Ενοτητα 20

12. $f: \mathbb{R} \to \mathbb{R}$ παρ/μη

$f'(x) \neq 3x^2 \quad \forall x \in \mathbb{R}$.

Νδο $f(x) = x^3$ εχει το πολυ 1 ριζα.

Εστω οτι η $g(x)$ εχει 2 ριζες

$\underline{\underline{g(x) = f(x) - x^3}}$

$g(x_1) = g(x_2) = 0$

Rolle $\exists\, \xi \in (x_1, x_2)$

τ.ω $g'(\xi) = 0$

$g'(x) = f'(x) - 3x^2$

$g'(\xi) = f'(\xi) - 3\xi^2 = 0$

$f'(\xi) = 3\xi^2$ Ατοπο!

η $g(x)$ εχει το πολυ 1 ριζα.

14. $f(0)=0$ $f(1)=1$

f δύο φορές παρ/μη $f''(x) \neq 2$

f συνεχής + παρ/μη

f' συνεχής + παρ/μη $g(x)=f(x)-x^2$

f''

(α) $g(0)=f(0)-0^2=0$

$g(1)=f(1)-1^2=1-1=0$ } Rolle ✓

(β) $f(x)=x^2$

$f(x)-x^2=0$

$g(x)=0$

Έστω ότι η $g(x)$ έχει 3 ρίζες

$g(x_1)=g(x_2)=g(x_3)=0$

$g'(\xi_1)=g'(\xi_2)=0$

$g''(\kappa)=0$

$g'(x)=f'(x)-2x$

$g''(x)=f''(x)-2$

$g''(\kappa)=0 \Rightarrow f''(\kappa)-2=0$

$f''(\kappa)=2$ Άτοπο!

Η $g(x)$ έχει το πολύ 2 ρίζες.

γ)

$f(x) = x^2$

$\underbrace{f(x) - x^2}_{g(x)} = 0$

$g(x) = 0$

$\left.\begin{aligned} g(0) &= 0 \\ g(1) &= 0 \end{aligned}\right\}$ έχω ήδη βρει 2 ρίζες

Δεν έχει άλλες γιατί έδειξα ότι έχει το πολύ 2 ρίζες.

23. $f(2)=0$ $\quad f(3)=0$.

(α) $G(x) = \dfrac{f(x)}{x-4}$ Rolle $[2,3]$

$$G(2) = \frac{f(2)}{2-4} = 0$$

$$G(3) = \frac{f(3)}{3-4} = 0$$

$\Big\}$ Rolle ✓

(β). Νδο $\exists \xi \in (2,3)$ τ.ω η εφαπτομενη στο ξ

να περνα $A(4,0)$

$$y - f(\xi) = f'(\xi)(x-\xi) \longrightarrow A(4,0)$$

$$0 - f(\xi) = f'(\xi)(4-\xi)$$

Αρκει νδο $\exists \xi \in (2,3)$ τ.ω $-f(\xi) = f'(\xi)(4-\xi)$

$$G'(\xi) = 0 \Rightarrow \frac{f'(\xi)(\xi-4) - f(\xi)}{\xi - 4} = 0$$

$$G'(x) = \frac{f'(x)(x-4) - f(x)}{x-4}$$

$$-f(\xi) + f'(\xi)(\xi-4) = 0$$

24. $3f(2) = 2f(3)$

(α) ΝΔΟ $\exists x_0 \in (2,3)$ τ.ω

$$f'(x_0) = \frac{f(x_0)}{x_0}$$

$$f'(x) = \frac{f(x)}{x}$$

$$\underbrace{x f'(x) - f(x)}_{g(x)} = 0$$

$$\frac{x f'(x) - f(x)}{x^2} = 0 \Rightarrow \Big(\underbrace{\frac{f(x)}{x}}_{\varphi(x)}\Big)' = 0$$

$$\varphi(2) = \frac{f(2)}{2}$$

$$\varphi(3) = \frac{f(3)}{3} = \frac{\frac{\cancel{3}f(2)}{2}}{\cancel{3}} = \frac{f(2)}{2} = \varphi(2)$$

Rolle $\exists \xi \in (2,3)$ τ.ω $\varphi'(\xi) = 0$.

$$\frac{\xi f'(\xi) - f(\xi)}{\xi^2} = 0 \Rightarrow f'(\xi) = \frac{f(\xi)}{\xi}.$$

(B) $y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow (0,0)$

$$0 - f(x_0) = f'(x_0)(0 - x_0)$$

$$-f(x_0) = -x_0 f'(x_0)$$

$$f(x_0) = x_0 f'(x_0)$$

$$\frac{f(x_0)}{x_0} = f'(x_0)$$

[illegible]

41. $f(x) = x^3 - 6\ln x$

$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(0,4)$

$4 - f(x_0) = f'(x_0)(0 - x_0)$

$4 - f(x_0) = -x_0 f'(x_0)$

$4 - f(x) = -x f'(x)$

$4 - (x^3 - 6\ln x) = -x\left(3x^2 - \frac{6}{x}\right)$

$4 - x^3 + 6\ln x = -3x^3 + 6$

$2x^3 + 6\ln x - 2 = 0$

$$\boxed{x^3 + 3\ln x - 1 = 0.}$$

$g(x) = x^3 + 3\ln x - 1 \quad , x > 0$

$g'(x) = 3x^2 + \frac{3}{x} > 0$

$g \nearrow$

Άρα $\quad x^3 + 3\ln x - 1 = 0$

$g(x) = 0$

$g(x) = g(1)$

g 1-1

$x = 1$

$y - f(1) = f'(1)(x-1)$

$y - 1 = -3(x-1)$

$y = -3x + 4$

42.

$f(x) = \eta\mu x \qquad x \in [0, \pi]$

$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\frac{\pi}{2} - f(x_0) = f'(x_0)\left(\frac{\pi}{2} - x_0\right)$

$\frac{\pi}{2} - \eta\mu x_0 = \sigma\upsilon\nu x_0 \left(\frac{\pi}{2} - x_0\right)$

$$\boxed{\frac{\pi}{2} - \eta\mu x - \sigma\upsilon\nu x\left(\frac{\pi}{2} - x\right) = 0}$$

$g(x) = \frac{\pi}{2} - \eta\mu x - \sigma\upsilon\nu x\left(\frac{\pi}{2} - x\right)$

$g(0) = \frac{\pi}{2} - 0 - 1 \cdot \left(\frac{\pi}{2}\right) = 0$

$g(\pi) = 0.$

Προφανείς ρίζες

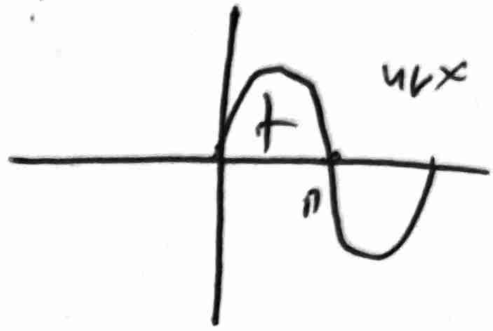
$x = 0 \qquad x = \pi$

$$g'(x) = -\sigma\upsilon\nu x - \left[(-\eta\mu x)\left(\frac{\pi}{2} - x\right) - \sigma\upsilon\nu x\right]$$

$$g'(x) = -\cancel{\sigma\upsilon\nu x} + \eta\mu x\left(\frac{\pi}{2} - x\right) + \cancel{\sigma\upsilon\nu x}$$

$$g'(x) = \underset{+}{\eta\mu x}\left(\frac{\pi}{2} - x\right)$$

$\rightarrow$ $g'(x) = 0 \Rightarrow x = \frac{\pi}{2}$



x	0	$\frac{\pi}{2}$	π
g'	$+$		$-$
g	↗		↘

Αφου εχω δυο διαστηματα μονοτονιας η Cf τεμνει τον x'x το πολυ 2 φορες. Εχω ηδη βρει 2 αρα

$x = 0$ $\quad$ $x = \pi$ μοναδικες ρι.

43. $f(x) = \ln(x-2) + 4 \ , \quad x > 2$

$g(x) = e^x - x^2$

(a) $y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(0,1)$

$$1 - f(x_0) = \frac{1}{x_0 - 2}(0 - x_0)$$

$$1 - f(x_0) = -\frac{x_0}{x_0 - 2}$$

$$1 - \ln(x_0 - 2) - 4 = -\frac{x_0}{x_0 - 2}$$

$$-3 - \ln(x_0 - 2) = -\frac{x_0}{x_0 - 2}$$

$$3 + \ln(x - 2) = \frac{x}{x - 2}$$

$$\boxed{3 + \ln(x-2) - \frac{x}{x-2} = 0}$$

$\varphi(x) = 0$

$\varphi(x) = \varphi(3)$

φ 1-1

$x = 3$

$$\varphi(x) = 3 + \ln|x-2| - \frac{x}{x-2}$$

$$\varphi'(x) = + \frac{1}{x-2} - \frac{x-2-x}{(x-2)^2}$$

$$\varphi'(x) = + \frac{x-2}{(x-2)^2} - \frac{-2}{(x-2)^2}$$

$$\varphi'(x) = \frac{x-2}{(x-2)^2} + \frac{2}{(x-2)^2} = \frac{x}{(x-2)^2} > 0$$

$\varphi \nearrow$

x	2		4	
φ'		+		−
φ		$\nearrow$		$\searrow$

$$\lim_{x \to 2^+} \varphi(x) = \lim_{x \to 2^+} 3 - \ln(x-2) - \frac{x}{x-2} =$$

$$= 3 - (-\infty) - (+\infty)$$

$y - f(3) = f'(3)(x-3)$

$y - 4 = x - 3$

$y = x + 1$

(13) $\begin{cases} g'(x) = 1 \\ g(x) = x+1 \end{cases}$

$\begin{cases} e^x - 2x = 1 \\ e^x - x^2 = x+1 \end{cases}$ $\Rightarrow e^x = 2x+1$

$2x + \cancel{1} - x^2 = x + \cancel{1}$

$x^2 - x = 0$

$x = 0$ $\cancel{x = 1}$

| 44. $f(x) = 2e^x - 1$, $x \geq 0$

$g(x) = -x^2$

$f'(x) = 2e^x$

$g'(x) = -2x$

$$\begin{cases} f'(a) = g'(B) \\ f(a) - a f'(a) = g(B) - B g'(B) \end{cases}$$

$$\begin{cases} 2e^a = -2B \quad \Rightarrow e^a = -B \Rightarrow B = -e^a \\ 2e^a - 1 - a\, 2e^a = -B^2 - B \cdot (-2B) \end{cases}$$

$$2e^a - 1 - 2ae^a = -B^2 + 2B^2$$

$$2e^a - 1 - 2ae^a = B^2$$

$$2e^a - 1 - 2ae^a = (-e^a)^2$$

$$2e^a - 1 - 2ae^a = e^{2a}$$

$$e^{2a} + 2ae^{a} - 2e^{a} + 1 = 0$$

$$\boxed{h(x) = e^{2x} + 2xe^{x} - 2e^{x} + 1}$$

$$h'(x) = 2e^{2x} + \cancel{2e^{x}} + 2xe^{x} - \cancel{2e^{x}}$$

$$h'(x) = 2e^{x}(x + e^{x}) > 0$$

$h \nearrow$

$$h(a) = 0$$

$$h(a) = h(0)$$

h 31-1

$$a = 0$$

$$y - f(0) = f'(0)(x - 0)$$

$$y - (-1) = 2(x - 0)$$

$$y = 2x - 1$$

Nδɔ $\qquad f'(x) < 0$

18. (B) $f(3) - \cos 2 < f(2) - \cos 3$

$$f(3) + \cos 3 < f(2) + \cos 2$$

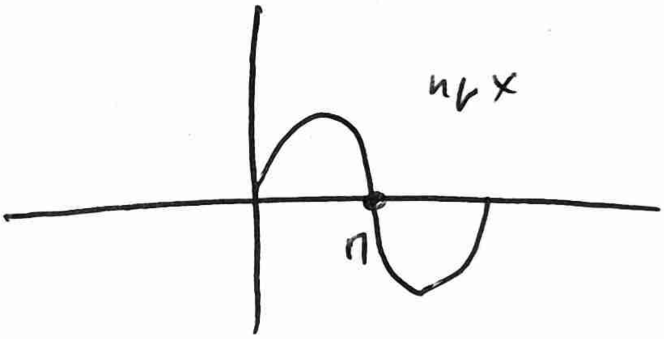
$$g(x) = f(x) + \cos x$$

$$g(3) < g(2)$$

$g\downarrow$

$3 > 2$

$$g'(x) = \underset{\ominus}{f'(x)} - \underset{\oplus}{\sin x} < 0$$



$g\downarrow$

(8) $f(2) - \cos 2 < f(1) - \cos 3$

$f(2) + \cos 3 < f(1) + \cos 2$

$\boxed{g(x) = f(x-1) + \cos x}$

$g(3) < g(2)$

$g \downarrow$

$3 > 2$ ✓

$g'(x) = f'(x-1) - \sin x < 0$

$\quad\quad\quad\quad (-) \quad\quad\quad (+)$

$g \downarrow$

| 19. (γ) $f(e) < \ln \frac{1+e^2}{5^{1-e}}$

$f'(x) < 0$

$f(2) = e\ln 5$

$f(e) < \ln(1+e^2) - \ln 5^{1-e}$

$f(e) < \ln(1+e^2) - (1-e)\ln 5$

$f(e) < \ln(1+e^2) - \ln 5 + e\ln 5$

$f(e) < \ln(1+e^2) - \ln 5 + f(2)$

$f(e) - \ln(1+e^2) < f(2) - \ln 5$.

$\varphi(x) = f(x) - \ln(x^2+1)$

$\varphi(e) < \varphi(2)$

φ ↘

$e > 2$ ✓

$$\varphi'(x) = \underbrace{f'(x)}_{-} - \frac{2x}{x^2+1} < 0$$

$$\varphi \swarrow$$

20. (B) $3^{x-1}+4^{x-1}=5^{x-1}$

$$\frac{3^{x-1}}{5^{x-1}}+\frac{4^{x-1}}{5^{x-1}}=1$$

$$\underbrace{\left(\frac{3}{5}\right)^{x-1}+\left(\frac{4}{5}\right)^{x-1}-1}_{f(x)}=0$$

$f(x)=0$

$f(x)=f(3)$

f 1-1

$\underline{\underline{x=3}}$

- $x_1<x_2 \Rightarrow x_1-1<x_2-1$

$$\left(\frac{3}{5}\right)^{x_1-1}>\left(\frac{3}{5}\right)^{x_2-1}$$

$\downarrow$

- $x_1<x_2 \Rightarrow \left(\frac{4}{5}\right)^{x_1-1}>\left(\frac{4}{5}\right)^{x_2-1}$

$f\downarrow$.

30. $f(1)=g(1)$

$f'(x) > g'(x) \quad \forall x \in \mathbb{R}$

Έστω $f(x)<g(x) \Rightarrow \underbrace{f(x)-g(x)}_{h(x)} < 0$

$h(x) < 0$

$h'(x) = f'(x) - g'(x) > 0$

$h \nearrow$

$h(x) < h(1)$

$h \nearrow$

$x > 1$.

Όταν $x<1 \Rightarrow f(x) > g(x)$

Όταν $x=1$ Τέμνονται

31. $f(0)=0$

$f'(x) < 2x - \text{συν}x \quad \forall x \in \mathbb{R}$

(α) Νδο $f(x) > x^2 - \text{ημ}x \quad \forall x<0$,

[redacted]

$$\underbrace{f(x) - x^2 + \text{ημ}x}_{\varphi(x)} > 0$$

$\varphi'(x) = f'(x) - 2x + \text{συν}x < 0$

$\varphi(x) \downarrow$

Αν $x<0 \Rightarrow \varphi(x) > \varphi(0)$

$\Rightarrow f(x) - x^2 + \text{ημ}x > 0$

$f(x) > x^2 - \text{ημ}x.$

(Β) $f(x) = x^2 - ημx$

$\boxed{\text{Προφανής ρίζα} \quad x=0}$

Αν $x<0$ $\quad f(x) > x^2 - ημx$.

Αν $x>0$ $\quad f(x) < x^2 - ημx$.

(γ) $\lim_{x \to +\infty} f(x)$

Αν $x<0$ $\quad f(x) > x^2 - ημx$

$\lim_{x \to +\infty} f(x) > \lim_{x \to +\infty} x^2 - ημx$ ..

$$-1 \le \text{ημ}x \le 1$$

$$1 \ge -\text{ημ}x \ge -1$$

$$\boxed{x^2+1 \ge x^2-\text{ημ}x \ge x^2-1}$$

$$\left.\begin{aligned} &\lim_{x\to+\infty} x^2+1 = +\infty \\ &\lim_{x\to+\infty} x^2-1 = +\infty \end{aligned}\right\}$$ Άρα Κ. Π.

$$\lim_{x\to+\infty} x^2-\text{ημ}x = +\infty.$$

$$\lim_{x\to+\infty} f(x) > +\infty$$

$$\lim_{x\to+\infty} f(x) = +\infty$$

34. $f:(0,+\infty)\to\mathbb{R}$ $\quad f(1)=0$

$\forall x>0$ ισχύει $x f'(x) < f(x) + x^2 συν x$

Νδο $f(x) < x ημ x$ $\quad \forall x>1$

$$\underbrace{f(x) - x ημ x}_{g(x)} < 0$$

$$g'(x) = f'(x) - (ημ x + x συν x)$$

$$g'(x) = f'(x) - ημ x - x συν x.$$

Δυστυχώς ΟΧΙ!

Β' τρόπος

$$x f'(x) < f(x) + x^2 \sigma\upsilon\nu x$$

$$x f'(x) - f(x) < x^2 \sigma\upsilon\nu x$$

$$\frac{x f'(x) - f(x)}{x^2} < \sigma\upsilon\nu x$$

$$\frac{x f'(x) - f(x)}{x^2} - \sigma\upsilon\nu x < 0$$

$$\Big(\underbrace{\frac{f(x)}{x} - \eta\mu x}_{\varphi(x)} \Big)' < 0.$$

$$\Rightarrow \varphi \downarrow$$

$$\forall x > \pi \quad \Rightarrow \quad \varphi(x) < \varphi(\pi)$$

$$\frac{f(x)}{x} - \eta\mu x < \frac{f(\pi)}{\pi}$$

$$\frac{f(x)}{x} - \eta\mu x < 0 \quad \Rightarrow \boxed{f(x) < x\eta\mu x}.$$