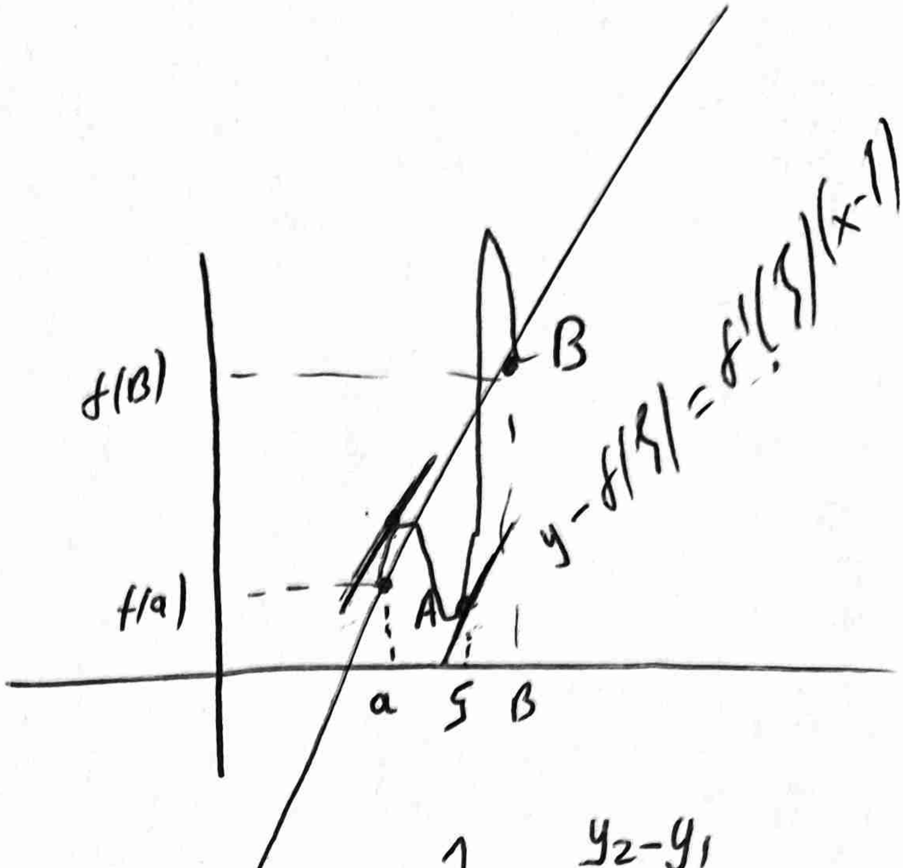
21

Το Θεωρημα Μεσου Τιμης.

ΘΜΤ

$\exists\, \xi \in (\alpha, \beta)$ τ.ω

$$f'(\xi) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$$



f συνεχής $[\alpha, \beta]$.

f παρ/μη (α, β).

ε_{AB} : $\lambda_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

$$\boxed{\lambda_{AB} = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}}$$

Bolzano

f συνεχ $[\alpha,\beta]$

$f(\alpha)f(\beta)<0$

$\Rightarrow \exists\, \xi\in(\alpha,\beta)$ τ.ω $f(\xi)=0$

Rolle

f συνεχ $[\alpha,\beta]$

f παρ/μη (α,β)

$f(\alpha)=f(\beta)$

$\Rightarrow \exists\, \xi\in(\alpha,\beta)$ τ.ω $f'(\xi)=0$

ΘΜΤ

f συνεχ $[\alpha,\beta]$

f παρ/μη (α,β)

$\Rightarrow \exists\, \xi\in(\alpha,\beta)$

τ.ω

$$f'(\xi)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$$

ΘΕΤ

f συνεχ $[\alpha,\beta]$

$f(\alpha)\neq f(\beta)$

$\forall\, \eta$ μεταξύ $(f(\alpha), f(\beta))$

$\exists\, \xi\in(\alpha,\beta)$ τ.ω

$f(\xi)=\eta$

ΘΜΕΤ

f συνεχ $[\alpha,\beta]$

$\Rightarrow m\le f(x)\le M$

$\forall x\in(\alpha,\beta)$.

2024 ΘΕΜΑ Γ

$$f(x) = \begin{cases} -2x+4+e^{\lambda}, & 0 \le x < 2 \\ -x^2+4x-3+\lambda, & x \ge 2 \end{cases}$$ συνεχής!

Γ1: Νδο $\lambda = 0$.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-2x+4+e^{\lambda}) = e^{\lambda}$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x^2+4x-3+\lambda) = 1+\lambda$.

Άρα πρέπει $e^{\lambda} = 1+\lambda$.

• $e^x \ge x+1$ Το "=" $x = 0$.

$\underline{\lambda = 0}$

Τ2 :

$$f(x) = \begin{cases} -2x+5, & 0 \le x < 2 \\ -x^2+4x-3, & x \geq 2 \end{cases}$$

$0 \le x < 2$

$f_1(x) = -2x+5$

$f_1'(x) = -2 < 0$

$x > 2$

$f_2(x) = -x^2+4x-3$

$f_2(x) = -2x+4$

$\rightarrow \; -2x+4=0$

$-2x=-4$

$x=2$

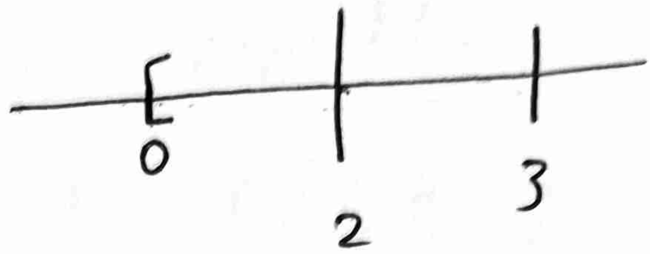
x	0	2
f_1'	$-$	$-$
f_2'	$+$	$-$
f'	$-$	$-$
f	$\searrow$	$\searrow$

Η $f \downarrow [0,+\infty)$

Το $A(0,5)$ Ο.Μ.

Γ3 : i) ΘΜΤ στο $[0,3]$;

Αφού η f συνεχής



είναι και συνεχής

στο $[0,3]$.

Η f παρ/μη $(0,2)$ και $(2,3)$ ως π.π.δ.

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{-2x+5-1}{x-2}=$$

$$=\lim_{x\to 2^-}\frac{-2x+4}{x-2}=\lim_{x\to 2^-}\frac{-2}{1}=-2$$

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{-x^2+4x-3-1}{x-2}$$

$$=\lim_{x\to 2^+}\frac{-x^2+4x-4}{x-2}=\lim_{x\to 2^+}\frac{-2x+4}{1}$$

Η f όχι παρ/μη στο 2 $=0$.

άρα δεν ισχύει το ΘΜΤ.

ii) Ψάχνω $\xi \in (0,3)$ ώστε η εφαπτομένη στο ξ να είναι παράλληλη $\Delta(0, f(0))$ και $E(3, f(3))$)

Αρκεί να βρω $\xi \in (0,3)$ τ.ω

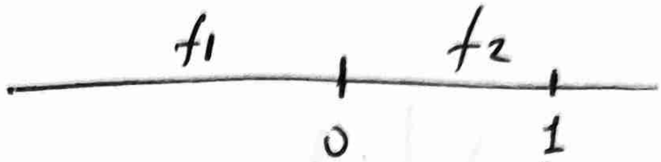
$$f'(\xi) = \frac{f(3) - f(0)}{3 - 0}.$$

$$f'(\xi) = \frac{0 - 5}{3} \Rightarrow f'(\xi) = -\frac{5}{3}.$$

$0 \leq x < 2$	$x > 2$
$f'(x) = -\frac{5}{3}$.	$f'(x) = -\frac{5}{3}$.
$-2 = -\frac{5}{3}$	$-2x + 4 = -\frac{5}{3}$
Αδύνατη.	$-2x = -\frac{5}{3} - 4$
	$-2x = -\frac{17}{3}$
	$x = \frac{17}{6}$

Ενοτητα 21

14. $f(x) = \begin{cases} x^3+\alpha, & x<0 \\ 0, & x=0 \\ x^2, & x>0 \end{cases}$



ⓐ Η f συνεχής $(0,1)$ ως π.σ.σ.

- $f(0) = 0$
- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$

✓

- $f(1) = 1$
- $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

✓

Συνεχής $[0,1]$.

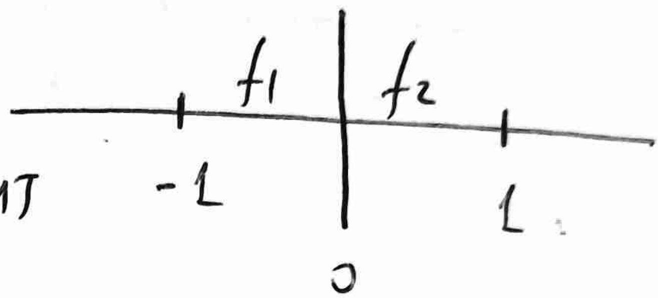
Η f παραγ/μη $(0,1)$ ως π.π.σ.

Δουλεύω το ΘΜΤ.

$\exists\, \xi \in (0,1)$ τ.ω. $f'(\xi) = \dfrac{f(1) - f(0)}{1 - 0}$

$$f'(\xi) = \frac{1-0}{1-0} \Rightarrow \underline{\underline{f'(\xi) = 1}}$$

(B)



Αφού ικανοποιούνται οι προυποθέσεις του ΘΜΤ στο $[-1, 1]$

Η f συνεχής στο 0!

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\lim_{x \to 0^-} x^3 + \alpha = \lim_{x \to 0^+} x^2$$

$$\underline{\underline{\alpha = 0}}$$

(γ) Νδο $\exists \xi \in (\alpha, \beta)$ η εφαπτομενη

της C_f στο ξ.

$$y - f(\xi) = f'(\xi)(x - \xi)$$

$$\varepsilon_{AB} : \lambda_{AB} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{f(1) - f(-1)}{2}.$$

$A(-1, f(-1))$

$B(1, f(1))$

Αρκει νδο $\exists \xi \in (\alpha, \beta)$

τ.ω $$f'(\xi) = \frac{f(1) - f(1)}{1 - (-1)}$$

Αυτο ισχυει απο πριν

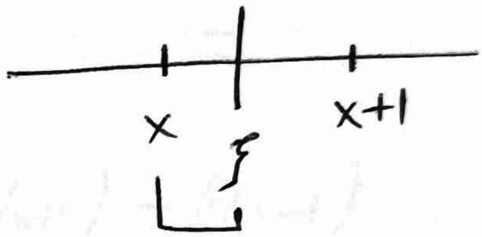
Βασικη Ασκηση

Ανισοτητων – ΘΜΤ

Εστω $f' \nearrow$.

1. Νδο $f'(x) + f(x) < f(x+1)$

$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x}$$



$$f'(\xi) = \frac{f(x+1) - f(x)}{1} = f(x+1) - f(x)$$

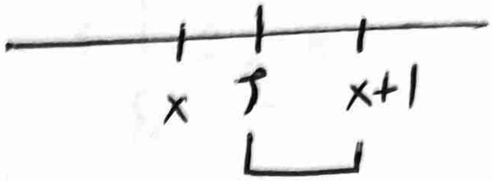
• $x < \xi \overset{f' \nearrow}{\Rightarrow} f'(x) < f'(\xi)$

$$f'(x) < f(x+1) - f(x)$$

$$f'(x) + f(x) < f(x+1)$$

2. Nδο $f(x+1) - f'(x+1) < f(x)$

$$f'(\xi) = \frac{f(x+1)-f(x)}{x+1-x} = f(x+1) - f(x)$$

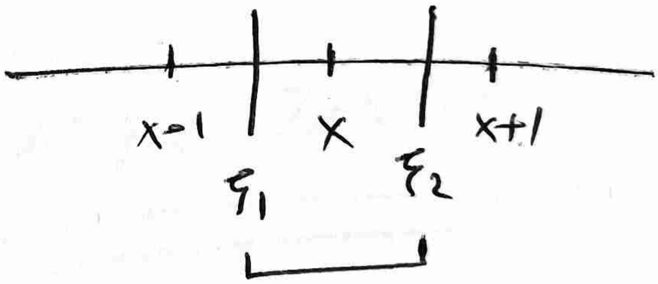


$$\xi < x+1 \overset{f'\uparrow}{\Longrightarrow} f'(\xi) < f'(x+1)$$

$$\underline{\underline{f(x+1) - f(x) < f'(x+1)}}$$

3. Nδο $2f(x) < f(x+1) + f(x-1)$

$$f'(\xi_1) = \frac{f(x)-f(x-1)}{x-(x-1)}$$



$$f'(\xi_1) = \frac{f(x)-f(x-1)}{1} = f(x) - f(x-1)$$

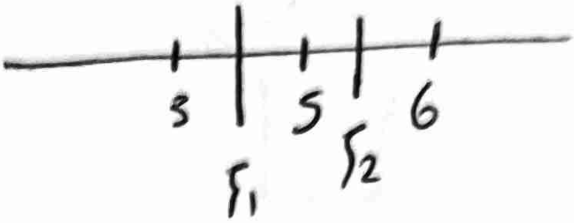
$$f'(\xi_2) = \frac{f(x+1)-f(x)}{x+1-x} = f(x+1) - f(x)$$

$$\xi_1 < \xi_2 \overset{f'\uparrow}{\Longrightarrow} f'(\xi_1) < f'(\xi_2) \Rightarrow f(x) - f(x-1) < f(x+1) - f(x)$$

$$\underline{2f(x) < f(x-1) + f(x+1)}$$

4. NdO $3f(5) < 2f(6)+f(3)$

$$f'(\xi_1) = \frac{f(5)-f(3)}{5-3} = \frac{f(5)-f(3)}{2}$$



$$f'(\xi_2) = \frac{f(6)-f(5)}{6-5} = f(6)-f(5)$$

$$\xi_1 < \xi_2 \xRightarrow{f'\nearrow} f'(\xi_1) < f'(\xi_2)$$

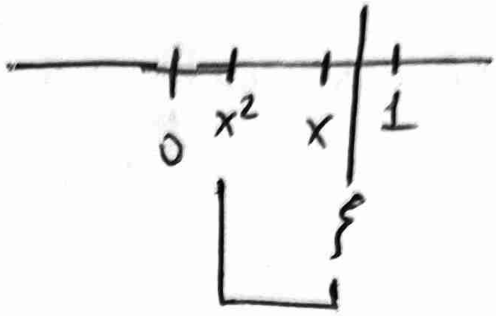
$$\Rightarrow \frac{f(5)-f(3)}{2} < f(6)-f(5)$$

$$f(5)-f(3) < 2f(6)-2f(5)$$

$$\underline{\underline{3f(5) < 2f(6)+f(3)}}$$

5. Νδο $f(x)<f(1)+(x-1)f'(x^2)$ $\quad x\in(0,1)$

$$f'(\xi)=\frac{f(1)-f(x)}{1-x}$$



$$x^2<\xi \overset{f'\uparrow}{\Rightarrow} f'(x^2)<f'(\xi) \Rightarrow f'(x^2)<\frac{f(1)-f(x)}{1-x}$$

$$\Rightarrow (1-x)f'(x^2) < f(1)-f(x)$$

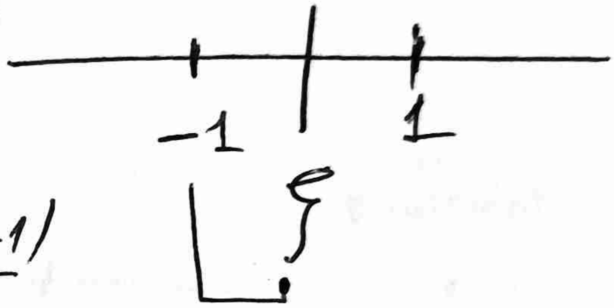
$$f(x) < f(1)+(x-1)f'(x^2)$$

6. Αν η ευθεία $\varepsilon: y = -2x+3$ εφάπτεται της C_f στο $A(-1, f(-1))$

ν.δ.ο $f(1) > 1$.

$$\begin{cases} f'(-1) = -2 \\ f(-1) = -2(-1)+3 \end{cases}$$

$$f(-1) = 5$$



$$f'(\xi) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{f(1)-f(-1)}{2}$$

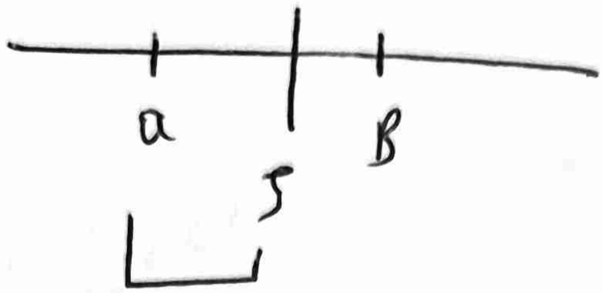
$$-1 < \xi \overset{f'\uparrow}{\Longrightarrow} f'(-1) < f'(\xi)$$

$$-2 < \frac{f(1)-f(-1)}{2}$$

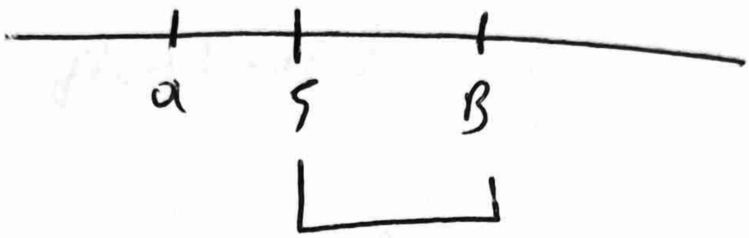
$$-4 < f(1) - 5$$

$$1 < f(1)$$

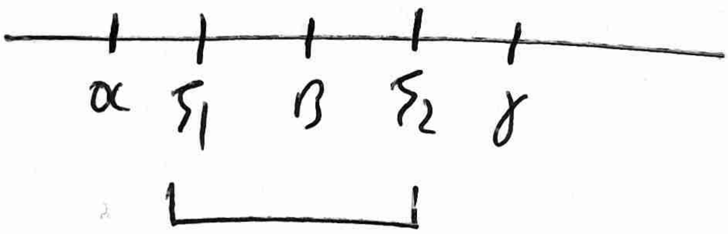
Περιπτωσεις

1. 

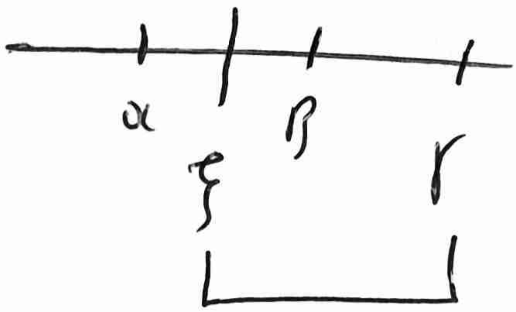
εχω $f'(\alpha)$

2. 

εχω $f'(\beta)$

3. 

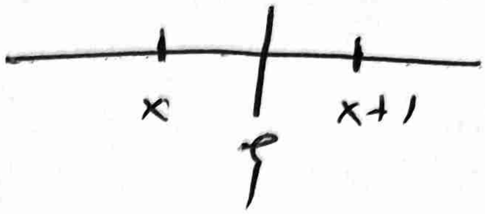
καθολου f'

4. 

εχω $f'(\gamma)$,

5. 8 NSO $\exists \xi \in (x, x+1)$

T.W $f(x) + f'(\xi) = f(x+1)$



$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x}$$

$$f'(\xi) = f(x+1) - f(x)$$

$$\underline{f(x) + f'(\xi) = f(x+1)}$$

Επομαο Μαθημα

21

2 γ

3

4

5 α β γ.

8

9

10

22