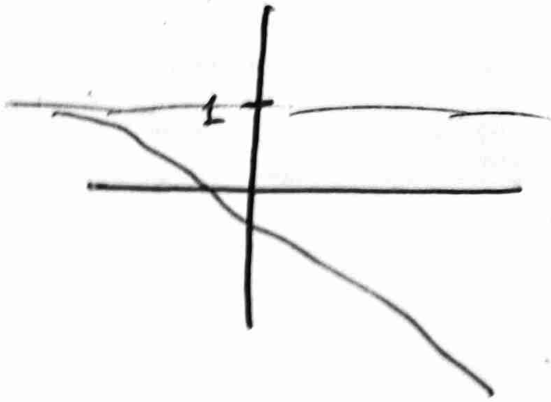
Ενότητα 14



23. $f: \mathbb{R} \to \mathbb{R}$ συνεχής.

$f \downarrow$

$\Sigma T_f = (-\infty, 1) \Rightarrow$ $\underline{\underline{f(x) < 1}}$

$\lim\limits_{x \to -\infty} f(x) = 1.$

$\lim\limits_{x \to +\infty} f(x) = -\infty$

(α) $\lim\limits_{x \to -\infty} \dfrac{x^2 + f(x)}{x+2} =$

$$= \lim_{x \to -\infty} \frac{\cancel{x^2}\left(1 + \frac{f(x)}{x^2}\right)}{\cancel{x}\left(1 + \frac{2}{x}\right)} = \lim_{x \to -\infty} \frac{x\left(1 + \frac{f(x)}{x^2}\right)}{1 + \frac{2}{x}} =$$

$$= \frac{-\infty(1+0)}{1+0} = -\infty.$$

(β) $\lim\limits_{x \to +\infty} \dfrac{x f(x) - x^2}{x-1} = \lim\limits_{x \to +\infty} \dfrac{\cancel{x}(f(x) - x)}{\cancel{x}\left(1 - \frac{1}{x}\right)} =$

$$= \lim_{x \to +\infty} \frac{f(x) - x}{1 - \frac{1}{x}} = \frac{-\infty - (+\infty)}{1 - 0} = -\infty.$$

(γ) $$\lim_{x\to-\infty}\frac{x^2+f(x)}{x-ημx}=\lim_{x\to-\infty}\frac{x+\frac{f(x)}{x}}{1-\frac{ημx}{x}}=\frac{-\infty+0}{1-0}=-\infty$$

$$-1\le ημx\le 1$$

$$-\frac{1}{x}\geq\frac{ημx}{x}\geq\frac{1}{x}$$

$$\left.\begin{aligned}&\lim_{x\to-\infty}-\frac{1}{x}=0\\&\lim_{x\to-\infty}\frac{1}{x}=0\end{aligned}\right\}\ \text{Από κ.π.}\ \lim_{x\to-\infty}\frac{ημx}{x}=0$$

(δ) $$\lim_{x\to+\infty}\frac{ημx}{f(x)}=0$$

$$-1\le ημx\le 1$$

$$\boxed{-\frac{1}{f(x)}\geq\frac{ημx}{f(x)}\geq\frac{1}{f(x)}}$$

$$\left.\begin{aligned}&\lim_{x\to+\infty}-\frac{1}{f(x)}=0\\&\lim_{x\to+\infty}\frac{1}{f(x)}=0\end{aligned}\right\}\ \text{Από κ.π.}\ \lim_{x\to+\infty}\frac{ημx}{f(x)}=0$$

25. $f(x) = x + \ln(1+e^x)$

$A_f = \mathbb{R}.$

(α) $x_1 < x_2 \Rightarrow 1+e^{x_1} < 1+e^{x_2}$

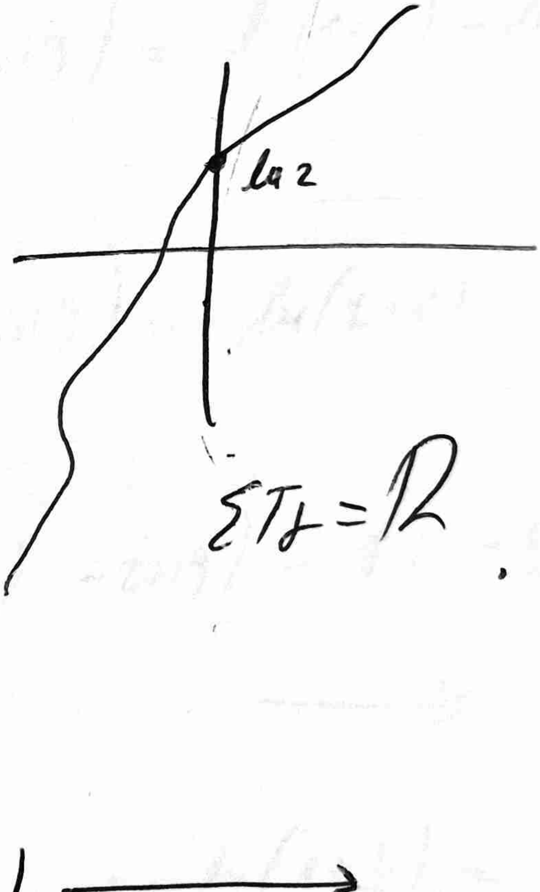
$\underline{\llcorner\ (+)}$ $\ln(1+e^{x_1}) < \ln(1+e^{x_2})$

$f(x_1) < f(x_2)$ $\quad f \nearrow$

$\lim\limits_{x \to -\infty} f(x) = -\infty + \ln 1 = -\infty + 0 = -\infty$

$\lim\limits_{x \to +\infty} f(x) = +\infty.$

x	$-\infty$ $\quad\quad$ $+\infty$
$f(x)$	$-\infty \nearrow +\infty$



$\Sigma T_f = \mathbb{R}.$

(β) $e^{f(f(x))} < 6$

$f(f(x)) < \ln 6$

$f(f(x)) < f(\ln 2) \longrightarrow$

$f(\ln 2) = \ln 2 + \ln(1 + e^{\ln 2}) = \ln 2 + \ln 3 = \ln 6$

$f(x) < \ln 2$

$f(x) < f(0)$

$f \nearrow$

$x < 0$

(γ) $f\left(\ln(e^{2x}+e^{x})-2019\right) = \ln\dfrac{1+e}{e^2}$

$f\left(\ln(e^{2x}+e^{x})-2019\right) = \ln(1+e) - \ln e^2$

$f\left(\ln(e^{2x}+e^{x})-2019\right) = \ln(1+e) - 2$

$f\left(\ln(e^{2x}+e^{x})-2019\right) = f(-1)$

$\xrightarrow{?} \longrightarrow$

$f(-1) = -1+\ln(1+e^{-1}) = -1+\ln\left(1+\frac{1}{e}\right) =$

$= -1 + \ln\left(\frac{e+1}{e}\right) = -1+\ln(e+1) - \ln e$

$= -1+\ln(e+1)-1 \Rightarrow -2+\ln(e+1)$

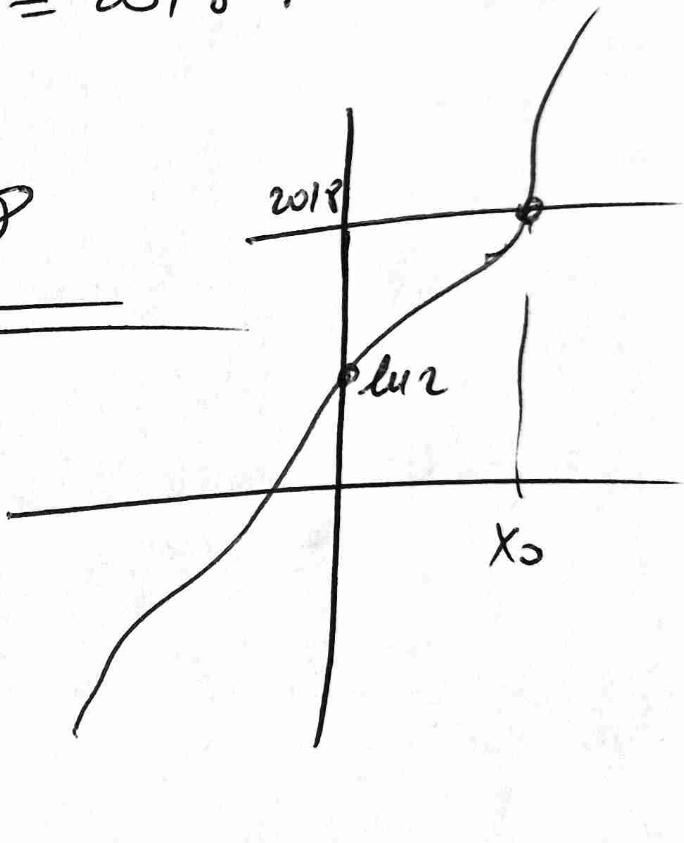
$$\ln(e^{2x}+e^{x})-2019=-1$$

$$\ln\left(e^{x}(e^{x}+1)\right)=2018$$

$$\ln e^{x}+\ln(e^{x}+1)=2018$$

$$x+\ln(e^{x}+1)=2018.$$

$$f(x)=2018$$



- f συνεχής
- $f \nearrow$
- $\Sigma T_f=\mathbb{R}$.

Το $2018\in\Sigma T_f$ άρα $\exists x_0$ τ.ω $f(x_0)=2018$

Το οποίο είναι μοναδικό λόγω μονοτονίας

131. $e^{2f(x)} + 2x\, e^{f(x)} = 1 \quad \forall x \in \mathbb{R}.$

ⓐ $e^{2f(x)} + 2x e^{f(x)} + x^2 = x^2 + 1$

$$\left(e^{f(x)} + x\right)^2 = \sqrt{x^2+1}^{\,2}$$

$$\left|e^{f(x)} + x\right| = \left|\overset{+}{\sqrt{x^2+1}}\right|$$

$$|\overset{+}{\varphi(x)}| = \sqrt{x^2+1}$$

$e^{2f(0)} = 1$

$2f(0) = 0$

$\underline{\underline{f(0) = 0}}$

Ρίζα $\varphi(x)$

$\varphi(x) = 0$

$|\varphi(x)| = 0$

$\sqrt{x^2+1} = 0$

Άτοπο.

$\varphi(x) \neq 0$ και συνεχής

$\varphi(x) > 0$ ή $\varphi(x) < 0$

$\varphi(0) = e^{f(0)} + 0 = e^0 = 1$

$\varphi(x) > 0$

$\varphi(x) = \sqrt{x^2+1}$

$e^{f(x)} + x = \sqrt{x^2+1}$

$e^{f(x)} = \sqrt{x^2+1} - x$

$f(x) = \ln\left(\sqrt{x^2+1} - x\right)$

(B). $f(x) = \ln\left(\sqrt{x^2+1} - x\right)$

$$f(-x) = \ln\left(\sqrt{x^2+1} + x\right) =$$

$$= \ln\left(\frac{\left(\sqrt{x^2+1} + x\right)\left(\sqrt{x^2+1} - x\right)}{\sqrt{x^2+1} - x}\right)$$

$$= \ln \frac{\sqrt{x^2+1}^{\,2} - x^2}{\sqrt{x^2+1} - x} =$$

$$= \ln \frac{1}{\sqrt{x^2+1} - x} = \ln 1 - \ln\left(\sqrt{x^2+1} - x\right)$$

$$= -f(x)$$

περιττή

(γ) $\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} \ln\left(\sqrt{x^2+1} - x\right) = +\infty$

$$\lim_{x\to +\infty} f(x) \underset{\substack{x\to+\infty \\ t\to-\infty}}{\overset{x=-t}{=\!=\!=}} \lim_{t\to -\infty} f(-t) =$$

$$= \lim_{t\to -\infty} -f(t) = -\lim_{t\to -\infty} f(t) = -(+\infty)$$

$$= -\infty.$$

$\Sigma T_f = \mathbb{R}$

(δ). $e^{2f(x)} - 2019 = 0.$

$e^{2f(x)} = 2019$

$2f(x) = \ln 2019$

$f(x) = \dfrac{\ln 2019}{2}$

- f συνεχής
- $\Sigma T_f = \mathbb{R}$

Το $\dfrac{\ln 2019}{2} \in \Sigma T_f$ άρα $\exists x_0$ τ.ω $f(x_0) = \dfrac{\ln 2019}{2}$

29. $f:\mathbb{R}\to\mathbb{R}$ συνεχής.

$f(x)\neq \sigma\upsilon\nu x \quad \forall x\in\mathbb{R}$.

(α) $\displaystyle\lim_{x\to+\infty}\frac{(f(0)-1)x^3+x+1}{(f(\pi)+1)x^2+1}=\lim_{x\to+\infty}\frac{(f(0)-1)x^3}{(f(\pi)+1)x^2}$

$\displaystyle=\lim_{x\to+\infty}\frac{(f(0)-1)}{(f(\pi)+1)}\cdot x=\frac{f(0)-1}{f(\pi)+1}(+\infty)$

$\displaystyle=\frac{h(0)}{h(\pi)}\cdot(+\infty)=\underline{\underline{+\infty}}$

Αφού $f(x)\neq\sigma\upsilon\nu x \Rightarrow \underbrace{f(x)-\sigma\upsilon\nu x}_{h(x)}\neq 0 \Rightarrow h(x)\neq 0$

$h(x)\neq 0$ και συνεχής. $\forall x\in\mathbb{R}$.

$\Rightarrow h(x)>0$ ή $h(x)<0$

Τα $h(0), h(\pi)$ ομόσημοι

$\displaystyle\Rightarrow\frac{h(0)}{h(\pi)}>0$

B

$f(0)=2$ $\qquad (f(x)-\eta\mu x)(f(x)+\eta\mu x) = 2\sigma\upsilon\nu x\, f(x)$

$$f^2(x) - \eta\mu^2 x = 2\sigma\upsilon\nu x\, f(x)$$

$$f^2(x) - 2\sigma\upsilon\nu x\, f(x) + \sigma\upsilon\nu^2 x = \sigma\upsilon\nu^2 x + \eta\mu^2 x$$

$$\left(f(x) - \sigma\upsilon\nu x\right)^2 = 1^2$$

$$\underbrace{|f(x) - \sigma\upsilon\nu x|}_{\varphi(x)} = |\overset{\oplus}{1}|$$

$$|\overset{\oplus}{\varphi(x)}| = 1 \Rightarrow \varphi(x) = 1$$

$$f(x) - \sigma\upsilon\nu x = 1$$

$$\underline{\underline{f(x) = \sigma\upsilon\nu x + 1}}$$

Ρίζα $\varphi(x)$

$\varphi(x)=0$

$|\varphi(x)|=0$

$1=0$

Άτοπο!

$\varphi(x)\neq 0$ και συνεχής

$\Rightarrow \varphi(x)>0$ ή $\varphi(x)<0$

$\varphi(0) = f(0) - \sigma\upsilon\nu 0 = 2-1=1$

$\varphi(x)>0$

(γ) $\lim\limits_{x \to +\infty} \dfrac{e^{-x}}{x+f(x)} = \lim\limits_{x \to +\infty} \dfrac{e^{-x}}{x+1+\sigma\upsilon\nu x} = 0.$

$$-1 \le \sigma\upsilon\nu x \le 1$$

$$0 \le 1+\sigma\upsilon\nu x \le 2$$

$$x \le x+1+\sigma\upsilon\nu x \le 2+x$$

$$\frac{1}{x} \geq \frac{1}{x+1+\sigma\upsilon\nu x} \geq \frac{1}{x+2}$$

$$\boxed{\frac{e^{-x}}{x} \geq \frac{e^{-x}}{x+1+\sigma\upsilon\nu x} \geq \frac{e^{-x}}{x+2}}$$

$$\lim_{x \to +\infty} \frac{e^{-x}}{x} = \lim_{x \to +\infty} e^{-x} \frac{1}{x} = 0 \cdot 0 = 0$$

$$\lim_{x \to +\infty} \frac{e^{-x}}{x+2} = 0$$

Άρα Κ.Π.

$$\lim_{x \to +\infty} \frac{e^{-x}}{x+1+\sigma\upsilon\nu x} = 0.$$

$f(x) = 1 + συν x$

(δ) Ν.δ.ο. $\exists!\, x_0 \in (1,3)$ τ.ω

$$f(x_0) = 1 + \frac{συν 1 + συν 2 + συν 3}{3}$$

$$f(x_0) = \frac{3 + συν 1 + συν 2 + συν 3}{3}$$

$$f(x_0) = \frac{1 + συν 1 + 1 + συν 2 + 1 + συν 3}{3}$$

$$f(x_0) = \frac{f(1) + f(2) + f(3)}{3}$$

Η f συνεχής $[1,3]$ άρα από Θ.Μ.Ε.Τ.

$$m \le f(x) \le M \quad \forall x \in [1,3]$$

$$\left.\begin{array}{l} m \le f(1) \le M \\ m \le f(2) \le M \\ m \le f(3) \le M \end{array}\right\} \overset{(+)}{\Rightarrow} 3m \le f(1) + f(2) + f(3) \le 3M$$

$$\Rightarrow m \le \frac{f(1) + f(2) + f(3)}{3} \le M.$$

Ο αριθμός $\frac{f(1)+f(2)+f(3)}{3} \in ΣΤ_f$

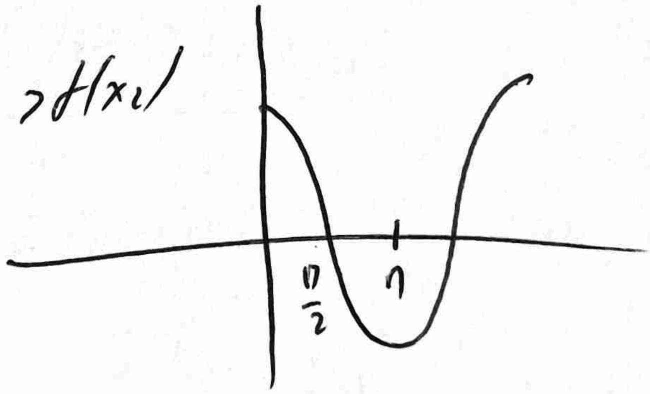
άρα $\exists\, x_0$ τ.ω $f(x_0) = \frac{f(1)+f(2)+f(3)}{3}$

$f(x) = 1 + συνx \quad x \in (1,3).$

• $x_1 < x_2 \Rightarrow συνx_1 > συνx_2$

$1 + συνx_1 > 1 + συνx_2$

$f(x_1) > f(x_2)$



$f \downarrow$ Το $συνx \downarrow (1,3)$

Το x_0 μοναδικό.

28.

$$f(x) = \begin{cases} \lambda(x-1) + e^x, & x \leq 1 \\ e + \ln x, & x > 1 \end{cases}$$

ΔΕΝ ισχύει το ΘΕΤ στο $[0, e]$

ΔΕΝ

f συνεχής.

(α) Νδο η f συνεχής

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \lambda(x-1) + e^x = e$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} e + \ln x = e$$

$$\lim_{x\to 1} f(x) = e$$

$f(1) = e$

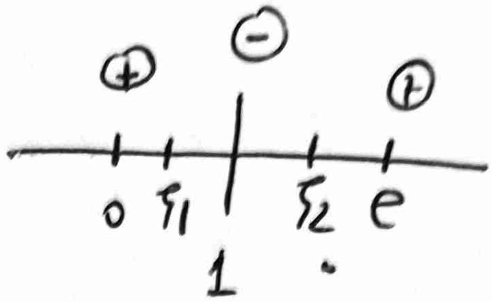
Συνεχής στο 1!

(β) Πρέπει $f(0) = f(e)$

$$-\lambda + 1 = 1 + e \Rightarrow \lambda = -e$$

$$f(x) = \begin{cases} -e(x-1) + e^x, & x \leq 1 \\ e + \ln x, & x > 1. \end{cases}$$

(γ) $f(x) = 3$



$g(x) = f(x) - 3$

Η $g(x)$ συνεχής στο $[0,1]$ και $[1,e]$ ως π.σ.σ.

$g(0) = f(0) - 3 = e + 1 > 0$

$g(1) = f(1) - 3 = e - 3 < 0$

$g(e) = f(e) - 3 = 1 + e - 3 = e - 2 > 0$

$g(0)\, g(1) < 0$ Bolzano $\exists \xi_1 \in (0,1)$ τ.ω $g(\xi_1) = 0$

$g(1)\, g(e) < 0$ Bolzano $\exists \xi_2 \in (1,e)$ τ.ω $g(\xi_2) = 0$

(δ) $\lim\limits_{x \to -\infty} \dfrac{\eta\mu x}{f(x)} = 0$.

$-1 \le \eta\mu x \le 1$

$$-\frac{1}{f(x)} \le \frac{\eta\mu x}{f(x)} \le \frac{1}{f(x)}$$

$\left.\begin{array}{l} \lim\limits_{x \to -\infty} -\dfrac{1}{f(x)} = 0 \\ \lim\limits_{x \to -\infty} \dfrac{1}{f(x)} = 0 \end{array}\right\}$ Από κ.π $\lim\limits_{x \to -\infty} \dfrac{\eta\mu x}{f(x)} = 0$

$\to \lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \left(-e(x-1) + e^x\right) = +\infty$

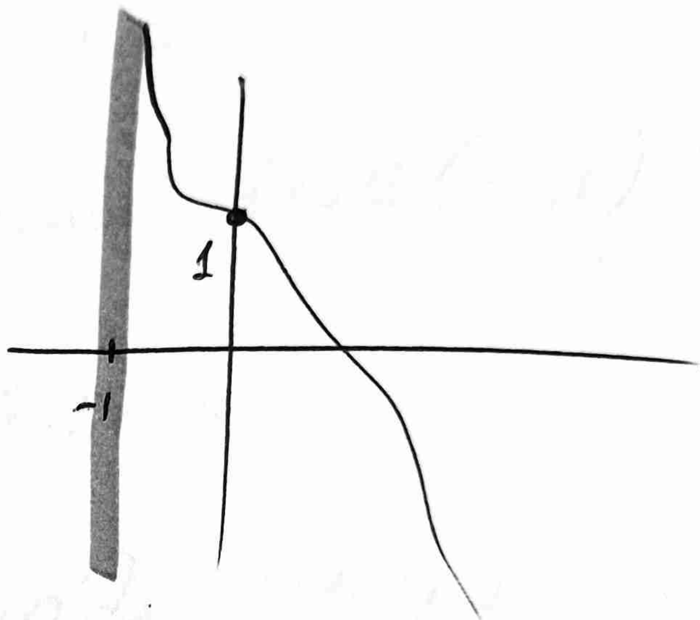
32 (α) $f(x) = e^{-x} - \ln(x+1)$, $x > -1$

$f \downarrow$

(β) Αφού $f \downarrow \Rightarrow f$ 1-1 άρα αντιστρ.

$D_{f^{-1}} = \Sigma T_f$.

x	-1 $\quad\quad$ $+\infty$
$f(x)$	$+\infty \searrow -\infty$



$\lim\limits_{x \to -1^+} f(x) = e - (-\infty) = +\infty$

$\lim\limits_{x \to +\infty} f(x) = e^{-\infty} - (+\infty) = -\infty$

$\Sigma T_f = \mathbb{R}$.

(β) $f^{-1}(x) = x$ $\quad (-\frac{1}{2}, 1)$

$f(f^{-1}(x)) = f(x)$

$x = f(x)$ $\quad \Rightarrow \underbrace{x - e^{-x} + \ln(x+1)}_{g(x)} = 0$.

Η $g(x)$ συνεχής $[-\frac{1}{2}, 1]$.

$$g(-\tfrac{1}{2}) = -\tfrac{1}{2} - e^{\frac{1}{2}} + \ln\tfrac{1}{2} < 0$$

$$g(1) = 1 + e^{-1} + \ln 2 > 0$$

$g(-\frac{1}{2})\, g(1) < 0$ Bolzano $\exists x_0 \in (-\frac{1}{2}, 1)$

τ.ω. $g(x_0) = 0$

$g \nearrow$ με γνήσια αύξουσα χαρακτηριστικά...

(δ) $$\frac{f(e^a-1)-1}{x-1} - \frac{f(\ln(1+a))-f(a)}{x-2} = 2019$$

$a>0$

$$\underbrace{(x-2)\left(f(e^a-1)-1\right) - (x-1)\left[f(\ln(1+a))-f(a)\right] - 2019(x-1)(x-2)}_{\varphi(x)} = 0$$

$\varphi(1) = 1 - f(e^a-1) < 0$

$\varphi(2) = f(a) - f(\ln(1+a)) < 0.$

- $|\ln(1+x)| \le |x| \Rightarrow |\ln(1+a)| \le |\overset{+}{a}|$

$|\ln(1+a)| \le a$

$f(\ln(1+a)) \geq f(a)$

$\cdot a > 0 \Rightarrow e^a > e^0 \Rightarrow e^a > 1$

$0 > 1 - e^a$

$f(0) < f(1-e^a)$

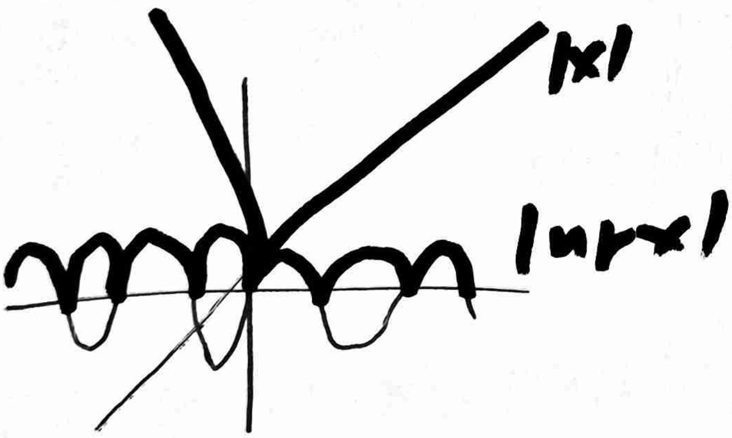
$1 < f(1-e^a)$.

Важни неравенства

1. $e^x \geq x+1$

2. $\ln x \leq x-1$

3. $|\sin x| \leq |x| \quad \forall x \in R.$



30. $f: \mathbb{R} \to \mathbb{R}$ συνεχής:

$$\lim_{x \to 0} \frac{f(x) + ημx - 2}{x^2 + x} = 1$$

• $e^{-x}\left(f^2(x) - 1\right) = 2f(x) - e^{x}$

(α) $\lim_{x \to 0} \frac{f(x) + ημx - 2}{x^2 + x} = 1 \Rightarrow \lim_{x \to 0} g(x) = 1.$

Θέτω $g(x) = \frac{f(x) + ημx - 2}{x^2 + x}$

$$\boxed{f(x) = g(x)\left(x^2 + x\right) - ημx + 2}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x)\left(x^2 + x\right) - ημx + 2$$

$$\lim_{x \to 0} f(x) = 2$$

f συνεχής

$$f(0) = 2.$$

B). $e^{-x}\left(f^2(x)-1\right)=2f(x)-e^x$

$\cancel{e^x\cdot e^{-x}}\left(f^2(x)-1\right)=2e^x f(x)-e^x e^x$

$f^2(x)=2e^x f(x)-e^{2x}+1$

$f^2(x)-2e^x f(x)+e^{2x}=1$

$\left(f(x)-e^x\right)^2=1^2$

$|f(x)-e^x|=|1|$

$|\varphi(x)|=1 \Rightarrow \varphi(x)=1$

$f(x)-e^x=1$

$f(x)=e^x+1$

Ρίζα $\varphi(x)$

$\varphi(x)=0$

$|\varphi(x)|=0$

$1=0$

Άτοπο

$\varphi(x)\neq 0$ και συνεχής

$\Rightarrow \varphi(x)>0$ ή $\varphi(x)<0$ $\forall x\in\mathbb{R}$.

$\varphi(0)=f(0)-e^0=2-1=1$

$\varphi(x)>0$

(γ) $f(x) = g(x)$

$e^x + 1 = \frac{1}{x} + 1$

$e^x - \frac{1}{x} = 0 \Rightarrow \underbrace{xe^x - 1}_{t(x)} = 0$

Η $t(x)$ συνεχής $[0,1]$ ως π.σ.σ.

$\left.\begin{array}{l} t(0) = -1 < 0 \\ t(1) = e > 0 \end{array}\right\}$ $\exists\, x_0 \in (0,1)$ ώστε $t(x_0) = 0$

$\left.\begin{array}{l} \bullet\ x_1 < x_2 \\ \bullet\ x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2} \end{array}\right\} \odot\ x_1 e^{x_1} < e^{x_2} x_2$

$x_1 e^{x_1} - 1 < x_2 e^{x_2} - 1$

$t(x_1) < t(x_2)$

$t \nearrow$

Θετικά ανισότητα στο $(0,1)$

Το x_0 μοναδικό

δ) $\lim_{x\to+\infty} \dfrac{f(x)-2^x}{f(x)-3^x} = \lim_{x\to+\infty} \dfrac{e^x+1-2^x}{e^x+1-3^x}$

$$= \lim_{x\to+\infty} \frac{e^x\left(1+\frac{1}{e^x}-\left(\frac{2}{e}\right)^x\right)}{3^x\left(\left(\frac{e}{3}\right)^x+\frac{1}{3^x}-1\right)}$$

$$= \lim_{x\to+\infty} \left(\frac{e}{3}\right)^x \frac{1+\frac{1}{e^x}-\left(\frac{2}{3}\right)^x}{\left(\frac{e}{3}\right)^x+\frac{1}{3^x}-1}$$

$$= 0 \cdot \frac{1+0-0}{0+0-1} = 0,$$