Ενότητα 20

2. ⓐ $f(x) = x^2 + 2x + 1$. $[-2, 0]$

Η f συνεχής $[-2, 0]$ ως π.σ.σ.

Η f παρ/μη $(-2, 0)$ ως π.π.σ.

$f(-2) = 1$
$f(0) = 1$ $\Big\}$ $f(-2) = f(0)$ Rolle

$\exists x_0 \in (-2, 0)$ τ.ω $\underline{f'(x_0) = 0}$

$f'(x) = 2x + 2$

$f'(x_0) = 0 \Rightarrow 2x_0 + 2 = 0$

$\underline{x_0 = -1}$

(B) $f(x) = \dfrac{e^x}{x}$ $\qquad [\ln 2, \ln 4]$

Η f συνεχής $[\ln 2, \ln 4]$ ως π.σ.σ

Η f παρ/μη $(\ln 2, \ln 4)$ ως π.π.σ.

$$f(\ln 2) = \frac{e^{\ln 2}}{\ln 2} = \frac{2}{\ln 2}$$

$$f(\ln 4) = \frac{e^{\ln 4}}{\ln 4} = \frac{4}{\ln 2^2} = \frac{4}{2\ln 2} = \frac{2}{\ln 2}$$

$f(\ln 2) = f(\ln 4) \Rightarrow$ Rolle $\exists\, \xi \in (\ln 2, \ln 4)$

τ.ω $f'(\xi) = 0$

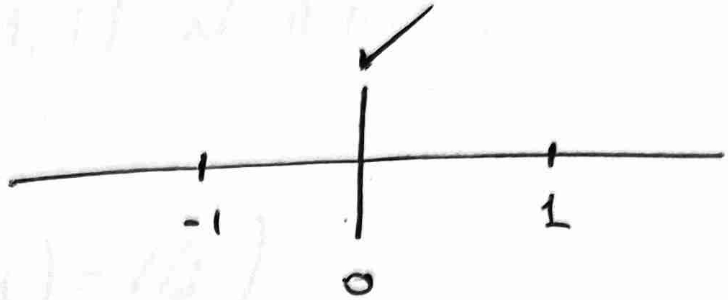
$$f'(x) = \frac{e^x x - e^x}{x^2} = 0$$

$$e^x x - e^x = 0$$

$$x - 1 = 0$$

$$\underline{\underline{x = 1}}$$

3. ⓐ $f(x) = \begin{cases} x^2, & x<0 \\ x^3, & x \geq 0 \end{cases}$ $[-1, 1]$



$$\left.\begin{array}{l}\lim_{x \to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0 \\ \\ \lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} x^3 = 0\end{array}\right\} \underline{\underline{\lim_{x\to 0} f(x) = 0}}$$

Συνεχής στο 0!

$\underline{\underline{f(0)=0}}$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^2-0}{x} = \lim_{x\to 0} x = 0$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^3}{x} = \lim_{x\to 0} x^2 = 0$$

$f'(0)=0$ Είναι παρ/μη στο 0!

Η f συνεχής $[-1,1]$ ως π.δ.σ.

Η f παρ/μη $(-1,1)$ ως π.π.σ.

$$\left.\begin{array}{l} f(-1)=1 \\ f(1)=1 \end{array}\right\} f(-1)=f(1)$$

Rolle $\exists\, \xi \in (-1,1)$ τ.ω $f'(\xi)=0$.

5. $f(x) = x^4 - x^3 + \lambda x^2 - \lambda x$

(α) $\left.\begin{array}{l} f(0)=0 \\ f(1)=0 \end{array}\right\}$ Rolle $\exists \xi \in (0,1)$ τ.ω $f'(\xi)=0$

Η f συνεχής $[0,1]$ ως π.σ.σ

Η f παρ/μη $(0,1)$ ως π.π.σ

(β) $f'(x) = 4x^3 - 3x^2 + 2\lambda x - \lambda$

$f'(\xi) = \underline{4\xi^3 - 3\xi^2 + 2\lambda\xi - \lambda = 0}$

6. $f(x) = \sigma\upsilon\nu x \cdot \ln x$

(α) Η εφαπτομένη της f στο ξ.

$y - f(\xi) = f'(\xi)(x-\xi) \; // \; x'x$

Αρκεί νδο $\exists \xi \in (1, \frac{\pi}{2})$ τ.ω $f'(\xi)=0$

$\left.\begin{array}{l} f(1)=0 \\ f(\frac{\pi}{2})=0 \end{array}\right\}$ Rolle $\exists \xi$ τ.ω $f'(\xi)=0$

4. $f(x) = (x+1)\,\eta\mu x$

ⓐ $f(-1)=0$, $f(0)=0$ $\Big\}$ $f(-1)=f(0)$

Η f συνεχής $[-1,0]$ ως π.δ.σ

Η f παραγ/μη $(-1,0)$ ως π.π.σ.

Άρα Rolle $\exists\, \xi \in (-1,0)$ τ.ω $f'(\xi)=0$.

ⓑ

$$f'(x) = \eta\mu x + (x+1)\,\sigma\upsilon\nu x = 0$$

$$f'(\xi) = \eta\mu\xi + (\xi+1)\,\sigma\upsilon\nu\xi = 0$$

$$\frac{\eta\mu\xi}{\sigma\upsilon\nu\xi} + \xi+1 = 0$$

$$\varepsilon\varphi\xi = -\xi - 1$$

(B) $f'(x) = -\eta\gamma x \ln x + \sigma w x \dfrac{1}{x}$

$$f'(\xi) = -\eta\gamma\xi \ln\xi + \frac{\sigma w \xi}{\xi} = 0$$

$$-\xi\eta\gamma\xi \ln\xi + \sigma w \xi = 0.$$

$$-\xi \ln\xi + \frac{\sigma w \xi}{\eta\gamma\xi} = 0$$

$$\sigma\varphi\xi = \xi \ln\xi$$

8. $g(x) = e^x f(x)$ $\qquad f(0) = f(\tfrac{3}{2}) = 0$

$$\left.\begin{aligned} g(0) &= e^0 f(0) = 0 \\ g(\tfrac{3}{2}) &= e^{3/2} f(\tfrac{3}{2}) = 0 \end{aligned}\right\} g(0) = g(\tfrac{3}{2})$$

Η $g(x)$ συνεχ. $[0, \tfrac{3}{2}]$ ως π.σ.σ

Η $g(x)$ παραγ/μη $(0, \tfrac{3}{2})$ ως π.π.σ.

Rolle $\exists\, \xi \in (0, \tfrac{3}{2})$ τ.ω. $g'(\xi) = 0$

$$g'(x) = e^x f(x) + e^x f'(x)$$

$$g'(\xi) = \cancel{e^{\xi}} f(\xi) + \cancel{e^{\xi}} f'(\xi) = 0$$

$$f'(\xi) = -f(\xi)$$

9. $f: [0,a] \to \mathbb{R}$ συνεχής.

$f: (0,a) \to \mathbb{R}$ παρ/μη

$f(a) = e f(0)$

$f(x) > 0$

$g(x) = \ln f(x) - ax$

$[0, a]$

Rolle ✓

(α) $g(0) = g(a)$

$\ln f(0) - a \cdot 0 = \ln f(a) - a^2$

$\ln f(0) = \ln e f(0) - a^2$

$\cancel{\ln f(0)} = \ln e + \cancel{\ln f(0)} - a^2$

$0 = 1 - a^2$

$a = 1$ $\quad \cancel{a = -1}$

(3) $g'(x) = \dfrac{f'(x)}{f(x)} - \alpha$

$$g'(\xi) = \frac{f'(\xi)}{f(\xi)} - \alpha = 0$$

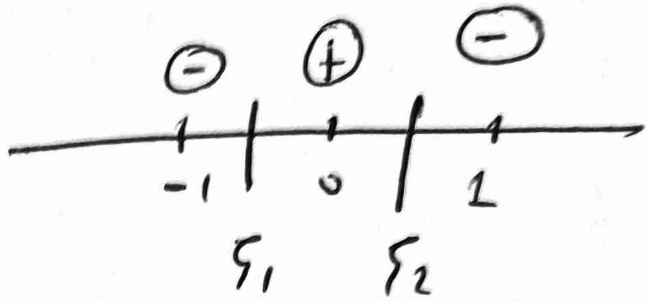
$$f'(\xi) = \alpha f(\xi)$$

$$f'(\xi) = f(\xi)$$

10. $f(x) = x^4 - 10x^3 - 15x^2 - x + 1$

ⓐ $$\frac{10x^3 + 15x^2 + x}{x^4 + 1} = 1$$

$\Rightarrow f(x) = 0$



$f(0) = 1$

$f(-1) = -2$

$f(1) = -44$

$f(-1)f(0) < 0$ Bolzano $\exists \xi_1 \in (-1, 0)$ τ.ω $f(\xi_1) = 0$

$f(0)f(1) < 0$ Bolzano $\exists \xi_2 \in (0, 1)$ τ.ω $f(\xi_2) = 0$

ⓑ $f(\xi_1) = f(\xi_2)$

Rolle $\exists \xi \in (\xi_1, \xi_2)$

τ.ω $f'(\xi) = 0$.

$f'(x) = 4x^3 - 30x^2 - 30x - 1$

$f'(\xi) = 0 \quad 4\xi^3 - 30\xi^2 - 30\xi - 1 = 0$

Ενότητα 24

28. $f(x) = \begin{cases} -x^3, & x < 0 \\ 2x - x^2, & x \geq 0 \end{cases}$

ⓐ $\lim\limits_{x \to 0^-} f(x) = 0$

$\lim\limits_{x \to 0^+} f(x) = 0$

$\Big\} \lim\limits_{x \to 0} f(x) = 0$

$f(0) = 0$

Συνεχής στο 0!

$\underline{x < 0}$

$f_1(x) = -x^3$

$f_1'(x) = -3x^2 \leq 0$

$\underline{x > 0}$

$f_2(x) = 2x - x^2$

$f_2'(x) = 2 - 2x$

$x = 1$

x		0		1	
f_1'	$-$		//////		//////
f_2'	//////		$+$	0	$-$
f'	$-$		$+$		$-$
f	↘		↗		↘

β) $f(-2x) > f(-x^2)$ $\qquad (0,+\infty)$

- $x>0 \Rightarrow -2x<0$
- $x>0 \Rightarrow x^2>0 \Rightarrow -x^2<0$

$\Big\}$ $\forall x<0$ η $f \downarrow$

$-2x < -x^2$

$2x > x^2 \qquad \Rightarrow \quad x^2-2x<0$

x		0		2	
x^2-2x	$+$		$-$		$+$

$x \in (0,2)$

γ) $f(e^x) = f(\frac{1}{2})$ $\qquad (-1,0)$

- $0>x>-1 \Rightarrow e^0>e^x>e^{-1} \Rightarrow 1>e^x>\frac{1}{e}$

$\forall x \in (0,1)$ η $f \uparrow \Rightarrow f$ 1-1.

$e^x = \frac{1}{2} \Rightarrow x = \ln\frac{1}{2} \qquad x = -\ln 2$

(δ) $f(x+1) > f(e^x)$ $\quad \forall x > 0$

- $x > 0 \Rightarrow x+1 > 1$
- $x > 0 \rightarrow e^x > e^0 \Rightarrow e^x > 1$

$\Big\}$ $\forall x > 1$ $f \downarrow$

$x+1 < e^x$

$x \in \mathbb{R}^*$

- $e^x \geq x+1$ $\quad$ To "=" για $x=0$

29. (a)

x		1	
f''	$-$		$-$
f'	$\searrow$ $+$	0	$\searrow$ $-$
f	$\nearrow$		$\searrow$

$f''(x)<0$

$x<1 \Rightarrow f'(x)>f'(1) \Rightarrow f'(x$

$x>1 \Rightarrow f'(x)<f'(1) \Rightarrow f'($

<0

(β) i) $f(2-x)=f(3)$ $\qquad (-\infty,0)$.

• $x<0 \Rightarrow -x>0 \Rightarrow 2-x>2$

$(\forall x>1 \quad n \quad f\downarrow)$

$2-x=3$

$x=-1$

ii). $f(e^{2-x})=f\left(\frac{1}{x-1}\right)$ $\qquad (1,2]$

• $1<x\leq 2 \Rightarrow -1>-x\geq -2 \Rightarrow 1>2-x\geq 0$

$e>e^{2-x}\geq 1$

• $1<x\leq 2 \Rightarrow 0<x-1\leq 1 \Rightarrow \frac{1}{x-1}\geq 1$,

$\forall x > 1 \quad u \quad f \downarrow$

$$e^{2-x} = \frac{1}{x-1}$$

$1>0$

x

$$e^{2-x} - \frac{1}{x-1} = 0$$

$$\frac{1}{e^{x-2}} - \frac{1}{x-1} = 0$$

$$\frac{x-1-e^{x-2}}{(x-1)\,e^{x-2}} = 0$$

$\bullet\ e^{x} \geq x+1 \longrightarrow \quad x=0$

$e^{x-2} \geq x-2+1 \qquad x-2=0$

$e^{x-2} \geq x-1 \qquad \underline{\underline{x=2}}$

$0 \geq x-1-e^{x-2}$

(γ) $f(e^{x-2}) = f\left(\frac{1}{x}\right)$ $(1,2)$

- $1<x<2 \Rightarrow -1<x-2<0 \Rightarrow e^{-1}<e^{x-2}<1$
- $1<x<2 \Rightarrow 1>\frac{1}{x}>\frac{1}{2}$

$\forall x<1$ η $f \nearrow$

$$e^{x-2} = \frac{1}{x}$$

$$\underbrace{e^{x-2} - \frac{1}{x}}_{g(x)} = 0 \Rightarrow g(x) = 0$$

$$g'(x) = e^{x-2} + \frac{1}{x^2} > 0 \qquad g \nearrow$$

$$\left.\begin{aligned} g(1) &= e^{-1} - 1 < 0 \\ g(2) &= e^{0} - \tfrac{1}{2} > 0 \end{aligned}\right\} \text{Bolzano}$$

$\exists! \xi \in (1,2)$ τ.ω. $g(\xi) = 0$

13. $f: \mathbb{R} \to \mathbb{R}$ παρ/μη

$f'(x) \neq -1 \quad \forall x \in \mathbb{R}$.

$f(x) = -x$ έχει το πολύ 1 ρίζα.

$\underbrace{f(x) + x}_{h(x)} = 0$

Έστω $h(x)$ έχει 2 ρίζες

$\underbrace{h(x_1) = h(x_2)}_{\text{Rolle}} = 0$

$h'(\xi) = 0$

$h'(x) = f'(x) + 1$

$h'(\xi) = f'(\xi) + 1 = 0$

$f'(\xi) = -1$ Άτοπο!

Άρα η $h(x)$ δεν μπορεί να έχει ρίζα.

Άρα έχει το πολύ 1.

| **15.** $f(x) = 3^x + x^2 - 3x - 1$

ⓐ $\left.\begin{array}{l} f(0) = 0 \\ f(1) = 0 \end{array}\right\}$ Rolle ✓

ⓑ Έστω ότι έχει τρεις ρίζες.

$$\underbrace{f(x_1) = f(x_2)}\ \underbrace{= f(x_3)} = 0$$

$$\underbrace{f'(\xi_1) = f'(\xi_2)} = 0$$

$$f''(\xi) = 0$$

$$f'(x) = 3^x \ln 3 + 2x - 3$$

$$f''(x) = 3^x \ln^2 3 + 2$$

$f''(\xi) = 0 \quad 3^x \ln^2 3 + 2 = 0$

Άτοπο

Έχει το πολύ 2 ρίζες.

ε) $g(x) = h(x)$

$3^x = -x^2 + 3x + 1$

$3^x + x^2 - 3x - 1 = 0$

$f(x) = 0$

$x = 0$ $x = 1$

προφανείς ρίζ.

Επομαο Μαθημι

Ενοτητα 20

12

14

23

24.

Ενοτητα 24

41

42

43

44.