# Ενοτητα 24

Συνεχεί.

**13.** $f(x) = \begin{cases} x\ln x + \alpha, & x>0 \\ \eta\mu x + \ln(1+\alpha^2), & x\le 0 \end{cases}$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(\cancel{x\ln x}^{\,0} + \alpha) = \alpha$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\left[\eta\mu x + \ln(1+\alpha^2)\right] = \ln(1+\alpha^2)$

$\longrightarrow \lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to 0^+} -\dfrac{\cancel{x^2}}{\cancel{x}} = 0$

$$\boxed{\lim_{x\to 0} x\ln x = 0 \quad \Theta.\ A}$$

$$\ln(1+a^2) = a$$

$$\ln(1+a^2) - a = 0 \longrightarrow \varphi(a) = 0$$

$$\varphi(a) = \varphi(0)$$

$\varphi$ 1-1

$$\boxed{\varphi(x) = \ln(1+x^2) - x}$$

$$\varphi'(x) = \frac{2x}{1+x^2} - 1$$

$$\underline{\underline{a = 0}}$$

$$\varphi'(x) = \frac{2x - 1 - x^2}{x^2+1}$$

$$\varphi'(x) = \frac{-(x^2 - 2x + 1)}{x^2+1}$$

$$\varphi'(x) = \frac{-(x-1)^2}{x^2+1} \leq 0$$

$$\varphi \searrow$$

**16.** $f(x) = e^x - x - 3\sigma\upsilon\nu x$

ⓐ Νδο η $\dfrac{e^x - x}{\sigma\upsilon\nu x} = 3$ εχει 1 ριζα ακριβώς $(0, \frac{\pi}{2})$.

$e^x - x = 3\sigma\upsilon\nu x$

$\underbrace{e^x - x - 3\sigma\upsilon\nu x}_{f(x)} = 0$

$f(0) = e^0 - 0 - 3 = 1 - 3 = -2 < 0$

$f\left(\frac{\pi}{2}\right) = e^{\pi/2} - \frac{\pi}{2} - \cancel{3\sigma\upsilon\nu\frac{\pi}{2}}^{\,0} = e^{\frac{\pi}{2}} - \frac{\pi}{2} > 0$

• $e^x \geq x+1 \Rightarrow e^{\frac{\pi}{2}} \geq \frac{\pi}{2} + 1 \Rightarrow e^{\frac{\pi}{2}} > \frac{\pi}{2}$

$e^{\frac{\pi}{2}} - \frac{\pi}{2} > 0$

$f(0)\, f\left(\frac{\pi}{2}\right) < 0$

Bolzano $\exists\, \xi \in \left(0, \frac{\pi}{2}\right)$ τ.ω $f(\xi) = 0$.

$$f'(x) = \overset{\oplus}{e^x - 1} + 3\,\overset{\oplus}{\eta\mu x} \qquad x \in \left(0, \frac{\pi}{2}\right)$$

$\cdot\ x > 0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1 \Rightarrow e^x - 1 > 0$

$f'(x) > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

$f \nearrow \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

Άρα $\exists!\ \xi \in \left(0, \frac{\pi}{2}\right)$ τ.ω $f(\xi) = 0$

(Β)

| $x$ | $0$ | $\xi$ | $\frac{\pi}{2}$ |
|---|---|---|---|
| $f(x)$ | $\nearrow$ $-$ | $0$ | $+$ $\nearrow$ |

$$\left.\begin{aligned} \lim_{x \to \xi^-} \frac{1}{f(x)} &= -\infty \\ \lim_{x \to \xi^+} \frac{1}{f(x)} &= +\infty \end{aligned}\right\}$$ Το όριο δεν υπάρχει!

Το $\xi$ μοναδικό!

**25.** $f(x) = x^6 - 6x + 7$

ⓐ. $f'(x) = 6x^5 - 6$

$\rightarrow f'(x) = 0 \Rightarrow 6x^5 - 6 = 0$

$6x^5 = 6$

$x^5 = 1$

$x = 1$

| $x$ | | $1$ | |
|---|---|---|---|
| $f'$ | $-$ | $0$ | $+$ |
| $f$ | $\searrow$ | | $\nearrow$ |

$f(x) \geq f(1)$

$f(x) \geq 2$

ⓑ i) $\frac{1}{5} > \frac{1}{7}$

$f \downarrow$

$f(\frac{1}{5}) < f(\frac{1}{7})$

ii). $\pi > 3$

$f \uparrow$

$f(\pi) > f(3)$

ⓒ Αν $x < 1$

$f \downarrow$

$f(x) > f(1)$

$f(x) > 2$

⑧

$$1 \le x < 2$$

$$f \nearrow$$

$$f(1) \le f(x) < f(2)$$

$$2 \le f(x) < 59$$

**26.** $f(x) = x^3 - 6x^2 + 9x - 3$

(α) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$\rightarrow f'(x) = 0 \Rightarrow x^2 - 4x + 3 = 0$

$x = 1$ $\quad x = 3$

| $x$ | | 1 | 3 | |
|---|---|---|---|---|
| $f'$ | + | − | + | |
| $f$ | ↗ | ↘ | ↗ | |

(β) $e < 3$

$f \downarrow$

$f(e) > f(3)$

(γ) i) $f(ημx) = f(συνx)$

$ημx \in (-1, 1)$ $\quad f \nearrow \Rightarrow f$ 1-1

$συνx \in (-1, 1)$ $\quad ημx = συνx$

$$\eta\mu x = \eta\mu\left(\frac{\pi}{2} - x\right)$$

$x = 2k\pi + \frac{\pi}{2} - x$ ή ~~$x = 2k\pi + \pi - \frac{\pi}{2} + x$~~

Αδύνατη

$2x = 2k\pi + \frac{\pi}{2}$

$$\boxed{x = k\pi + \frac{\pi}{4}}$$

11). $f(3+x^2) < f(3+2x^2)$

$\left.\begin{array}{l} \cdot\ x^2+3 > 3 \\ \cdot\ 2x^2+3 > 3 \end{array}\right\} \forall x > 3$ η $f \nearrow$

$f \nearrow$

$3+x^2 < 3+2x^2$

$0 < x^2$

$x \in (-\infty, 0) \cup (0, +\infty)$

**27.** $f(x) = \ln x - x \quad , x>0$

(α) $f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$

$f'(x)=0 \Rightarrow \frac{1-x}{x} = 0 \Rightarrow 1-x=0$

$x=1$

| $x$ | $0$ | $1$ | |
|---|---|---|---|
| $1-x$ | $+$ | $0$ | $-$ |
| $x$ | $+$ | | $+$ |
| $f'$ | $+$ | | $-$ |
| $f$ | ↗ | | ↘ |

$f(x) \leq f(1)$

$f(x) \leq -1$

(β) $e\,\eta\mu x < e^{\eta\mu x} \quad \forall x \in (0,\pi)$

$\ln(e\,\eta\mu x) < \ln e^{\eta\mu x}$

$1 + \ln(\eta\mu x) < \eta\mu x$

$\ln(\eta\mu x) - \eta\mu x < -1$

$f(\eta\mu x) < f(1)$

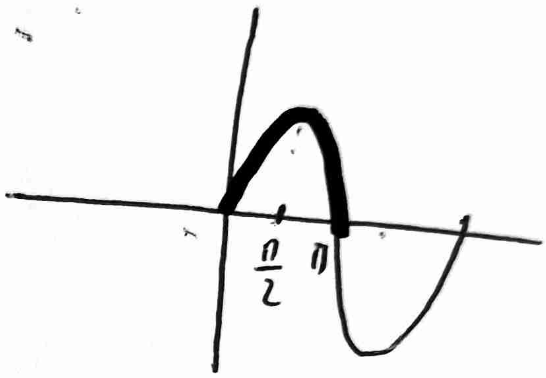
• $f(\eta\mu x) < f(1)$

$f \nearrow \quad \forall x \leq 1$

$\eta\mu x \in (-1, 1)$

$\eta\mu x < 1$

$x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$




(γ) $\ln \dfrac{x^2+3}{2x^2+1} = 2 - x^2$

$\ln(x^2+3) - \ln(2x^2+1) = 2 - x^2$

$\ln(x^2+3) - (x^2+3) = \ln(2x^2+1) - (2x^2+1)$

$f(x^2+3) = f(2x^2+1)$.

$$\left.\begin{array}{l} \circ\ x^2+3>3 \\ \circ\ 2x^2+1>1 \end{array}\right\} \forall x>1 \text{ η } f \downarrow$$

$$f(x^2+3) = f(2x^2+1)$$

$f$ 1-1

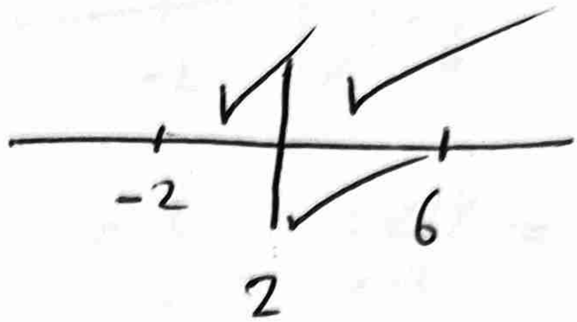
$$x^2+3 = 2x^2+1$$

$$2 = x^2$$

$$x = \pm\sqrt{2}$$

1Π

**16.** (a) $f(x) = \begin{cases} x^2-6x+3, & x>2 \\ -2x-1, & x\leq 2 \end{cases}$




Η $f$ συνεχής $[-2,2)$ και $(2,6]$ ως π.σ.σ

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} -2x-1 = -5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2-6x+3 = -5$$

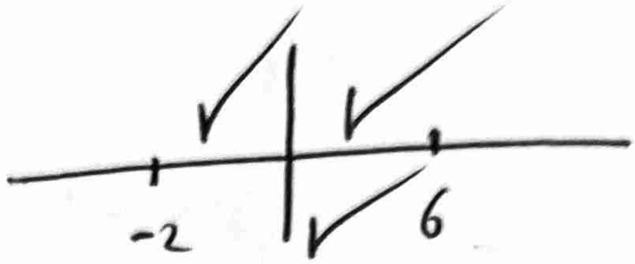
$\lim_{x\to 2} f(x) = -5$

$f(2) = -5$

Συνεχής στο 2!

**Συνεχής στο $[-2,6]$**

Η f παρ/μη $(-2,2)$ και $(2,6)$ ως π.π.σ.




$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{-2x-1+5}{x-2}=\lim_{x\to 2^-}\frac{-2x+4}{x-2}=$$

$$=\lim_{x\to 2^-}\frac{-2}{1}=-2,$$

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{x^2-6x+3+5}{x-2}=\lim_{x\to 2^+}\frac{x^2-6x+8}{x-2}$$

$$=\lim_{x\to 2^+}\frac{2x-6}{1}=-2$$

Η f παρ/μη στο 2.

Η f παρ/μη $(-2,6)$

$$\left.\begin{array}{l} f(-2)=3 \\ \\ f(6)=3 \end{array}\right\} f(-2)=f(6)$$

Ικανοποιούνται οι προυποθέσεις του Rolle στο $[-2,6]$.

$\exists\ \xi \in (-2,6)$ τ.ω $f'(\xi)=0$.

| $x<2$ | $x>2$ |
| --- | --- |
| $f'(x)=0$ | $f'(x)=0$ |
| $-2=0$ | $2x-6=0$ |
| Άτοπο | $x=3$ |

(B) $f(x) = 1 + ημ3x$ $\quad [0,π]$.

H $f$ συνεχής $[0,π]$ ως π.σ.σ

H $f$ παραγ/μη $(0,π)$ ως π.π.σ.

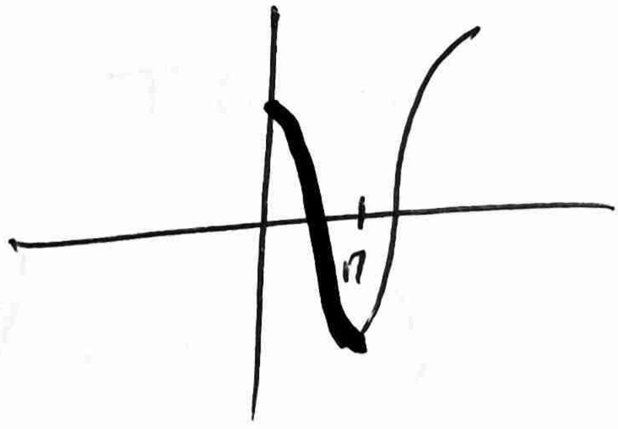
$$\left.\begin{array}{l} f(0)=1 \\ f(π)=1 \end{array}\right\} \text{Rolle } \exists ξ \in (0,π)$$

τ.ω. $f'(ξ)=0$.

$f'(x) = συν3x \cdot 3$

$f'(ξ)=0 \implies 3συν3ξ = 0$

$συν3ξ = 0$




$3ξ = \frac{π}{2}$

$ξ = \frac{π}{6}$

**7.** $f : [0,\pi] \to \mathbb{R}$ συνεχής $[0,\pi]$
παρ/μη $(0,\pi)$.

Rolle $[0,\pi]$.

(α) $g(x) = f(x)\,ημx$

$$\left.\begin{aligned} g(0) &= f(0)\,ημ0 = 0 \\ g(\pi) &= f(\pi)\,ημ\pi = 0 \end{aligned}\right\} \checkmark \quad g(0) = g(\pi)$$

→ Η $g(x)$ συνεχής $[0,\pi]$ ως π.σ.σ

→ Η $g(x)$ παρ/μη $(0,\pi)$ ως π.π.σ.

Ικανοποιείται ο Rolle για την $g(x)$ στο $[0,\pi]$

$\exists\, \xi \in (0,\pi)$ τ.ω

$$g'(\xi) = 0.$$

(β). Νδο η εξισωση

$$f'(x) = -f(x)\,\sigma\varphi x \quad \text{εχει}$$

τουλ. 1 ριζα $(0,\pi)$.

$$\boxed{\begin{array}{c} \exists\, \xi \in (0,\pi) \ \text{τ.ω} \ g'(\xi) = 0 \\ \checkmark \ \text{Rolle} \ \checkmark \end{array}}$$

$$g'(x) = f'(x)\,\eta\mu x + f(x)\,\sigma\upsilon\nu x$$

$$g'(\xi) = f'(\xi)\,\eta\mu \xi + f(\xi)\,\sigma\upsilon\nu \xi = 0$$

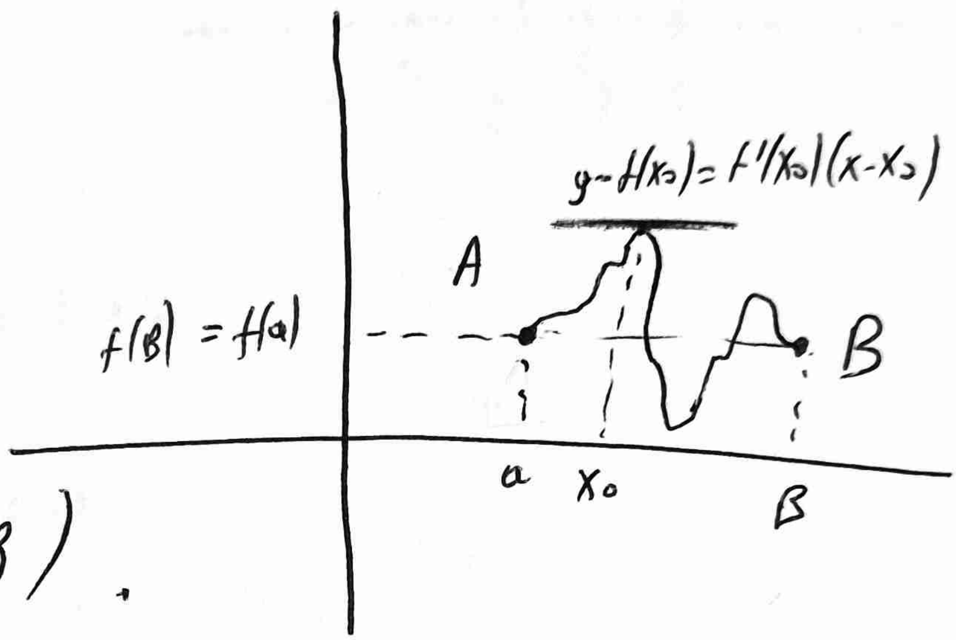
$\xi \in (0,\pi)$

$$f'(\xi) + f(\xi)\frac{\sigma\upsilon\nu\xi}{\eta\mu\xi} = 0$$

$$f'(\xi) + f(\xi)\,\sigma\varphi\xi = 0$$

$$\checkmark \quad f'(\xi) = -f(\xi)\,\sigma\varphi\xi \quad \checkmark$$

# Το Θεωρημα του Rolle.



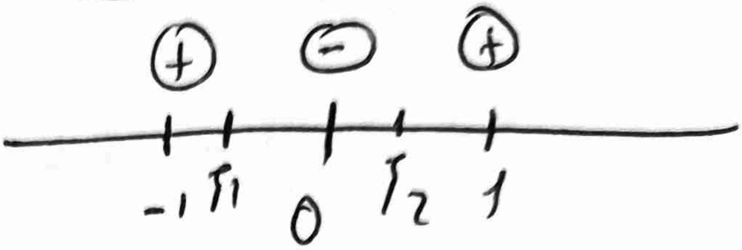

- $f(a)=f(B)$.
- $f$ συνεχης στο $[a,B]$
- $f$ παρ/μη στο $(a,B)$

$\exists\, x_0 \in (a,B)$ τ.ω $f'(x_0)=0$.

**11.** (a) ΝΔΟ $2x^4+\lambda^2(x^2-1)=x$ , $\lambda\neq 0$

$$\boxed{f(x)=2x^4+\lambda^2(x^2-1)-x}$$

$(-1,1)$




$f(-1)=2+1=3>0$

$f(0)=-\lambda^2<0$

$f(1)=1>0$

$f(-1)f(0)<0$ Bolzano $\exists\, \xi_1\in(-1,0)$ τω $f(\xi_1)=0$

$f(0)f(1)<0$ Bolzano $\exists\, \xi_2\in(0,1)$ τ.ω $f(\xi_2)=0.$

(B) Νδο $8x^3+2\lambda^2 x=1$ $(-1,1)$

$$\boxed{\varphi(x)=8x^3+2\lambda^2 x-1}$$

$\varphi(-1)=-8-2\lambda^2-1<0$

$\varphi(1)=8+2\lambda^2-1=7+2\lambda^2>0$

$\varphi(-1)\varphi(1)<0$ Bolzano $\exists \rho\in(-1,1)$

τ.ω $\varphi(\rho)=0$.

β' τρόπος

$\underbrace{f(\xi_1)=f(\xi_2)}=0$

Rolle $\exists \xi\in(\xi_1,\xi_2)$

τ.ω $f'(\xi)=0$

$f'(x)=8x^3+2\lambda^2 x-1$

$f'(\xi)=8\xi^3+2\lambda^2\xi-1=0$ ✓

# Επομαο Μαθημα

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