# Ενότητα 14

**33.** $f: \mathbb{R} \to \mathbb{R}$ συνεχής!

$$x f(x) = x^2 + 3ημx$$

(α) $$f(x) = \frac{x^2 + 3ημx}{x}, \quad x \neq 0$$

$$\boxed{f(x) = x + \frac{3ημx}{x} \qquad x \neq 0}$$

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0}\left(x + \frac{3ημx}{x}\right) = 0 + 3 \cdot 1 = 3$$

$$\boxed{f(x) = \begin{cases} x + \dfrac{3ημx}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}}$$

(B) Νδο $|f(x) - x| < 3 \quad \forall x \in \mathbb{R}^*$

$$-3 < f(x) - x < 3$$

$$\boxed{x-3 < f(x) < x+3}$$

$$x - 3 < x + \frac{\eta\mu x}{x} < x + 3$$

$$-3 < \frac{\eta\mu x}{x} < 3$$

$$\left|\frac{\eta\mu x}{x}\right| < 3$$

Ιδιότητα

$|x| < \theta$

$\Leftrightarrow$

$-\theta < x < \theta$

- $|\eta\mu x| \leq |x|$

$$\frac{|\eta\mu x|}{|x|} \leq 1 \Rightarrow \left|\frac{\eta\mu x}{x}\right| \leq 1$$

$$\left|\frac{\eta\mu x}{x}\right| < 3$$

(γ) Ν.δ.ο. $f(x)=6x+1$ έχει ρίζα $(0,1)$

$$\underbrace{f(x)-6x-1}_{\varphi(x)}=0 \quad \left( \text{όπου } f(x) \text{ από α ερωτ.} \right)$$

$\varphi(0)=f(0)-1=3-1=2>0$

$\varphi(1)=f(1)-7=1+3\eta\mu1-7=3\eta\mu1-6=$

$=3(\eta\mu1-2)<0$

$\varphi(0)\varphi(1)<0$

Bolzano $\exists\xi\in(0,1)$ τ.ω. $\varphi(\xi)=0$.

(δ) Nδο $\exists x_0 \in [0,3]$ τ.ω $6f(x_0) = 3 + 2f(1) + 3f(3)$

$$\boxed{6f(x) = f(0) + 2f(1) + 3f(3)}$$

Η $f$ συνεχής στο $[0,3]$ $\forall x \in [0,3]$

ΘΜΕΤ $m \leq f(x) \leq M$

$m \leq f(0) \leq M$

$m \leq f(1) \leq M \Rightarrow 2m \leq 2f(1) \leq 2M$

$m \leq f(3) \leq M \Rightarrow 3m \leq 3f(3) \leq 3M$

$(+)$

$$6m \leq f(0) + 2f(1) + 3f(3) \leq 6M$$

$$m \leq \frac{f(0) + 2f(1) + 3f(3)}{6} \leq M.$$

ο αριθμός $\frac{f(0)+2f(1)+3f(3)}{6}$ ανήκει στο $\Sigma T_f$

αρα $\exists x_0$ τ.ω $f(x_0) = \frac{f(0)+2f(1)+3f(3)}{6}$.

## Συνεχής στο $x_0$

$$f(x_0) = \lim_{x \to x_0} f(x) \left\langle \begin{array}{l} \lim\limits_{x \to x_0^-} f(x) \\[2em] \lim\limits_{x \to x_0^+} f(x) \end{array} \right.$$

## Παρ/μη στο $x_0$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \ell \in \mathbb{R}.$$

**14.** $f(x) = \begin{cases} x^3, & x \leq 1 \\ ax+\beta, & x > 1 \end{cases}$ παρ/μη στο 1 !

Αφού $f$ παρ/μη στο 1 είναι συνεχής στο 1

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} x^3 = \lim_{x \to 1^+} ax+\beta .$$

$$\boxed{1 = a+\beta .}$$

$$\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$\lim_{x \to 1^-} \frac{x^3-1}{x-1} = \lim_{x \to 1^+} \frac{(ax+\beta)-1}{x-1}$$

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = \lim_{x \to 1^+} \frac{ax+\beta-1}{x-1}$$

$$3 = \lim_{x \to 1^+} \frac{ax + B - 1}{x - 1}$$

όπου $B = 1 - a$

$$3 = \lim_{x \to 1^+} \frac{ax + 1 - a - 1}{x - 1}$$

$$3 = \lim_{x \to 1^+} \frac{ax - a}{x - 1}$$

$$3 = \lim_{x \to 1^+} \frac{a\cancel{(x-1)}}{\cancel{x-1}}$$

$3 = a$ $\quad$ $B = -2$

**12.** ⓐ $f(x) = \begin{cases} x^2+1, & x<1 \\ \ln x, & x\geq 1 \end{cases}$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^2+1) = 2$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \ln x = 0$

Όχι συνεχής στο 1.

Όχι παρ/μη στο $\underline{1}$.

ⓑ $f(x) = |x-2| + 2x - 1$, $\underline{\underline{x_0 = 2}}$

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} |x-2| + 2x - 1 = 3$

$f(2) = 3$

Συνεχής στο 2!

$\lim\limits_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2} \frac{|x-2|+2x-1-3}{x-2} = \lim\limits_{x\to 2} \frac{|x-2|+2x-4}{x-2}$

$\to \lim\limits_{x\to 2^-} \frac{-\cancel{(x-2)} + 2\cancel{(x-2)}}{\cancel{x-2}} = \frac{-1+2}{1} = 1$

$\to \lim\limits_{x\to 2^+} \frac{\cancel{(x-2)} + 2\cancel{(x-2)}}{\cancel{x-2}} = \frac{1+2}{1} = 3$

όχι παρ/μη στο 2.

$$\boxed{f(x) = x^3 - 2x^2 + 3x - 1}$$

Είναι παρ/μη στο 1,

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{x^3 - 2x^2 + 3x - 1 - 1}{x-1}$$

$$= \lim_{x\to 1} \frac{x^3 - 2x^2 + 3x - 2}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 - x + 2)}{\cancel{x-1}}$$

$$= 2.$$

| | | | | |
|---|---|---|---|---|
| 1 | -2 | 3 | -2 | 1 |
| | 1 | -1 | 2 | |
| 1 | -1 | 2 | 0 | |

$f'(1) = 2$

Η $f(x)$ είναι παρ/μη στο 1 και η παραγωγ/ στο 1 είναι 1 $\Rightarrow$ $\underline{f'(1) = 2}$.

# Παραγωγίσιμη

## Ασκηση 1

1. $f(x) = \ln x - \sqrt{x} + \eta\mu x$

$f'(x) = (\ln x)' - (\sqrt{x})' + (\eta\mu x)'$

$\boxed{f'(x) = \frac{1}{x} - \frac{1}{2\sqrt{x}} + \sigma\upsilon\nu x}$

2. $f(x) = 3e^x - 2\ln x + 3\varepsilon\varphi x$

$f'(x) = (3e^x)' - (2\ln x)' + (3\varepsilon\varphi x)'$

$f'(x) = 3\cdot(e^x)' - 2(\ln x)' + 3(\varepsilon\varphi x)'$

$\boxed{f'(x) = 3\cdot e^x - 2\frac{1}{x} + 3\frac{1}{\sigma\upsilon\nu^2 x}}$

3. $f(x) = 2x^2 - 5x + 6$

$f'(x) = (2x^2)' - (5x)' + (6)'$

$f'(x) = 2(x^2)' - 5\cdot(x)' + 0$

$f'(x) = 2\cdot 2x - 5\cdot 1$

$\boxed{f'(x) = 4x - 5}$

4. $f(x) = \frac{x^3}{3} - \frac{1}{2}x^2 - 5x - \frac{3}{x}$

$$f'(x) = \left(\frac{x^3}{3}\right)' - \left(\frac{1}{2}x^2\right)' - (5x)' - \left(\frac{3}{x}\right)'$$

$$f'(x) = \frac{1}{3} \cdot (x^3)' - \frac{1}{2} \cdot (x^2)' - 5(x)' + 3 \cdot \left(\frac{1}{x}\right)'$$

$$f'(x) = \frac{1}{\cancel{3}} \cancel{3}x^2 + \frac{1}{\cancel{2}} \cancel{2}x - 5 \cdot 1 - 3 \cdot \left(-\frac{1}{x^2}\right)$$

$$\boxed{f'(x) = x^2 + x - 5 + \frac{3}{x^2}}$$

# Άσκηση 2

**1.** $f(x) = (x^2-3x)\ln x$

$f'(x) = (x^2-3x)'\ln x + (x^2-3x)(\ln x)'$

$$\boxed{f'(x) = (2x-3)\ln x + (x^2-3x)\frac{1}{x}}$$

**2.** $f(x) = \text{ημ}x \cdot \text{συν}x$

$f'(x) = (\text{ημ}x)'\,\text{συν}x + \text{ημ}x\,(\text{συν}x)'$

$f'(x) = \text{συν}x\,\text{συν}x + \text{ημ}x\,(-\text{ημ}x)$

$$\boxed{f'(x) = \text{συν}^2x - \text{ημ}^2x}$$

**3.** $f(x) = \ln x\sqrt{x}$

$f'(x) = (\ln x)'\sqrt{x} + \ln x(\sqrt{x})'$

$$\boxed{f'(x) = \frac{1}{x}\sqrt{x} + \ln x\frac{1}{2\sqrt{x}}}$$

# Άσκηση 3

**1.** $f(x) = \dfrac{x^2-2x}{x^2+1}$

$$f'(x) = \frac{(x^2-2x)'(x^2+1) - (x^2-2x)(x^2+1)'}{(x^2+1)^2}$$

$$\boxed{f'(x) = \frac{(2x-2)(x^2+1) - (x^2-2x)\,2x}{(x^2+1)^2}}$$

**2.** $f(x) = \dfrac{\ln x}{ημx}$

$$f'(x) = \frac{(\ln x)'\,ημx - \ln x\,(ημx)'}{ημ^2 x}$$

$$\boxed{f'(x) = \frac{\frac{1}{x}\,ημx - \ln x\,συνx}{ημ^2 x}}$$

# Άσκηση 4 — Σύνθετη

**1.** $f(x) = e^{x^2-1}$

$$(e^x)' = e^x$$

$$(e^{\square})' = e^{\square}(\square)'$$

$$f'(x) = e^{x^2-1} \cdot (x^2-1)'$$

**2.** $f(x) = ημ(\ln x)$

$$(ημx)' = συνx$$

$$f'(x) = συν(\ln x)(\ln x)'$$

$$f'(x) = \frac{συν(\ln x)}{x}$$

**3.** $f(x) = \sqrt{x^2+1}$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (x^2+1)'$$

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}}$$

# Ασκηση 5

**1.** $f(x) = \underbrace{x\ln x}_{1}\,\underbrace{e^x}_{2}$

Τριπλό γινόμενο

$$f'(x) = (x\ln x)'e^x + x\ln x\,(e^x)'$$

$$f'(x) = \left((x)'\ln x + x(\ln x)'\right)e^x + x\ln x\,e^x$$

$$\boxed{f'(x) = \left(\ln x + x\cdot\frac{1}{x}\right)e^x + x\ln x\,e^x}$$

**2.** $f(x) = x^{\ln x}$

$$x^{\ln x} = e^{\ln x^{x}} = e^{x\ln x} \Rightarrow \underline{\underline{x^{\ln x} = e^{x\ln x}}}$$

$$f'(x) = \left(x^{\ln x}\right)' = \left(e^{x\ln x}\right)' = e^{x\ln x}(x\ln x)'$$

$$f'(x) = x^{\ln x}\left((x)'\ln x + x(\ln x)'\right)$$

$$f'(x) = x^{\ln x}\left(\ln x + x\cdot\frac{1}{x}\right) \Rightarrow \boxed{f'(x) = x^{\ln x}(\ln x + 1)}$$

**3.** $f(x) = \sqrt[4]{x^3}$ , $x \geq 0$

$f(x) = x^{\frac{3}{4}}$ $\qquad \sqrt[\nu]{x^{v}} = x^{\frac{v}{\nu}}$

$f'(x) = \frac{3}{4} x^{\frac{3}{4}-1}$

$f'(x) = \frac{3}{4} x^{-\frac{1}{4}} = \frac{3}{4} \frac{1}{x^{1/4}} = \frac{3}{4} \frac{1}{\sqrt[4]{x}}$

$$\boxed{f'(x) = \frac{3}{4\sqrt[4]{x}}}$$

**4.** $f(x) = \sqrt[3]{x^4}$ , $x \in \mathbb{R}$.

$$f(x) = |x|^{\frac{4}{3}} = \begin{cases} (-x)^{\frac{4}{3}}, & x < 0 \\ x^{\frac{4}{3}}, & x \geq 0 \end{cases}$$

$x < 0$

$$f(x) = (-x)^{\frac{4}{3}}$$

$$f'(x) = \frac{4}{3}(-x)^{\frac{4}{3}-1}(-x)'$$

$$f'(x) = -\frac{4}{3}(-x)^{\frac{1}{3}}$$

$$\boxed{f'(x) = -\frac{4}{3}\sqrt[3]{-x}}$$

$x > 0$

$$f(x) = x^{4/3}$$

$$f'(x) = \frac{4}{3}x^{1/3}$$

$$\boxed{f'(x) = \frac{4}{3}\sqrt[3]{x}}$$

$$\boxed{f'(x) = \begin{cases} -\frac{4}{3}\sqrt[3]{-x}, & x < 0 \\ \frac{4}{3}\sqrt[3]{x}, & x \geq 0 \end{cases}}$$

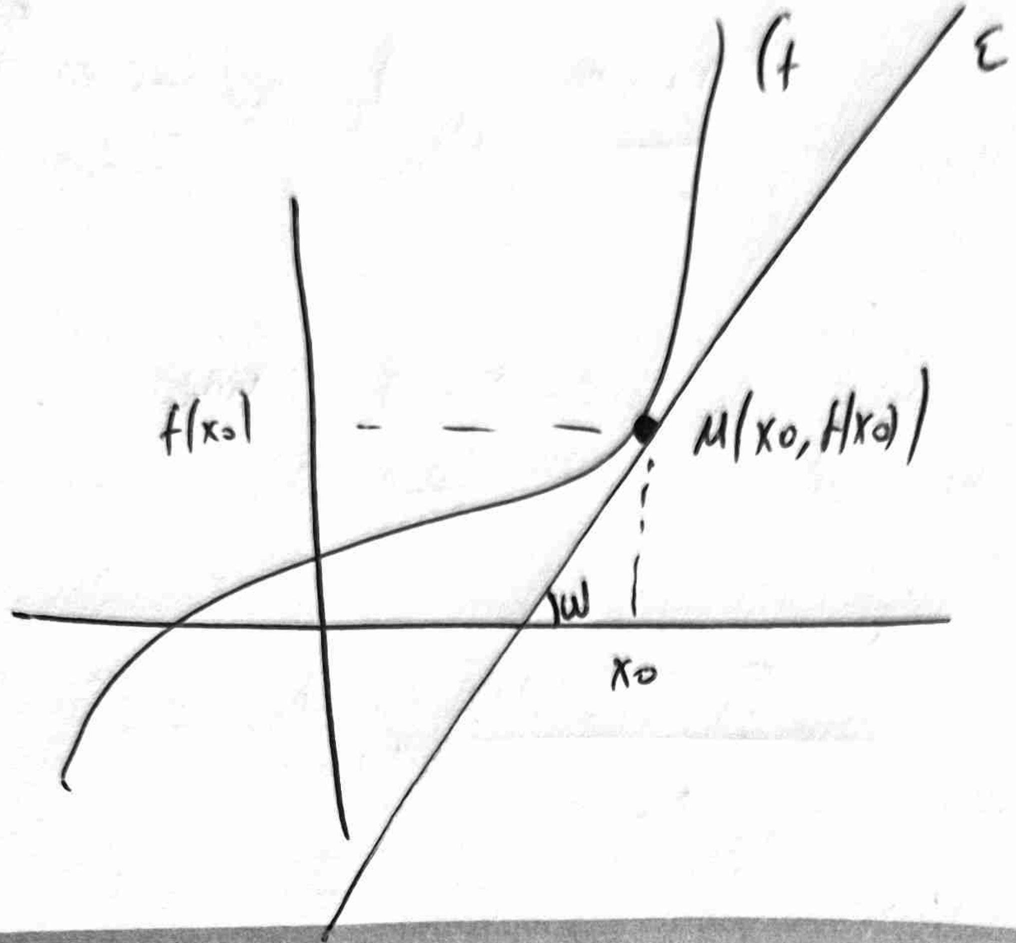
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(-x)^{4/3} - 0}{x - 0} =$$

$$= \lim_{x \to 0^-} \frac{\sqrt[3]{(-x)^4}}{x} = \lim_{x \to 0^-} \frac{\sqrt[3]{x^4}}{\sqrt[3]{(-x)^3}} =$$

$$= \lim_{x \to 0^-} \sqrt[3]{\frac{x^4}{(-x)^3}} = \lim_{x \to 0^-} \sqrt[3]{-x} = 0$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^{4/3}-0}{x-0} = \lim_{x\to 0} \frac{\sqrt[3]{x^4}}{x}$$

$$= \lim_{x\to 0^+} \frac{\sqrt[3]{x^4}}{\sqrt[3]{x}} = \lim_{x\to 0^+} \sqrt[3]{x^3} = 0 .$$




Εφαπτομένη στο $x_0$ ε: $y - f(x_0) = f'(x_0)(x - x_0)$

- $f'(x_0)$: κλίση ή συντελεστής διεύθυνσης.
- $εφ\hat{ω} = f'(x_0)$

$εφ45 = 1$
$εφ135 = -1$

- $ε \parallel ζ \Rightarrow f'(x_0) = λ_ζ$ κλίση της ζ.
- $ε \parallel x'x \Rightarrow f'(x_0) = 0$.

# Ασκηση 1

Εφαπτομενη
ΣΤΟ $M(x_0, f(x_0))$

**1.** Να βρεθει η εφαπτομενη ε της $f(x)$ στο $A(1, f(1))$ αν $f(x) = \frac{x^2-1}{x^2}$

$$\varepsilon: y - f(1) = f'(1)(x-1)$$

$$f'(x) = \frac{(x^2-1)'x^2 - (x^2-1)(x^2)'}{(x^2)^2}$$

$$f'(x) = \frac{2x \cdot x^2 - 2x(x^2-1)}{x^4} = \frac{\cancel{2x^3} - \cancel{2x^3} + 2x}{x^4}$$

$$f'(x) = \frac{2}{x^3}$$

$\rightarrow f(1) = \frac{1-1}{1} = 0$

$\rightarrow f'(1) = \frac{2}{1^3} = 2$

$$\varepsilon: y - 0 = 2(x-1) \qquad \underline{\underline{y = 2x - 2}}$$

**2.** $f(x) = \begin{cases} x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$

ⓐ Εφαπτομένη στο $x_0 = 1$

$y - f(1) = f'(1)(x-1) \Rightarrow y - 1 = 3(x-1)$

- $f'(x) = 3x^2$
- $f(1) = 1^3 = 1$
- $f'(1) = 3 \cdot 1^2 = 3$

$y = 3x - 2$

ⓑ Εφαπτομένη στο $x_0 = -1$

$y - f(-1) = f'(-1)(x - (-1)) \longrightarrow y - 1 = -2(x+1)$

- $f'(x) = 2x$
- $f(-1) = 1$
- $f'(-1) = 2(-1) = -2$

$y = -2x - 2 + 1$

$y = -2x - 1$

① Εφαπτομενη στο $x_0=0$

$$y-f(0)=f'(0)(x-0)$$

• $f(0)=0$

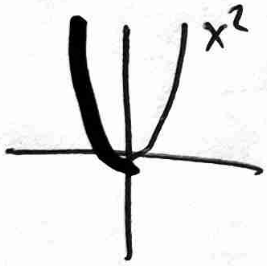
$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2-0}{x}=0$$

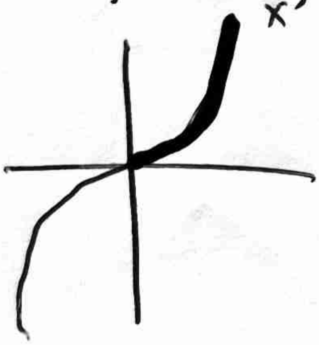
$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^3-0}{x}=0$$

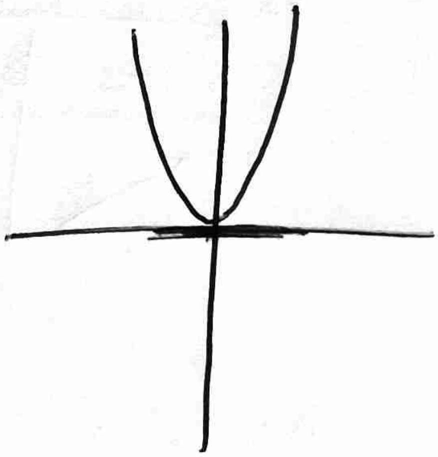
• $f'(0)=0$.

$$y-0=0(x-0)$$

$$y=0 \Rightarrow \underline{\underline{x'x}}$$



# Ασκηση 2

Δινεται $f(x) = -x^2 + 3x$

Παραγωγη $f'(x) = -2x + 3$

Να βρεθει η εφαπτομενη ε της $f(x)$ η οποια.

(α) Ειναι παραλληλη στον $x'x$

$\varepsilon: y - f(x_0) = f'(x_0)(x - x_0) \ // \ x'x$.

Απαιτω $f'(x_0) = 0$

$-2x_0 + 3 = 0$

$2x_0 = 3 \Rightarrow x_0 = \frac{3}{2}$.

$y - f(\frac{3}{2}) = f'(\frac{3}{2})(x - \frac{3}{2})$

$y - \frac{9}{4} = 0(x - \frac{3}{2})$

$\varepsilon: y = \frac{9}{4}$

(Β) Είναι παράλληλη Ι: $2x-3y-1=0$

$\varepsilon: y-f(x_0)=f'(x_0)(x-x_0) \parallel 2x-3y-1=0$.

$\downarrow$

Απαιτώ $f'(x_0)=\lambda_I$ $\quad -3y=-2x+1$

$f'(x_0)=\frac{2}{3}$ $\quad y=\frac{2}{3}x-\frac{1}{3}$

$-2x_0+3=\frac{2}{3}$

$-6x_0+9=2$

$-6x_0=-7$

$x_0=\frac{7}{6}$

$$\boxed{y-f\left(\frac{7}{6}\right)=f'\left(\frac{7}{6}\right)\left(x-\frac{7}{6}\right)}$$

κτλ.

(γ) Είναι κάθετη στην $T: y = 2x - 5$.

$$\varepsilon: y - f(x_0) = f'(x_0)(x - x_0) \perp y = 2x - 5.$$

Απαιτώ. $f'(x_0) \cdot 2 = -1$

$$f'(x_0) = -\frac{1}{2}$$

$$-2x_0 + 3 = -\frac{1}{2}$$

$$-4x_0 + 6 = -1$$

$$-4x_0 = -7$$

$$x_0 = \frac{7}{4}.$$

$$\boxed{y - f\left(\frac{7}{4}\right) = f'\left(\frac{7}{4}\right)\left(x - \frac{7}{4}\right)}$$

κτλ.

(δ) σχηματίζει γωνία 135° με τον x'x.

$\varepsilon: y - f(x_0) = f'(x_0)(x - x_0)$ έχει γωνία 135°

$\Rightarrow f'(x_0) = \varepsilon\varphi 135$

$f'(x_0) = -1$

$-2x_0 + 3 = -1$

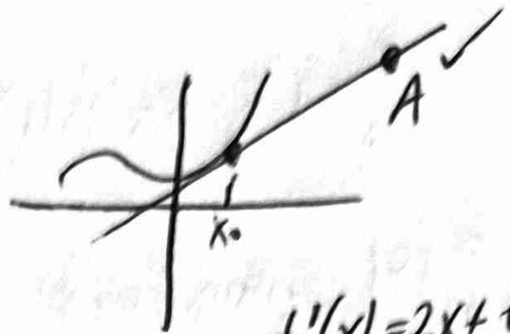
$-2x_0 = -4$

$x_0 = 2$.

$y - f(2) = f'(2)(x - 2)$

$\boxed{y = -x + 4}$

# Ασκηση 3




Δινεται $f(x) = x^2 + 6x$.  $f'(x) = 2x + 6$

Να Βραθει η εφαπτομενη της $C_f$

η οποια διερχεται $A(-3, -10)$.

$\varepsilon: \ y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(-3, -10)$

$-10 - f(x_0) = f'(x_0)(-3 - x_0)$

$-10 - (x_0^2 + 6x_0) = (2x_0 + 6)(-3 - x_0)$

$-10 - x_0^2 - \cancel{6x_0} = -\cancel{6x_0} - 2x_0^2 - 18 - 6x_0$

$x_0^2 + 6x_0 + 8 = 0$

$x_0 = -4$  $x_0 = -2$

$y - f(-4) = f'(-4)(x + 4)$  κ'  $y - f(-2) = f'(-2)(x + 2)$.

# κανονες παραγωγισης

1. $(c)' = 0$
2. $(x)' = 1$
3. $(x^{\nu})' = \nu x^{\nu - 1}$
4. $(e^x)' = e^x$
5. $(\ln x)' = \frac{1}{x}$
6. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$
7. $(\eta\mu x)' = \sigma\upsilon\nu x$
8. $(\sigma\upsilon\nu x)' = -\eta\mu x$
9. $(\epsilon\varphi x)' = \frac{1}{\sigma\upsilon\nu^2 x}$
10. $(\sigma\varphi x)' = -\frac{1}{\eta\mu^2 x}$
11. $(\alpha^x)' = \alpha^x \ln \alpha$
12. $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$
13. $(\ln|x|)' = \frac{1}{x}$

## Προσοχη

1. $[c f(x)]' = c \cdot f'(x)$
2. $[f(x) + g(x)]' = f'(x) + g'(x)$
3. $(f(x) \cdot g(x))' = f'(x) g(x) + f(x) g'(x)$
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}$

# Επομαο Μαθημα

Δευτερα 9

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**1.** Διαβασμα προετοιμασια διαγωνισματος για τυχον απορiες.

**2.** Ενοτητα 15
5 α β γ δ ε
13.

+ Ασκησει extra

# Ασκησεις

1. $f(x) = 3x^3 - 2x^2 + 5x - 8$

2. $f(x) = 4\ln x - 3\sigma\upsilon\nu x - 7\eta\mu x$

3. $f(x) = x^2 \ln x - e^x \eta\mu x$

4. $f(x) = \frac{4}{x^2} - (2x^2 - 4)\ln x$

5. $f(x) = e^{x^2} - x^2 \ln(2x+1)$.

6. $f(x) = x^3 - 2x^2 \eta\mu x - \frac{1}{x}$

7. $f(x) = \frac{x^2 - 4x}{x^2 - 1}$

8. $f(x) = \ln(e^x - 1) - e^{x^2 - x}$

9. $f(x) = (x+1)\sqrt{x} - x^2 \ln x$

10. $f(x) = \frac{e^x \eta\mu x}{\sigma\upsilon\nu x}$

11. $f(x) = \sqrt{\ln x - 1} - \ln\left(\sqrt{x^2 - 1}\right)$.