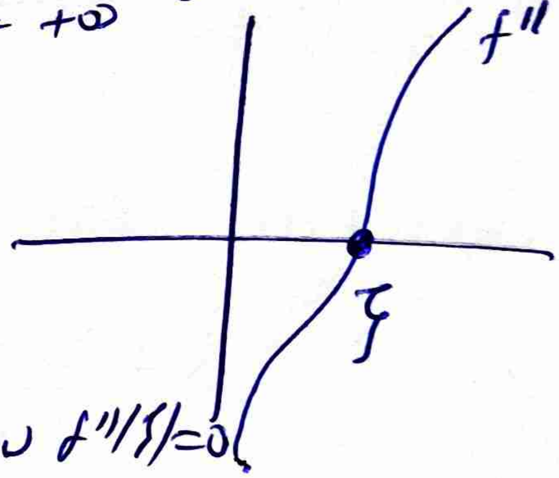
$f''(x) = e^x - 1 - \frac{1}{x^2}$

$f'''(x) = e^x - \frac{-2x}{x^4} = e^x + \frac{2}{x^3} > 0$

| $x$ | $0 \qquad +\infty$ |
|---|---|
| $f'''$ | $+$ |
| $f''$ | $-\infty \nearrow$ |
| $f$ | |

$\Sigma T_{f''}$

$\left.\begin{array}{l} \lim_{x \to 0^+} f''(x) = -\infty \\ \lim_{x \to +\infty} f''(x) = +\infty \end{array}\right\} \Sigma T_{f''} = \mathbb{R}.$




- $f''$ συνεχής
- $f'' \nearrow$
- $\Sigma T_{f''} = \mathbb{R}$

το $0 \in \Sigma T_{f''}$ άρα $\exists! \, \xi$ τ.ω $f''(\xi) = 0$.

| $x$ | | $\xi$ | |
|---|---|---|---|
| $f''$ | $\nearrow \; -$ | $0$ | $+ \nearrow$ |
| $f$ | $\frown$ | | $\smile$ |