Ενότητα 26

2. ⓐ $f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x$

$f'(x) = 3x(x-2)$

$\rightarrow f'(x) = 0 \Rightarrow 3x(x-2) = 0$

$3x = 0$ ή $x - 2 = 0$

$x = 0$ $x = 2$

x		0		2	
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

$A(0, f(0))$ T.M

$B(2, f(2))$ T.E.

2. (β) $f(x) = x^3 - 6x^2 + 9x - 2$

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 3(x^2 - 4x + 3)$

x		1		3	
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

$A(1, f(1))$ T.M

$f'(x) = 0 \Rightarrow x^2 - 4x + 3 = 0$

$x = 1$ $\quad \boxed{x = 3}$

$B(3, f(3))$ T.E.

(γ) $f(x) = 3x^4 - 4x^3 + 1$

$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$

$\longrightarrow f'(x) = 0 \Rightarrow 12x^2 = 0$ ή $x - 1 = 0$

$x = 0$ $\qquad x = 1$

x		0		1	
$12x^2$	$+$	0	$+$		$+$
$x-1$	$-$		$-$	0	$+$
f'	$-$		$-$		$+$
f	$\searrow$		$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$f(x) \geq 0$

$A(1, 0)$ Ο.Ε

3. (γ) $f(x) = x - 2\sqrt{x}$, $x \geq 0$

$$f'(x) = 1 - \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x}} = \frac{\sqrt{x}-1}{\sqrt{x}}$$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{\sqrt{x}-1}{\sqrt{x}} = 0 \Rightarrow \sqrt{x} = 1$

$x = 1$

x	0	1	$+\infty$
$\sqrt{x}-1$	$-$	$+$	
$\sqrt{x}$	$+$	$+$	
f'	$-$	$+$	
f	$\searrow$	$\nearrow$	

$A(1,-1)$

$f(x) \geq f(1)$

$f(x) \geq -1$

Ο.Ε.

3. B) $f(x) = \dfrac{x}{x^2+9}$ $D_f = \mathbb{R}$

$$f'(x) = \frac{x^2+9 - x \cdot 2x}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{9-x^2}{(x^2+9)^2} = 0 \Rightarrow 9-x^2 = 0$

$x = 3$ $x = -3$

x		-3		3	
$9-x^2$	$-$		$+$		$-$
$(x^2+9)^2$	$+$		$+$		$+$
f'	$-$		$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$

$A(-3, f(-3))$ T. E

$B(3, f(3))$ T. M.

3. ⓐ $f(x) = x + \dfrac{1}{x-1}$, $x \neq 1$

$$f'(x) = 1 + \frac{-1}{(x-1)^2} = \frac{(x-1)^2 - 1}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{x^2 - 2x}{(x-1)^2}}$$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{x^2-2x}{(x-1)^2} = 0 \Rightarrow x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ $x = 2$.

x		0		1		2	
x^2-2x	$+$		$-$	$\|\|$	$-$		$+$
$(x-1)^2$	$+$		$+$	$\|\|$	$+$		$+$
f'	$+$		$-$	$\|\|$	$-$		$+$
f	$\nearrow$		$\searrow$	$\|\|$	$\searrow$		$\nearrow$

$A(0, f(0))$ T.M

$B(2, f(2))$ T. E

4. ⓐ $f(x) = x^2 e^{1-x}$, $x \in \mathbb{R}$

$$f'(x) = 2x\, e^{1-x} - x^2\, e^{1-x}$$

$$\boxed{f'(x) = e^{1-x}\left(2x - x^2\right)}$$

$\longrightarrow$ $f'(x) = 0 \Rightarrow 2x - x^2 = 0$

$$x(2-x) = 0$$

$x = 0$ $\qquad$ $2 - x = 0$

$x = 2$

x		0	2
e^{1-x}	$+$	$+$	$+$
$2x - x^2$	$-$	$+$	$-$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

$A(0, f(0))$ / T. E.

$B(2, f(2))$ / T. M.

4. (B) $f(x) = (x-1)\ln x \quad , x > 0$

$$f'(x) = \ln x + (x-1)\frac{1}{x}$$

<u>α' τρόπος</u>

$$f'(x) = \frac{x\ln x + x - 1}{x}$$

$g(x) = x\ln x + x - 1 \qquad g(1) = 0$

$g'(x) = \ln x + 1 + 1$

$g'(x) = \ln x + 2$

$\to g'(x) = 0 \Rightarrow \ln x + 2 = 0$

$\ln x = -2$

$x = e^{-2}$

x	0	e^{-2}	1
g'	$-$	0 $+$	$+$
g	$-$ $\searrow$ a	$-$ $\nearrow$ 0	$\nearrow$ $+$
x	$+$	$+$	$+$
f'	$-$	$-$	$+$
f	$\searrow$	$\searrow$	$\nearrow$

$g(x) \geq g(e^{-2})$

$g(x) \geq e^{-2}(-2) + e^{-2} - 1$

$g(x) \geq -e^{-2} - 1$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \overset{0}{\cancel{x\ln x}} + x - 1 = -1$$

$$\to \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

Β' τρόπος

$$f'(x) = \ln x + (x-1)\frac{1}{x}$$

$$f'(x) = \ln x + 1 - \frac{1}{x} \qquad f'(1) = 0$$

$$f''(x) = \frac{1}{x} + \frac{1}{x^2} > 0$$

x	0		1	
f''		$+$		$+$
f'		$-$	0	$+$
f		↘		↗

4. (γ) $f(x) = \ln(x+1) - e^x + 2 \quad , x > -1$

$$f'(x) = \frac{1}{x+1} - e^x \quad , \quad f'(0) = 0$$

$$f''(x) = -\frac{1}{(x+1)^2} - e^x < 0$$

x	-1	0
f''	$-$	$-$
f'	$\searrow$ $+$	0 $\searrow$ $-$
f	$\nearrow$	$\searrow$

$x < 0 \Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 0$

$x > 0 \Rightarrow f'(x) < f'(0) \Rightarrow f'(x) < 0.$

(δ) $f(x) = \dfrac{\ln x}{x^2} \quad , x > 0$

$$f'(x) = \frac{\frac{1}{x} x^2 - \ln x \cdot 2x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

$\to f'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow 1 = 2\ln x \Rightarrow 1 = \ln x^2$

$x^2 = e$

$x = \pm\sqrt{e}$

x	0	$\sqrt{e}$
$1 - 2\ln x$	$+$	$-$
x^3	$+$	$+$
f'	$+$	$-$
f	$\nearrow$	$\searrow$

5. ⓐ $f(x) = x^2 + x\cos x - \sin x$

$$f'(x) = 2x + \cancel{\cos x} - x\sin x - \cancel{\cos x}$$

$$f'(x) = x\,(\underset{+}{2 - \sin x})$$

$\rightarrow f'(x) = 0 \Rightarrow x = 0$

x		0	
x	$-$		$+$
$2-\sin x$	$+$		$+$
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

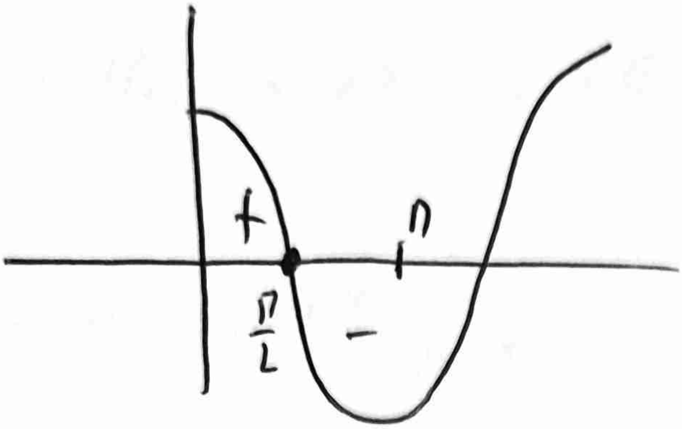
$\underline{\underline{f(x) \geq 0}}$

ⓑ . $f(x) = \dfrac{\sin x - \cos x}{e^x}$, $x \in [0, \pi]$

$$f'(x) = \frac{(\cos x + \sin x)e^x - (\sin x - \cos x)e^x}{e^{2x}}$$

$$f'(x) = \frac{\cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x}{e^x} = \frac{2\cos x}{e^x}$$

$f'(x) = 0 \Rightarrow \dfrac{2\cos x}{e^x} = 0 \Rightarrow 2\cos x = 0.$



x	0	$\frac{\pi}{2}$	π
$2\sigma\upsilon\nu x$	$+$		$-$
e^x	$+$		$+$
f'	$+$		$-$
f	$\nearrow$		$\searrow$

(γ) $f(x) = x^2 - \sigma\upsilon\nu x$

$f'(x) = 2x + \eta\mu x$. $\qquad f'(0) = 0$.

$f''(x) = 2 + \sigma\upsilon\nu x > 0$

x		0	
f''	$+$		$+$
f'	$\nearrow -$	0	$+ \nearrow$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

$f(x) \geq -1$

(δ) $f(x) = x\,\varepsilon\varphi x \quad , \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

$$f'(x) = \varepsilon\varphi x + x \cdot \frac{1}{\sigma\upsilon\nu^2 x}$$

$$f'(x) = \frac{\eta\mu x}{\sigma\upsilon\nu x} + \frac{x}{\sigma\upsilon\nu^2 x}$$

$$\boxed{f'(x) = \frac{\eta\mu x\,\sigma\upsilon\nu x + x}{\sigma\upsilon\nu^2 x}}$$

$$\boxed{\varphi(x) = \eta\mu x\,\sigma\upsilon\nu x + x \qquad \varphi(0)=0}$$

$$\varphi'(x) = \sigma\upsilon\nu^2 x - \eta\mu^2 x + 1$$

$$\varphi'(x) = \sigma\upsilon\nu^2 x + \sigma\upsilon\nu^2 x > 0$$

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$
φ'	$+$		$+$
φ	$\nearrow$ $-$	0	$+$ $\nearrow$
$\sigma\upsilon\nu^2 x$	$+$		$+$
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$$f(x) \geq f(0)$$

$$f(x) > 0$$

6. (a) $f(x) = \begin{cases} e^{x+1} - x, & x \leq 0 \\ x^3 + e, & x > 0 \end{cases}$

$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} e^{x+1} - x = e \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x^3 + e = e \end{aligned}\right\} \lim_{x \to 0} f(x) = e$$

$f(0) = e$

Συνεχ στο 0!

$\underline{x < 0}$

$f_1(x) = e^{x+1} - x$

$f_1'(x) = e^{x+1} - 1$

$\rightarrow e^{x+1} - 1 = 0$

$e^{x+1} = 1$

$x + 1 = 0$

$\boxed{x = -1}$

$\underline{x > 0}$

$f_2(x) = x^3 + e$

$f_2'(x) = 3x^2 \geq 0$

x		-1		0	
f_1'	$-$	0	$+$		////////
f_2'	////////		////////		$+$
f'	$-$		$+$		$+$
f	↘		↗		↗

$f(x) \geq f(-1)$

$f(x) \geq 2$

(B) $f(x) = \begin{cases} ex - e^x + 1, & 0 \leq x \leq 1 \\ x - 2\ln x, & x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ex - e^x + 1 = 1$

$\lim_{x\to 1^+} f(x) = 1$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \lim_{x\to 1} f(x) = 1$

$f(1) = 1$ ✓

$0 \leq x < 1$

$f_1(x) = ex - e^x + 1$

$f_1'(x) = e - e^x$

$\to e - e^x = 0$

$e = e^x$

$\underline{\underline{x = 1}}$

$x > 1$

$f_2(x) = x - 2\ln x$

$f_2'(x) = 1 - \frac{2}{x}$

$f_2'(x) = \frac{x-2}{x}$

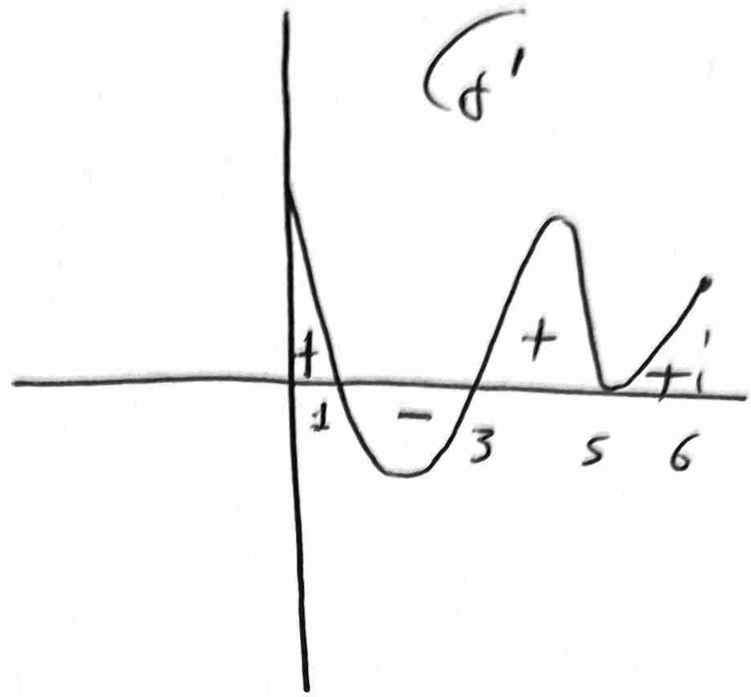
$\to f_2'(x) = 0$

$x - 2 = 0$

$x = 2$

x		1		2	
f_1'	+		//////		//////
f_2'	//////		−	0	+
f'	+		−		+
f	↗		↘		↗

7.



x		1	3	5
f'	$+$	$-$	$+$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$

9. (a) $e^{x^2} + \ln(x^2+1) = 1$

Θετω $x^2 = u$

$\underbrace{e^u + \ln(u+1) - 1}_{f(u)} = 0$ $\quad D_f = (-1, +\infty)$

$f(u) = 0$

$f'(u) = e^u + \dfrac{1}{u+1} > 0$ $\quad f(u) = f(0)$

f 1-1

$f \nearrow$

$u = 0$

$x^2 = 0$

$x = 0$

(B) $\ln \dfrac{x^2+1}{x+1} > e^x - e^{x^2}$

$\ln(x^2+1) - \ln(x+1) > e^x - e^{x^2}$

$\ln(x^2+1) + e^{x^2} > e^x + \ln(x+1)$

$\boxed{f(x) = \ln(x+1) + e^x}$

$f(x^2) > f(x)$

$$x^2 > x$$

$$x^2 - x > 0$$

x		0	1	
$x^2 - x$	$+$	$-$	$+$	

$$x \in (-\infty, 0) \cup (1, +\infty)$$

(8) $$e^{x^2-1} + \ln x = e^{x-1}$$

$$\boxed{f(x) = e^x + \ln(x+1) - 1}$$

$$e^{x^2-1} + \ln \frac{x^2}{x} = e^{x-1}$$

$$e^{x^2-1} + \ln x^2 - \ln x = e^{x-1}$$

$$e^{x^2-1} + \ln x^2 = e^{x-1} + \ln x$$

$$f(x^2-1) = f(x-1)$$

$$f \circ 1-1$$

$$x^2 \cancel{-1} = x \cancel{-1} \longrightarrow x^2 = x$$

$$x(x-1) = 0$$

$$x = 0 \quad x = 1$$

10. (α) $f(x) = e^x + x + \sigma\upsilon\nu x$

$f'(x) = e^x + \underbrace{1 - \eta\mu x}_{\oplus} > 0 \quad f \nearrow$

(Β) i) ν.δ.ο. $f(\varepsilon\varphi x) - e^x > x + \sigma\upsilon\nu x \quad \forall x \in (0, \frac{\pi}{2})$

$$f(\varepsilon\varphi x) > x + e^x + \sigma\upsilon\nu x$$

$$f(\varepsilon\varphi x) \overset{f\nearrow}{>} f(x)$$

$$\boxed{\text{Αρκεί ν.δ.ο. } \varepsilon\varphi x > x. \quad \forall x \in [0, \tfrac{\pi}{2})}$$

$$\underbrace{\varepsilon\varphi x - x}_{\varphi(x)} > 0$$

$$\varphi'(x) = \frac{1}{\sigma\upsilon\nu^2 x} - 1 = \frac{1 - \sigma\upsilon\nu^2 x}{\sigma\upsilon\nu^2 x} = \frac{\eta\mu^2 x}{\sigma\upsilon\nu^2 x} > 0$$

$\varphi \nearrow$

$\forall x \geq 0 \Rightarrow \varphi(x) \geq \varphi(0) \Rightarrow \underline{\underline{\varepsilon\varphi x - x \geq 0}}$

11). $f(e^{x-1}-1) > f(\ln x^{x}) \quad \forall x > 1$

$f \nearrow$

$$e^{x-1}-1 > \ln x^{x}$$

$$e^{x-1}-1 > x\ln x.$$

$$\boxed{e^{x-1}-1-x\ln x > 0}$$

$\varphi(x) = e^{x-1}-1-x\ln x$

$\varphi'(x) = e^{x-1}-\ln x-1 \qquad \varphi'(1)=0$

$\varphi''(x) = e^{x-1}-\dfrac{1}{x} \qquad \varphi''(1)=0$

$\varphi'''(x) = e^{x-1}+\dfrac{1}{x^2} > 0,$

x	0		1	
φ'''		$+$		$+$
φ''		$\nearrow$ $-$	0	$+$ $\nearrow$
φ'		$\searrow$ $+$	0	$+$ $\nearrow$
φ		$+$		$+$

$$\forall x > 1 \Rightarrow \varphi(x) > \varphi(1)$$

$$\underline{e^{x-1} - 1 - x\ln x > 0}$$

iii). $f(e^x) > f(1+\eta\mu x) \quad \forall x>0$

$f \nearrow$

$$e^x > 1+\eta\mu x$$

$$\underbrace{e^x - 1 - \eta\mu x}_{g(x)} > 0 ,$$

$$g'(x) = e^x - \sigma\upsilon\nu x$$

$$g'(x) = \underbrace{e^x - 1}_{+} + \underbrace{1 - \sigma\upsilon\nu x}_{+} > 0 \qquad g \nearrow$$

$$x>0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1 \Rightarrow e^x - 1 > 0$$

$$\forall x>0 \Rightarrow g(x) > g(0) \Rightarrow e^x - 1 - \eta\mu x > 0$$

14. (1) $f(x) = \ln x$

$g(x) = x - \frac{1}{x}$

Έστω $f(x) < g(x) \Rightarrow \ln x < x - \frac{1}{x}$

$$\underbrace{\ln x - x + \frac{1}{x}}_{\varphi(x)} < 0$$

$\varphi(x) < 0$

$\varphi(x) < \varphi(1)$

$\varphi \downarrow$

$x > 1$

$$\varphi'(x) = \frac{1}{x} - 1 - \frac{1}{x^2}$$

$$\varphi'(x) = \frac{x - x^2 - 1}{x^2}$$

$$\varphi'(x) = \frac{\overset{\ominus}{-x^2 + x - 1}}{\underset{\oplus}{x^2}} < 0$$

$\varphi \downarrow$

Συμπέρασμα

Αν $x > 1$ C_f κάτω C_g

Αν $x < 1$ C_f πάνω C_g

Αν $x = 1$ τέμνονται.

14. (α) $f(x) = x^2 + 3\ln x$

$g(x) = 1 + \ln x$

Έστω $f(x) < g(x) \Rightarrow x^2 + 3\ln x < 1 + \ln x$

$$\underbrace{x^2 + 2\ln x - 1}_{h(x)} < 0$$

$h'(x) = 2x + \frac{2}{x} > 0$

$h \nearrow$

$h(x) < 0$

$h(x) < h(1)$

$h \nearrow$

$x < 1$

Συμπέρασμα

$x < 1$ και C_f κάτω C_g

$x > 1$ και C_f πάνω C_g.

$x = 1$ Τέμνονται.

15. (α) $x^3+x=\sigma\upsilon\nu x$ $\quad\Delta=(0,\frac{\pi}{2})$.

$$\underbrace{x^3+x-\sigma\upsilon\nu x}_{f(x)}=0$$

$$\left.\begin{aligned} f(0)&=-1<0\\ f(\tfrac{\pi}{2})&=\frac{\pi^3}{8}+\frac{\pi}{2}>0\end{aligned}\right\}\ f(0)\,f(\tfrac{\pi}{2})<0$$

Η f συνεχής $[0,\frac{\pi}{2}]$ ως π.σ.σ.

Bolzano $\exists\,\xi\in(0,\frac{\pi}{2})$ τ.ω $f(\xi)=0$.

$$f'(x)=3x^2+\underbrace{1+\eta\mu x}_{+}>0$$

$f\nearrow$

$$\left.\text{Το}\right\}\ \xi \text{ μοναδικό.}$$

(γ) $e^x = x^4$ $\quad \Delta = (-1,0)$.

$$\underbrace{e^x - x^4}_{f(x)} = 0$$

$$\left.\begin{array}{l} f(-1) = \frac{1}{e} - 1 < 0 \\ f(0) = e^0 = 1 > 0 \end{array}\right\} f(-1)f(0) < 0$$

f συνεχής στο $[-1,0]$ ως π.σ.σ.

Bolzano $\exists x_0 \in (-1,0)$ τ.ω. $f(x_0) = 0$.

$$f'(x) = e^x \underbrace{-4x^3}_{+} > 0 \quad \forall x \in (-1,0)$$

f ↗

- $-1 < x < 0$

$-1 < x^3 < 0$

$4 > -4x^3 > 0$

x_0 μοναδική.

Επομενο Μαθημα

24

16

13

25

26

27