Θεμα 4ο

- $f:\mathbb{R} \to \mathbb{R}$ παρ/μη
- $f(x) \neq 0 \quad \forall x \in \mathbb{R}$
- $f'(x) = -2x f^2(x) \quad \forall x \in \mathbb{R}$.

(α) Ν.δ.ο $g(x) = \frac{1}{f(x)} - x^2$ σταθερή

$$g'(x) = \frac{-f'(x)}{f^2(x)} - 2x = -\frac{-2x\cancel{f^2(x)}}{\cancel{f^2(x)}} - 2x = 0$$

άρα $g(x) = c$

(β) Αν $f(0) = 1$ τοτε $g(x) = c \Leftrightarrow \frac{1}{f(x)} - x^2 = c$

$\xrightarrow{x=0}$ $\frac{1}{f(0)} = c \Leftrightarrow \underline{\underline{c = 1}}$

$$\frac{1}{f(x)} - x^2 = 1$$

$$\frac{1}{f(x)} = x^2 + 1 \Leftrightarrow \boxed{f(x) = \frac{1}{x^2+1}}$$

(γ) $f'(x) = -\frac{2x}{(x^2+1)^2}$

x		0	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

$$f''(x) = -\frac{2(x^2+1)^2 - 2x\cdot 2(x^2+1)2x}{(x^2+1)^4}$$

$$f''(x) = -\frac{2(x^2+1) - 8x^2}{(x^2+1)^3} = -\frac{2x^2+2-8x^2}{(x^2+1)^3}$$

$$f''(x) = \frac{6x^2-2}{(x^2+1)^3} = 2\,\frac{3x^2-1}{(x^2+1)^3}$$

$\longrightarrow 3x^2 - 1 = 0 \iff 3x^2 = 1 \iff x^2 = \frac{1}{3} \iff x = \pm\frac{\sqrt{3}}{3}$

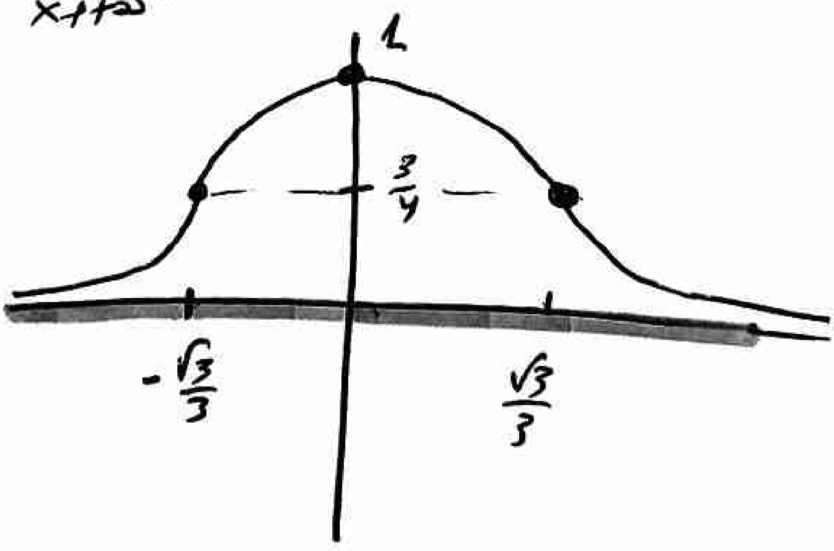
x		$-\frac{\sqrt{3}}{3}$		$\frac{\sqrt{3}}{3}$	
f''	$+$	0	$-$	0	$+$
f	$\cup$		$\cap$		$\cup$

(8) $\lim_{x\to-\infty} f(x) = 0$

$\lim_{x\to+\infty} f(x) = 0$ $\Big\}$ E: $y=0$

$+\infty$

$-\infty$



Μορφη Euler: $f'(x)+g(x)f(x)=h(x)$

Θεμα 41

- $f: (0,+\infty) \to \mathbb{R}$ παρ/μη.
- $f(1)=1$
- $x f(x) = 1 - x^2 f'(x) \quad \forall x>0$

ⓐ Τυπο $f(x)$.

$$x f(x) + x^2 f'(x) = 1.$$

$$f(x) + x f'(x) = \frac{1}{x}$$

$$(x f(x))' = (\ln x)' \iff x f(x) = \ln x + C$$

$x=1$

$$f(1) = C = 1.$$

$$x f(x) = \ln x + 1$$

$$\boxed{f(x) = \frac{\ln x + 1}{x}}$$

β' τροπος

$$x^2 f'(x) + x f(x) = 1$$

$$\boxed{f'(x) + \frac{1}{x} f(x) = \frac{1}{x^2}}$$

$$g(x) = \frac{1}{x}$$

$$G(x) = \ln x$$

$$e^{G(x)} = e^{\ln x} = x$$

$$x f'(x) + f(x) = \frac{1}{x}$$

$$(x f(x))' = (\ln x)'$$

ⓑ. $$f'(x) = \frac{\frac{1}{x} x - (\ln x + 1)}{x^2} = \frac{1 - \ln x - 1}{x^2} = \frac{-\ln x}{x^2}$$

$\to -\ln x = 0 \iff \ln x = 0 \iff x = 1$

x	0	1	$+\infty$
f'	$+$	0	$-$
f	↗		↘

$$f(x) \le f(1)$$

$$f(x) \le 1$$

$$f'(x) = \frac{-\ln x}{x^2}$$

$$f''(x) = -\frac{\frac{1}{x}x^2 - \ln x \cdot 2x}{x^4}$$

$$f''(x) = -\frac{x - \ln x \cdot 2x}{x^4} = -\frac{1-2\ln x}{x^3}$$

$$f''(x) = \frac{2\ln x - 1}{x^3}.$$

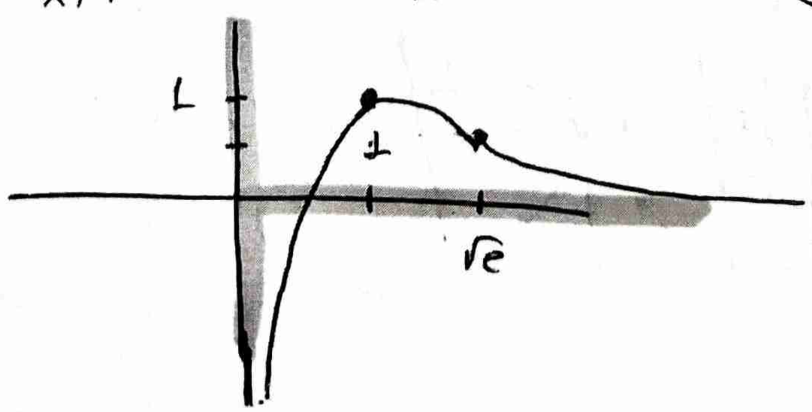
$\longrightarrow 2\ln x - 1 = 0 \iff \ln x = \frac{1}{2} \iff x = e^{1/2} = \sqrt{e}.$

x	0	$\sqrt{e}$	
f''	$-$	0	$+$
f	$\frown$		$\smile$

① $\cdot \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{1+\ln x}{x} = \lim\limits_{x\to 0^+} (1+\ln x)\cdot\dfrac{1}{x} =$

$= -\infty \cdot (+\infty) = -\infty$ $\boxed{\varepsilon_1 : x = 0}$

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} \dfrac{1+\ln x}{x} = \lim\limits_{x\to +\infty} \dfrac{1}{x} = 0$ $\varepsilon_2 : y = 0$ $+\infty$



(δ) $$E = \int_1^2 \overset{\oplus}{|H(x)|}\,dx = \int_1^2 \left|\overset{\oplus}{\frac{1+\ln x}{x}}\right| dx$$

$$= \int_1^2 \frac{1+\ln x}{x}\,dx \;\underset{\frac{1}{x}dx = dt}{\overset{\ln x = t}{=\!=\!=}}\; \int_0^{\ln 2} (1+t)\,dt$$

$$= (t)_0^{\ln 2} + \frac{1}{2}(t^2)_0^{\ln 2} = \ln 2 + \frac{1}{2}\ln^2 2.$$

(ε).

x		1		$\sqrt{e}$	
$H(x)$	↗		↘		↘

(B)

Θέμα 42

- $f:(0,+\infty) \longrightarrow \mathbb{R}$
- $\lim_{x\to 1} \frac{f(x)}{\ln x} = 1$
- $1 + x^2 f''(x) = 0 \ , \ \forall x>0$

ⓐ ν.δ.ο. $f(1)=0 \ , \ f'(1)=1$

$$\lim_{x\to 1} \frac{f(x)}{\ln x} = 1$$

Θέτω $g(x) = \frac{f(x)}{\ln x}$ τότε $\lim_{x\to 1} g(x) = 1$

$$g(x) = \frac{f(x)}{\ln x} \Leftrightarrow f(x) = g(x)\ln x$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)\ln x = 0$$

Άρα f συνεχής $\underline{\underline{f(1)=0}}$

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{g(x)\ln x}{x-1} = \lim_{x\to 1} g(x)\frac{\ln x}{x-1}$$

$$= 1\cdot 1 = 1$$

$$\rightarrow \lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1}{x} = 1$$

$f'(1)=1$.

(B). $1+x^2 f''(x)=0$

$x^2 f''(x) = -1 \Leftrightarrow f''(x) = -\frac{1}{x^2} \Leftrightarrow f''(x) = \left(\frac{1}{x}\right)'$

$f'(x) = \frac{1}{x} + c$

$\underline{x=1}$

$f'(1) = 1 + c$

$1 = 1 + c$

$c = 0$

$f'(x) = \frac{1}{x}$

$f'(x) = (\ln x)'$

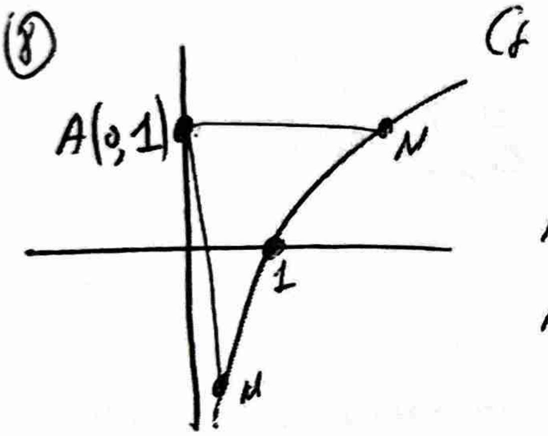
$f(x) = \ln x + c$

$\underline{x=1}$

$f(1) = c$

$\underline{\underline{c=0}}$ $\qquad$ $\underline{\underline{f(x) = \ln x}}$

(γ)



$AM = d(A,M) = \sqrt{(y-1)^2 + (x-0)^2}$

$A(0,1)$ $\qquad$ $d(x) = \sqrt{(\ln x - 1)^2 + x^2}$

$M(x,y)$ $\qquad$ Συνάρτηση απόστασης

$$d'(x) = \frac{2(\ln x - 1)\frac{1}{x} + 2x}{2\sqrt{(\ln x - 1)^2 + x^2}} = \frac{\ln x - 1 + x^2}{x\sqrt{(\ln x - 1)^2 + x^2}}$$

Θεωρώ $\boxed{\varphi(x) = x^2 + \ln x - 1}$ $\varphi(1)=0$

$\varphi'(x) = 2x + \frac{1}{x} > 0$

x	0	1	
φ'	$+$		$+$
φ	$\nearrow$	0	$\nearrow$
d'	$-$		$+$
d	$\searrow$		$\nearrow$

Άρα $d(x) \geq d(1) \Leftrightarrow d(x) \geq \sqrt{2}$.

$M(1,0)$.

(δ). $\varepsilon: \; y - f(1) = f'(1)(x-1)$

$$y - 0 = 1(x-1)$$

$$\varepsilon: \; y = x-1$$

άρα $\lambda_\varepsilon = 1$.

$$\lambda_{AM} = \frac{0-1}{1-0} = -1$$

$A(0,1)$ άρα $\lambda_{AM} \cdot \lambda_\varepsilon = -1$

$M(1,0)$ $\varepsilon \perp AM$.

Θεμα 49

- $f(x) = \ln x \quad , x>0$
- $g(x) = \frac{1}{x} - 1 \quad , x \neq 0$

(α) i) $h(x) = (f \circ g)(x) = f(g(x)) = \ln\left(\frac{1}{x} - 1\right) = \ln\left(\frac{1-x}{x}\right)$

$x \in D_g$ και $g(x) \in D_f$

$x \neq 0$ $\quad \frac{1}{x} - 1 > 0 \iff \frac{1-x}{x} > 0$

x		0		1	
$1-x$	+		+	0	−
x	−	0	+		+
$\frac{1-x}{x}$	−		+		−

$D_h = (0,1)$.

ii). $h(x_1) = h(x_2) \iff \ln\left(\frac{1-x_1}{x_1}\right) = \ln\left(\frac{1-x_2}{x_2}\right)$

$\iff \frac{1-x_1}{x_1} = \frac{1-x_2}{x_2} \iff (1-x_1)x_2 = x_1(1-x_2)$

$\iff x_2 - x_1x_2 = x_1 - x_1x_2 \iff x_1 = x_2$ $\qquad h$ 1-1.

αρα αντιστρεφεται.

$h(x) = y \iff \ln\left(\frac{1}{x} - 1\right) = y \iff \frac{1}{x} - 1 = e^y$

$\iff \frac{1}{x} = e^y + 1 \iff \frac{1}{e^y+1} = x \iff h^{-1}(x) = \frac{1}{e^x+1}$.

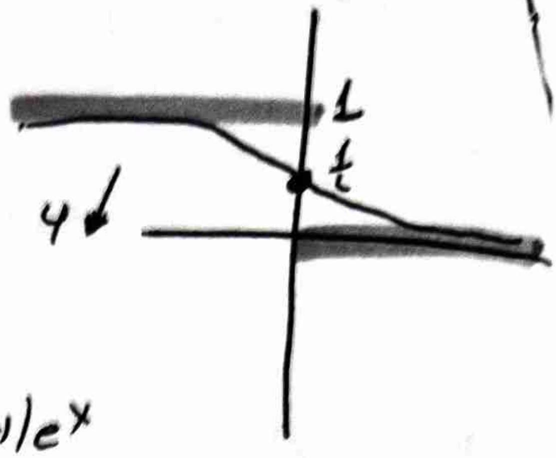
$D_{h^{-1}} = \mathbb{R}$

Τελος

$0 < x < 1 \iff 0 < \frac{1}{e^y+1} < 1 \implies \frac{1}{e^y+1} < 1 \implies 1 < e^y + 1$

$0 < e^y$

(β) $\varphi(x) = \dfrac{1}{e^x+1}$



$$\varphi'(x) = -\frac{e^x}{(e^x+1)^2} < 0 \qquad \varphi \downarrow$$

$$\varphi''(x) = -\frac{e^x(e^x+1)^2 - e^x\, 2(e^x+1)e^x}{(e^x+1)^4}$$

$$\varphi''(x) = -\frac{e^x(e^x+1)\left[e^x+1-2e^x\right)}{(e^x+1)^4}$$

$$\varphi''(x) = -\frac{\overset{\oplus}{e^x}(1-e^x)}{\underset{\oplus}{(e^x+1)^3}}$$

x		0	
φ''	$-$	0	$+$
φ	$\frown$		$\smile$

(γ). $\lim\limits_{x\to+\infty} \varphi(x) = 0$ — $\varepsilon_1 : y = 0$ (+∞)

$\lim\limits_{x\to-\infty} \varphi(x) = 1$ — $\varepsilon_2 : y = 1$ (−∞)

(δ). $E = \displaystyle\int_0^1 |\varphi(x)|\,dx = \int_0^1 \left|\overset{\oplus}{\frac{1}{e^x+1}}\right| dx = \int_0^1 \frac{1}{e^x+1}\,dx =$

$$= \int_0^1 \frac{e^x}{e^x(e^x+1)}\,dx \;\underset{e^x dx = dt}{\overset{e^x+1=t}{=\!=}}\; \int_2^{e+1} \frac{1}{t(t+1)}\,dt \;\overset{(*)}{=\!=}$$

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

$$1 = A(t+1) + Bt$$

$$1 = At + A + Bt$$

$$1 = (A+B)t + A$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases}$$

$B=-1$.

Άρα $\overset{*}{=}$ $\displaystyle\int_2^{e+1} \frac{1}{t} - \frac{1}{t+1}\, dt$

$$= \left(\ln|t|\right)_2^{e+1} - \left(\ln|t+1|\right)_2^{e+1} =$$

$$= \ln(e+1) - \ln 2 - \left(\ln(e+2) - \ln 3\right).$$

$$= \ln\frac{e+1}{2} - \ln\frac{e+2}{3} =$$

$$= \ln\frac{\frac{e+1}{2}}{\frac{e+2}{3}} = \ln\frac{3e+3}{2e+4}$$

Θεμα 50

- $f(x) = x^3+2x-1$

ⓐ $f'(x) = 3x^2+2>0$ $\quad f \nearrow$ αρα 1-1

$\left.\begin{array}{l} f(0)=-1 \\ f(1)=2 \end{array}\right\} f(0)f(1)<0$ Bolzano $\exists \xi>0$

τ.ω $f(\xi)=0$

ⓑ $f(x) > \dfrac{12}{2+f^2(x)} \iff 2f(x)+f^3(x)>12$

$f^3(x)+2f(x)-12>0 \iff f^3(x)+2f(x)-1>12-1$

$f(f(x))>11 \iff f(f(x))>f(2)$

$f\nearrow$

$f(x)>2$

$f(x)>f(1)$

$f\nearrow$

$x>1$.

(γ) $\cdot \lim_{x\to 0} \dfrac{\ln x}{f(2x)-f(x)} = \lim_{x\to 0^+} \ln x \cdot \dfrac{1}{f(2x)-f(x)}$

$\oplus$

$= (-\infty)\cdot(+\infty) = -\infty$

- $x<2x \Rightarrow f(x)<f(2x) \Rightarrow f(2x)-f(x)>0$

(δ) $\displaystyle\int_1^2 \frac{f(x)}{x^2} = \int_1^2 \frac{x^3+2x-1}{x^2}\,dx = \int_1^2 x+\frac{2}{x}-\frac{1}{x^2}\,dx$

$\displaystyle = \int_1^2 x\,dx + 2\int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{1}{x^2}\,dx$

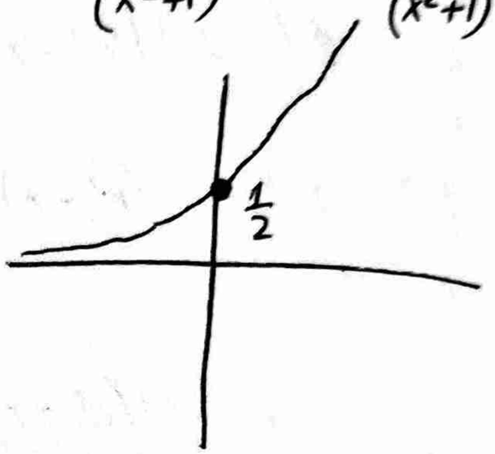
$\displaystyle = \frac{1}{2}\left(x^2\right)_1^2 + 2\left(\ln x\right)_1^2 + \left(\frac{1}{x}\right)_1^2$

$\displaystyle = \frac{3}{2} + 2\ln 2 + \frac{1}{2} - 1 \quad = 1+2\ln 2$.

Θεμα 51

• $f(x) = \dfrac{e^x}{x^2+1}$ $\quad D_f = \mathbb{R}$.

ⓐ $f'(x) = \dfrac{e^x(x^2+1) - e^x 2x}{(x^2+1)^2} = \dfrac{e^x(x^2-2x+1)}{(x^2+1)^2} = \dfrac{e^x(x-1)^2}{(x^2+1)^2}$



$f'(x) \geq 0$ $\quad f \nearrow$

$D_{f^{-1}} = \Sigma T_f = (0, +\infty)$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{e^x}{x^2+1} = 0$

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{e^x}{x^2+1} = \lim\limits_{x \to +\infty} \dfrac{e^x}{2x} = \lim\limits_{x \to +\infty} \dfrac{e^x}{2} = +\infty$

ⓑ • $f(x) + \ln 2 > 1 + \ln(1 + f^2(x))$

$e^{f(x) + \ln 2} > e^{1 + \ln(1 + f^2(x))}$

$e^{f(x)} \cdot e^{\ln 2} > e^1 \cdot e^{\ln(1 + f^2(x))}$

$2e^{f(x)} > e(1 + f^2(x)) \Leftrightarrow \dfrac{e^{f(x)}}{1 + f^2(x)} > \dfrac{e}{2}$

$f(f(x)) > f(1)$

$f \uparrow$

$f(x) > 1$

$f(x) > f(0)$

$f \uparrow \quad \underline{\underline{x > 0}}$

γ. $f\left(f^{-1}(e^x - x) - f^{-1}(1)\right) = 1,$

$f\left(f^{-1}(e^x - x) - f^{-1}(1)\right) = f(0)$

f 1-1

$f^{-1}(e^x - x) - f^{-1}(1) = 0$

$f^{-1}(e^x - x) = f^{-1}(1)$

$f\left(f^{-1}(e^x - x)\right) = f\left(f^{-1}(1)\right)$ f συνάρτηση

$e^x - x = 1 \Rightarrow e^x = x + 1$ $\quad x = 0$

γιατί $e^x \geq x + 1$

Το "=" για $x = 0$

δ) $\lim\limits_{x \to 1} \dfrac{\ln(x-1)}{f(x^2) - f(x)} = \lim\limits_{x \to 1^+} \ln(x-1) \dfrac{1}{f(x^2) - f(x)} = (-\infty)(+\infty)$

$\underline{\underline{= -\infty}}$

• $x^2 > x \Rightarrow f(x^2) > f(x) \Rightarrow f(x^2) - f(x) > 0$

Θεμα 53

$$f(x) = \begin{cases} -x^2+x+1, & x<1 \\ e^{1-\lambda x}, & x \geq 1 \end{cases}$$

(α) $f(0) = f(1) \Leftrightarrow 1 = e^{1-\lambda} \Leftrightarrow \underline{\underline{\lambda = 1}}$

$$\boxed{f(x) = \begin{cases} -x^2+x+1, & x<1 \\ e^{1-x}, & x \geq 1 \end{cases}}$$

(β) $\lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^-} \dfrac{-x^2+x+1-1}{x-1} =$

$= \lim\limits_{x\to 1^-} \dfrac{-x^2+x}{x-1} = \lim\limits_{x\to 1^-} \dfrac{-2x+1}{1} = -1.$

$\lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+} \dfrac{e^{1-x}-1}{x-1} = \lim\limits_{x\to 1^+} \dfrac{-e^{1-x}}{1} = -1.$

Αρα $f'(1) = -1$

αρα η $\varepsilon\varphi\hat{\omega} = f'(1) = -1$

$\underline{\underline{\hat{\omega} = 135}}$

(γ) $\underline{x<1}$ $\qquad\qquad$ $\underline{x>1}$

$f_1(x) = -x^2+x+1$ $\qquad\qquad$ $f_2(x) = e^{1-x}$

$f_1'(x) = -2x+1$ $\qquad\qquad$ $f_2'(x) = -e^{1-x} < 0$

$\longrightarrow -2x+1=0 \Leftrightarrow \boxed{x=\frac{1}{2}}$

x		$\frac{1}{2}$		1	
f_1'	$+$	0	$-$		//////////
f_2'	///		/////		$-$
f'	$+$		$-$		$-$
f	$\nearrow$		$\searrow$		$\searrow$

$f(x) \le f\left(\frac{1}{2}\right)$

$f(x) \le -\frac{1}{4}+\frac{1}{2}+1$

$f(x) \le \frac{5}{4}$

(δ) Αφού $D_f=\mathbb{R}$ δεν έχω κατακόρυφη ασύμπτωτη

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}\left(-x^2+x+1\right) = -\infty$ $\qquad$ Δεν έχει οριζόντια στο $-\infty$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} e^{1-x} = 0$ $\qquad$ $\varepsilon: y=0$ ($+\infty$)

$\lim\limits_{x\to-\infty} \frac{f(x)}{x} = \lim\limits_{x\to-\infty} \frac{-x^2+x+1}{x} = \lim\limits_{x\to-\infty} \frac{-x^2}{x} = +\infty$

Δεν έχει πλάγια στο $-\infty$.

$x<1$

$f_1'(x) = -2x+1$

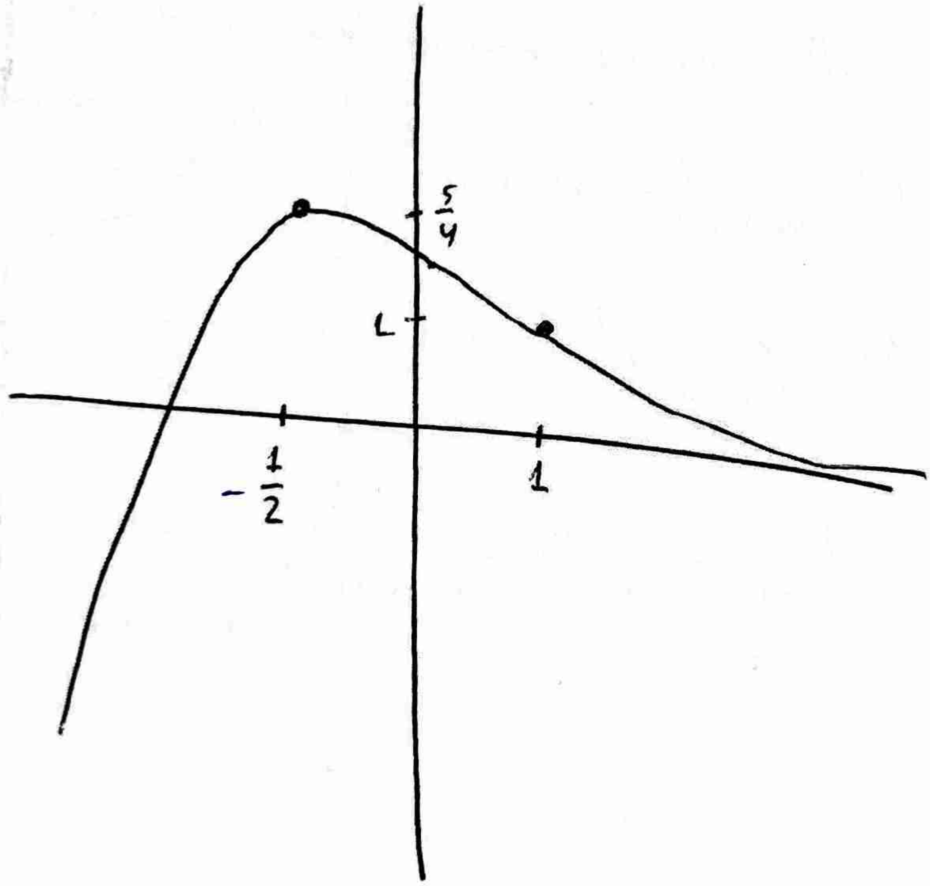
$f_1''(x) = -2$

$x>1$

$f_2'(x) = -e^{1-x}$

$f_2''(x) = e^{1-x} > 0$

x		1	
f_1''	$-$		//////////
f_2''	//////////		$+$
f''	$-$		$+$
f	$\curvearrowright$		$\circlearrowleft$



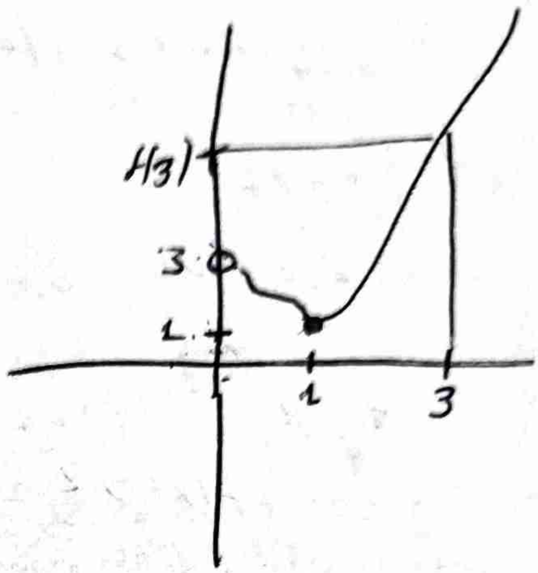
Θεμα 54

• $f(x) = x\ln x + x^2 - 3x + 3$, $x>0$

ⓐ $f'(x) = \ln x + 1 + 2x - 3 = \ln x + 2x - 2$ $\quad f'(1)=0$

$f''(x) = \frac{1}{x} + 2 > 0$

x	0		1	
f''		$+$		$+$
f'		$\nearrow$	0	$\nearrow$
f		$\searrow$		$\nearrow$



$f(x) \geq f(1)$

$f(x) \geq 1$.

ⓑ νδο η εξίσωση $f(x)=f(3)$ έχει μοναδική λύση

$\lim_{x\to 0^+} f(x) = 3$

$\lim_{x\to +\infty} f(x) = +\infty$

Αν $x>1$

$f(x)=f(3)$

f 1-1

$\boxed{x=3}$

Αν $x<1$

Το $f(3)$ δεν ανήκει στο $\Sigma T f$.

$f(3) = 3\ln 3 + 9 - 9 + 3 = 3(\ln 3 + 1) > 3$

• $3>1 \Rightarrow \ln 3 > \ln 1 \Rightarrow \ln 3 > 0$

$\ln 3 + 1 > 1$

$\Leftrightarrow f(3) > 3$

⑧ Ανίσωση $f(1+f(x)) > \ln 4e$

$f(1+f(x)) > \ln 4 + 1.$

$f(1+f(x)) > f(2)$

$\bullet\ 2>1$

$\bullet\ f(x) \geq 1 \Rightarrow f(x)+1 > 1$

$\Big\}\ \forall x>1\ f\nearrow$

$f \nearrow$

$1+f(x) > 2 \Rightarrow f(x) > 1.$

$x \in (0,1) \cup (1,+\infty)$.

⑤. $\lim_{x\to 1} \dfrac{\ln x}{(x-1)(f(x)-1)} =$

$= \lim_{x\to 1} \dfrac{\ln x}{x-1} \dfrac{1}{f(x)-1} = 1\cdot(+\infty) = +\infty,$

$\bullet \lim_{x\to 1} \dfrac{\ln x}{x-1} = \lim_{x\to 1} \dfrac{\frac{1}{x}}{1} = 1$

Θεμα 56

• $f: (0,+\infty) \longrightarrow R$

• $f(x) = \dfrac{x}{\ln(1+x)}, \quad x > 0$

(α) $f'(x) = \dfrac{\ln(x+1) - \dfrac{x}{x+1}}{\ln^2(x+1)} = \dfrac{(x+1)\ln(x+1) - x}{(x+1)\ln^2(x+1)}$

$\boxed{g(x) = (x+1)\ln(x+1) - x}$

$g'(x) = \ln(x+1) + 1 - 1 = \ln(x+1) > 0$

$\forall x > 0 \Rightarrow x+1 > 1 \Rightarrow \ln(x+1) > 0$

x	0 $\quad\quad$ $+\infty$
g'	$+$
g	$+$ ↗
f'	$+$
f	1 ↗ $+\infty$

$x > 0 \Rightarrow g(x) > g(0) \Rightarrow g(x) > 0$

Αφού $f \nearrow \Rightarrow f$ 1-1 άρα αντιστρέφεται

$D_{f^{-1}} = \Sigma T_f$.

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{x}{\ln(x+1)} = \lim\limits_{x\to 0^+} \dfrac{1}{\frac{1}{x+1}} = 1$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{\ln(x+1)} = \lim_{x\to+\infty} \frac{1}{\frac{1}{x+1}} = +\infty$$

$D_{f^{-1}} = \Sigma T_f = (1, +\infty)$.

(β) ν.δ.ο. $f(x) < 2^{f(x)} - 1$, $\forall x > 0$.

$$f(x)+1 < 2^{f(x)} \Leftrightarrow \ln(f(x)+1) < f(x)\cdot \ln 2$$

$$\Leftrightarrow \frac{1}{\ln 2} < \frac{f(x)}{\ln(f(x)+1)} \Leftrightarrow f(1) < f(f(x))$$

$f\uparrow$

$1 < f(x)$

που ισχύει!

(γ) εξίσωση $f(x^2) + \ln x = f(x)$

Προφανής ρίζα το 1.

$x > 1$

- $x^2 > x \Rightarrow f(x^2) > f(x)$
- $\ln x > 0$

$\oplus$ $f(x^2) + \ln x > f(x)$

$0 < x < 1$

- $x^2 < x \Rightarrow f(x^2) < f(x)$
- $\ln x < 0$ $\quad \oplus$

$f(x^2) + \ln x < f(x)$

(δ) νδο $\int_{e}^{e^4} \frac{f(\sqrt{\ln x})}{x}\,dx < \frac{6}{\ln 3}$.

$$\boxed{\begin{aligned} \sqrt{\ln x} &= t \\ \ln x &= t^2 \\ \frac{1}{x}\,dx &= 2t\,dt \end{aligned}}$$

$$\Leftrightarrow \underbrace{\int_1^2 f(t)\,dt < \frac{6}{\ln 3}}_{\text{Αρκεί νδο}}$$

Όταν $1 < x < 2 \Leftrightarrow f(1) < f(x) < f(2)$

$$f(x) < \frac{2}{\ln 3}$$

$$\int_1^2 f(x)\,dx < \int_1^2 \frac{2}{\ln 3}\,dx$$

$$\int_1^2 f(x)\,dx < \frac{2}{\ln 3}(x)_1^2$$

$$\int_1^2 f(x)\,dx < \frac{2}{\ln 3} < \frac{6}{\ln 3}$$

✓

Θέμα 57

(α) $f'(1)=0$ και $f(1)=0$

$f(x)=\alpha e^{x-1}+\beta\ln x-1$

$f(1)=\alpha-1=0 \Leftrightarrow \underline{\alpha=1}$

$f'(x)=\alpha e^{x-1}+\frac{\beta}{x}$

$f'(1)=\alpha+\beta=0 \Leftrightarrow 1+\beta=0 \Leftrightarrow \underline{\beta=-1}$

$\boxed{f(x)=e^{x-1}-\ln x-1}$ $D_f=(0,+\infty)$.

(β) $f'(x)=e^{x-1}-\frac{1}{x}$ $f'(1)=0$

$f''(x)=e^{x-1}+\frac{1}{x^2}>0$

x		1	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

$f(x)\geq f(1)$

$f(x)\geq 0$

(γ) $E: C_g, x'x, x=2$. όπου $g(x) = x f(x)$, $x>0$.

$g(x)=0 \Leftrightarrow x f(x) = 0 \Leftrightarrow \boxed{x=0}$ ή $f(x)=0$.

Απορ/ται $\quad \boxed{x=1}$

$$E = \int_1^2 |g(x)|\,dx = \int_1^2 |x f(x)|\,dx = \int_1^2 x f(x)\,dx$$

$$= \int_1^2 x(e^{x-1} - \ln x - 1)\,dx = \int_1^2 x e^{x-1} - x\ln x - x\,dx$$

$$= \int_1^2 x e^{x-1}\,dx - \int_1^2 x\ln x\,dx - \int_1^2 x\,dx. \quad (*)$$

$$\rightarrow \int_1^2 x e^{x-1}\,dx = \int_1^2 x (e^{x-1})'\,dx = (xe^{x-1})_1^2 - \int_1^2 e^{x-1}\,dx$$

$$= 2e - 1 - (e^{x-1})_1^2 = 2e - 1 - (e-1) = 2e - 1 - e + 1$$

$$= e$$

$$\rightarrow \int_1^2 x\ln x\,dx = \int_1^2 \left(\frac{x^2}{2}\right)' \ln x\,dx =$$

$$= \left(\frac{x^2}{2}\ln x\right)_1^2 - \int_1^2 \frac{x^2}{2}\frac{1}{x}\,dx = 2\ln 2 - \frac{1}{2}\int_1^2 x\,dx$$

$$= 2\ln 2 - \frac{1}{2}\frac{1}{2}(x^2)_1^2 = \ln 4 - \frac{3}{4}$$

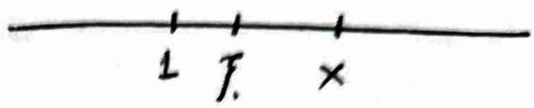
$$\rightarrow \int_1^2 x\,dx = \frac{1}{2}(x^2)_1^2 = \frac{3}{2}.$$

(*)

$$e - \ln 4 + \frac{3}{4} - \frac{3}{2} = e - \ln 4 - \frac{3}{4}$$

(δ). Ν.δ.ο. $f'(x) > \dfrac{H(x)}{x-1}$, $\forall x > 1$.

$$f'(\xi) = \frac{H(x) - H(1)}{x-1}$$



$$\xi < x \overset{f' \uparrow}{\underset{\text{αύξ}}{\Longleftrightarrow}} f'(\xi) < f'(x) \Leftrightarrow \frac{H(x) - H(1)}{x-1} < f'(x)$$

f κυρτή

$$\frac{H(x) - 0}{x-1} < f'(x)$$

$$f'(x) > \frac{H(x)}{x-1}.$$

Θεμα 58

- $f(x) = \dfrac{1+\ln^2 x}{x}$

ⓐ $f'(x) = \dfrac{2\ln x \frac{1}{x} x - 1 - \ln^2 x}{x^2} = \dfrac{2\ln x - 1 - \ln^2 x}{x^2}$

$f'(x) = -\dfrac{\ln^2 x - 2\ln x + 1}{x^2} = -\dfrac{(\ln x - 1)^2}{x^2} \leq 0$

$f \downarrow$

ⓑ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{1+\ln^2 x}{x} = \lim\limits_{x\to 0^+} (1+\ln^2 x)\, \dfrac{1}{x}$

$= (+\infty)\cdot(+\infty) = +\infty.$ $\boxed{\varepsilon_1 : x = 0}$

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} \dfrac{1+\ln^2 x}{x} = \lim\limits_{x\to +\infty} \dfrac{2\ln x \frac{1}{x}}{1}$

$= \lim\limits_{x\to +\infty} \dfrac{2\frac{1}{x}}{1} = 0$ $\quad \Sigma T_f = (0, +\infty).$

$\boxed{\varepsilon_2 : y = 0 \quad (+\infty)}$

(γ) $f'(x) = -\dfrac{(\ln x - 1)^2}{x^2}$

$$f''(x) = -\frac{2(\ln x-1)\frac{1}{x}x^2 - 2x(\ln x-1)^2}{x^4}$$

$$f''(x) = -\frac{2(\ln x-1)x - 2x(\ln x-1)^2}{x^4} = -\frac{2(\ln x-1) - 2(\ln x-1)^2}{x^3}$$

$$f''(x) = -\frac{2(\ln x-1)(1-\ln x+1)}{x^3} = \frac{2\ln x(\ln x-1)}{x^3}$$

$\longrightarrow f''(x)=0 \iff \ln x = 0$ ή $\ln x - 1 = 0$

$x=1$ $\quad$ $x=e$.

x	0		1		e	
$\ln x$		$-$	0	$+$		$+$
$\ln x-1$		$-$		$-$	0	$+$
f''		$+$		$-$		$+$
f		$\cup$		$\cap$		$\cup$

(δ) $E = \int_1^e |f(x)|\,dx = \int_1^e \left|\dfrac{1+\ln^2 x}{x}\right| dx = \int_1^e \dfrac{1+\ln^2 x}{x}\,dx$

$\ln x = t$
$\frac{1}{x}dx = dt$

$$= \int_0^1 (1+t^2)\,dt = (t)_0^1 + \frac{1}{3}(t^3)_0^1 =$$

$$= 1 + \frac{1}{3} = \frac{4}{3}.$$

Επομενα Μαθηματα

Εχουμε δει 2016

Λυνετε

2025, 2024, 2023, 2022

2021, 2020.

Αυτα θα τα δουμε

Παρηπα και παρακατω