22.

Ενοτητα 27

$f(x) = x^4 - 6x^2 + 4x$

$\varepsilon : y = 8x$

$y - f(x_0) = f'(x_0)(x - x_0) \parallel y = 8x$

Εξίσωση $f'(x) = 8$

$4x^3 - 12x + 4 = 8$

$4x^3 - 12x - 4 = 0$

$x^3 - 3x - 1 = 0$

$g(x) = x^3 - 3x - 1$

$g'(x) = 3x^2 - 3 = 3(x^2 - 1)$

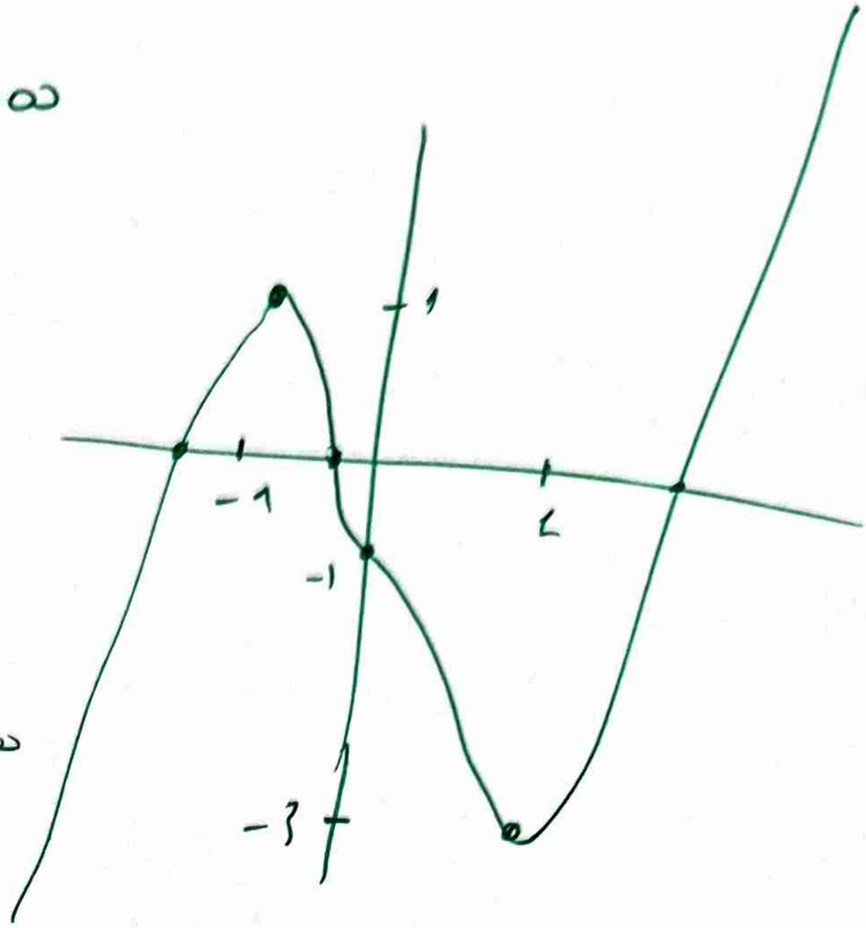
x		-1		1	
g'	+		-		+
g	↗		↘		↗

$\lim_{x\to-\infty} g(x) = -\infty$

$g(-1) = 1$

$g(1) = -3$

$\lim_{x\to+\infty} g(x) = +\infty$



$x < -1$

- $g \nearrow$
- g συνεχής
- $\Sigma T_g = (-\infty, 1]$

Το $0 \in \Sigma T_g$

άρα $\exists! \, \xi_1$ τ.ω

$g(\xi_1) = 0$

$-1 \le x \le 1$

- $g \searrow$

g συνεχής

$\Sigma T_g = [-3, 1]$

Το $0 \in \Sigma T_g$

άρα $\exists! \, \xi_2$

τ.ω $g(\xi_2) = 0$

$x > 1$

- $g \nearrow$

g συνεχής

$\Sigma T_g = [-3, +\infty)$

Το $0 \in \Sigma T_g$

άρα $\exists!$

ξ_3 τ.ω

$g(\xi_3) = 0$

52. $f(x) = e^{x-1} - \ln x$, $x > 0$

(a) $f'(x) = e^{x-1} - \dfrac{1}{x}$, $f'(1) = 0$

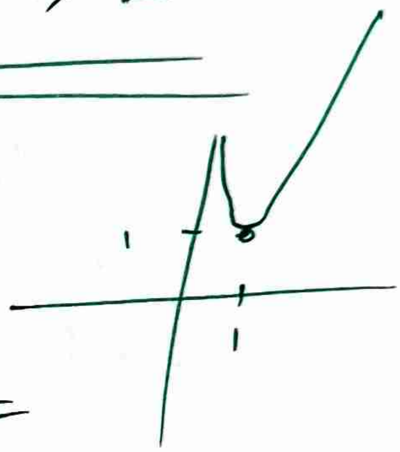
$f''(x) = e^{x-1} + \dfrac{1}{x^2} > 0$

x	0		1	
f''		$+$		$+$
f'		$\nearrow$ $-$	0	$+$ $\nearrow$
f	$+\infty$	$\searrow$	1	$\nearrow$ $+\infty$

$f(x) \geq f(1)$

$f(x) \geq 1$

(b) $\lim\limits_{x\to 0^+} f(x) = +\infty$



$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} e^{x-1} - \ln x =$

$\Sigma T_f = [1, +\infty)$

$= \lim\limits_{x\to+\infty} e^{x-1}\left(1 - \dfrac{\ln x}{e^{x-1}}\right) = +\infty$

$\to \lim\limits_{x\to+\infty} \dfrac{\ln x}{e^{x-1}} = \lim\limits_{x\to+\infty} \dfrac{1}{x e^{x-1}} = 0$

(γ) Νδο $\dfrac{f(\alpha)-1}{x} + \dfrac{f\left(e^{\beta-1}-\ln\beta\right)-1}{x-1} = 0$

έχει ρίζα μοναδική $(0,1)$

$$\varphi(x) = (x-1)\left(f(\alpha)-1\right) + x\left(f\left(e^{\beta-1}-\ln\beta\right)-1\right)$$

$$\varphi(0) = -\left(f(\alpha)-1\right) = 1-f(\alpha) < 0$$

$$\varphi(1) = f\left(e^{\beta-1}-\ln\beta\right)-1 \;;\; 0$$

$$f\left(f(\beta)\right)-1 \qquad .\, f(\beta) \geq 1$$

Αφού $f(x) \geq 1 \quad \forall x > 0$

$$f(\alpha) > 1 \Rightarrow 1-f(\alpha) < 0$$

$$f\left(e^{\beta-1}-\ln\beta\right) > 1 \Rightarrow f\left(e^{\beta-1}-\ln\beta\right)-1 > 0$$

$\varphi(0)\varphi(1) < 0$ Bolzano $\exists \xi$ τω $\varphi(\xi) = 0$

Η φ(x) ειναι 1ου βαθμου πολυωνυμο αρα εχει 1 ριζα το πολυ, βρηκα ηδη 1 αρα ειναι και μοναδικη.

(δ) $$\lim_{x \to 1} \frac{\ln x}{x-1} \cdot \frac{1}{f(x)-1} = 1\,(+\infty) = +\infty$$

$$\to \lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{1}{x} = 1$$

31. $f(x) = x \ln(x+1) + 1 \quad , x > -1$

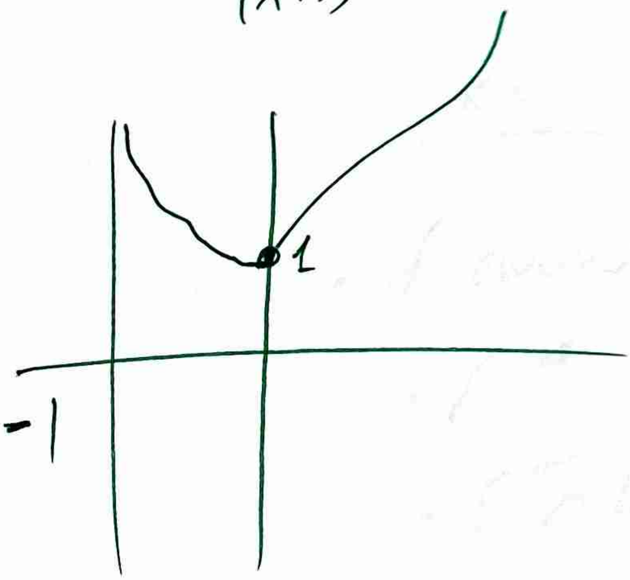
ⓐ $f'(x) = \ln(x+1) + x \dfrac{1}{x+1}$

$f'(x) = \ln(x+1) + \dfrac{x}{x+1}$ $\qquad f'(0) = 0$

$f''(x) = \dfrac{1}{x+1} + \dfrac{x+1-x}{(x+1)^2}$

$f''(x) = \dfrac{x+1+1}{(x+1)^2} = \dfrac{x+2}{(x+1)^2} > 0$

x	-1		0		$+\infty$
f''		$+$		$+$	
f'		$\nearrow$ $-$	0	$+$ $\nearrow$	
f	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$



$\lim\limits_{x \to -1^+} f(x) = +\infty$

$f(0) = 1$

$\lim\limits_{x \to +\infty} f(x) = +\infty$

$\Sigma T_f = [1, +\infty)$.

$f(x) \geq 1$.

(B) Νδο $(x+1)^x = e$ $(-1,+\infty)$.

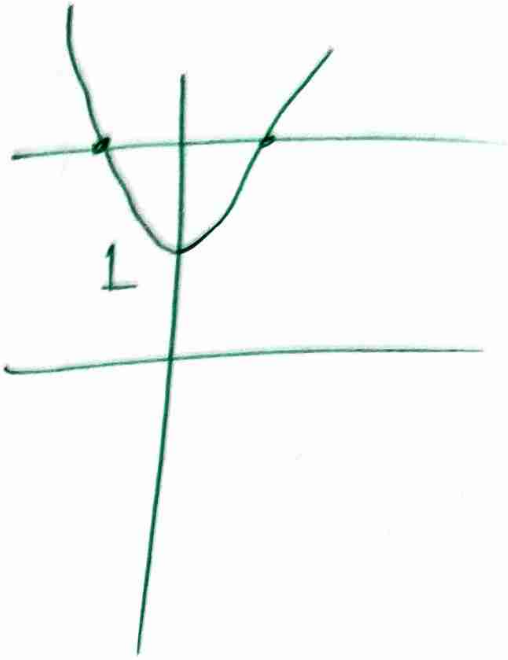
Εχει 2 ρίζα ετεροσημα:

$\ln (x+1)^x = \ln e$

$x \ln(x+1) = 1$

$x \ln(x+1) + 1 = 2$

$f(x) = 2.$



$x < 0$

- f συνεχ.
- $f \downarrow$
- $\Sigma T_f = [1,+\infty)$

Το $2 \in \Sigma T_f$.

άρα $\exists!\ \xi_1 < 0$

τ.ω $f(\xi_1) = 2$

$x \geq 0$

- f συνεχ.
- $f \uparrow$
- $\Sigma T_f = [1,+\infty)$

Το $2 \in \Sigma T_f$

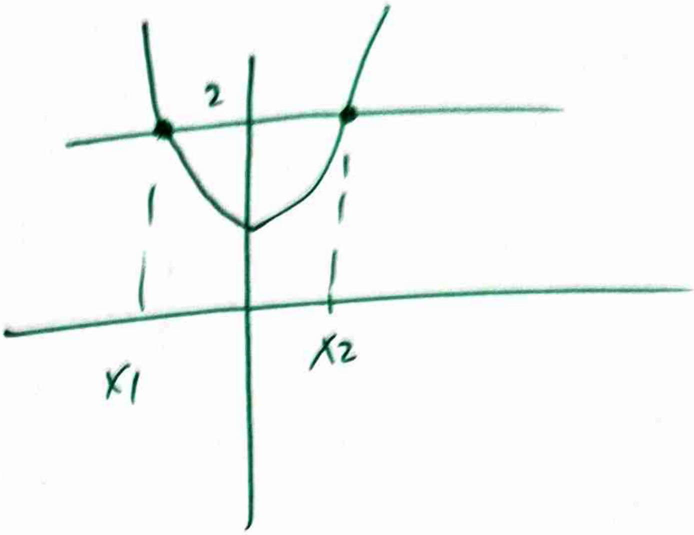
άρα $\exists!\ \xi_2 > 0$.

τ.ω $f(\xi_2) = 2$

① Ν.δ.ο. $\exists!\, x_0 \in (0, x_2)$ τ.ω. $f'(x_0) + f(x_0) =$

$f(x_1) = 2$

$f(x_2) = 2$



$$\boxed{\varphi(x) = f'(x) + f(x) - 2}$$ $(0, x_2)$

$\varphi(0) = f'(0) + f(0) - 2 = 0 + 1 - 2 = -1$

$\varphi(x_2) = \underbrace{f'(x_2)}_{+} + \cancel{f(x_2) - 2} = f'(x_2)$

$\varphi(x_2) > 0$

$\varphi(0)\varphi(x_2) < 0$ Bolzano $\exists\, \xi \in (0, x_2)$

τω $\varphi(\xi) = 0$

$\varphi'(x) = \underbrace{f''(x)}_{+} + \underbrace{f'(x)}_{+} > 0$ $\varphi \nearrow$

27. (α) $f(x) = e^x + x^2$

$f'(x) = e^x + 2x$

$f''(x) = e^x + 2$

x	$-\infty$ $\qquad$ $+\infty$
f''	$+$
f'	$-\infty \nearrow$
f	

$\Sigma T_{f'}$

- f' συνεχ.
- $f' \nearrow$
- $\Sigma T_{f'} = \mathbb{R}$

$\lim_{x \to -\infty} f'(x) = -\infty$

$\lim_{x \to +\infty} f'(x) = +\infty$

Το $0 \in \Sigma T_{f'}$

άρα $\exists!\, x_0$ τ.ω

$f'(x_0) = 0$

x		x_0	
f''	$+$		$+$
f'	$\nearrow -$	0	$+ \nearrow$
f	$\searrow$		$\nearrow$

(β) i) Έστω $x_0 \geq -\frac{1}{3}$

$f'(x_0) \geq f'(-\frac{1}{3})$

$0 \geq e^{-\frac{1}{3}} + 2(-\frac{1}{3})$

$$0 \geqslant e^{-\frac{1}{3}} - \frac{2}{3}$$

$$\frac{2}{3} \geqslant e^{-\frac{1}{3}}$$

$$\frac{2}{3} \geqslant \frac{1}{e^{1/3}}$$

$$\frac{2}{3} \geqslant \frac{1}{\sqrt[3]{e}}$$

$$\left(\frac{2}{3}\right)^3 \geqslant \left(\frac{1}{\sqrt[3]{e}}\right)^3$$

$$\frac{8}{27} \geqslant \frac{1}{e}$$

$$8e \geqslant 27$$

Άτοπο!

Άρα $x < -\frac{1}{3}$.

iii) νδο $f(x_0) = x_0^2 - 2x_0$

Γνωρίζω $f(x_0) = e^{x_0} + x_0^2$

$f'(x_0) = 0$

$e^{x_0} + 2x_0 = 0$

$e^{x_0} = -2x_0$

$f(x_0) = -2x_0 + x_0^2$

$f(x_0) = x_0^2 - 2x_0$

25. $f(x) = (x+1)\ln(x+1)$

$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(0,-1)$

$-1 - f(x) = f'(x)(-x)$

$-1 - f(x) = -x f'(x)$

$-1 - (x+1)\ln(x+1) = -x\left[\ln(x+1) + 1\right]$

$-1 - \cancel{x\ln(x+1)} - \ln(x+1) = \cancel{-x\ln(x+1)} - x$

$-1 - \ln(x+1) = -x$

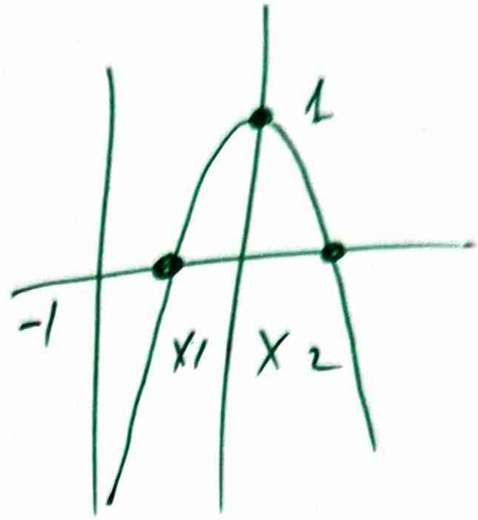
$1 + \ln(x+1) = x$

$\ln(x+1) - x + 1 = 0$

$\varphi(x) = \ln(x+1) - x + 1$

$\varphi'(x) = \frac{1}{x+1} - 1 = \frac{1 - x - 1}{x+1} = \frac{-x}{x+1}$

x	-1		0	
φ'		$+$		$-$
φ	$-\infty$	$\nearrow$	1	$\searrow$ $-\infty$



$$\lim_{x\to -1^+} \varphi(x) = -\infty$$

$$\varphi(0) = 1$$

$$\lim_{x\to +\infty} \varphi(x) = \lim_{x\to +\infty} \ln(x+1) - x + 1 =$$

$$= \lim_{x\to +\infty} x\left(\frac{\ln(x+1)}{x} - 1 + \frac{1}{x}\right) = -\infty$$

$$\rightarrow \lim_{x\to +\infty} \frac{\ln(x+1)}{x} = \lim_{x\to +\infty} \frac{1}{x+1} = 0$$

$x < 0$

- $\varphi \nearrow$
- φ συνεχ.
- $\Sigma T\varphi = (-\infty, 1)$

Το $0 \in \Sigma T\varphi$ άρα $\exists!\, x_1$ τ.ω $\varphi(x_1) = 0$.

$x \geq 0$

- $\varphi \searrow$
- φ συνεχ.
- $\Sigma T\varphi = (-\infty, 1]$

Το $0 \in \Sigma T\varphi$ άρα $\exists!\, x_2$ τ.ω $\varphi(x_2) = 0$

24.

$f(x) = x^5 + 10x^2 + x - 1$

$f'(x) = 5x^4 + 20x + 1$

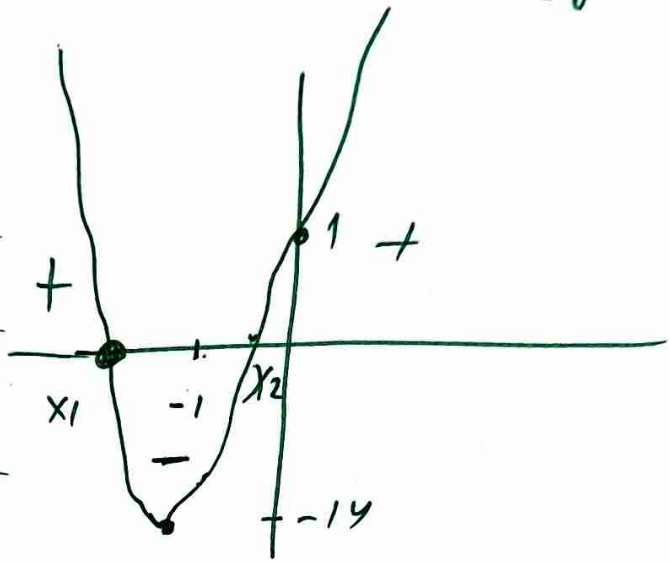
$f''(x) = 20x^3 + 20 = 20(x^3+1)$

$\longrightarrow x^3 + 1 = 0$

$\underline{\underline{x = -1}}$

$C_{f'}$

x	$-\infty$	-1	$+\infty$
f''	$-$	0	$+$
f'	$+\infty \searrow$	-14	$\nearrow +\infty$
f			



$\lim\limits_{x\to-\infty} f'(x) = +\infty$

$f'(-1) = -14$

$\lim\limits_{x\to+\infty} f'(x) = +\infty$

$\underline{x < -1}$

- f' συνεχής
- $f' \downarrow$
- $\Sigma T_{f'} = [-14, +\infty)$

Το $0 \in \Sigma T_{f'}$

άρα $\exists\, x_1$ τ.ω

$f'(x_1) = 0$

$\underline{x > -1}$

- f' συνεχής
- $f' \uparrow$
- $\Sigma T_{f'} = (-14, +\infty)$
- Το $0 \in \Sigma T_{f'}$

άρα $\exists\, x_2$ τ.ω

$f'(x_2) = 0$

x		x_1		-1		x_2	
f''	$-$		$-$	0	$+$		$+$
f'	$+$ $\searrow$	0	$\searrow$ $-$		$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

23. $f(x)=e^x+x^2-2x+3 \quad , x\in\mathbb{R}$

$f'(x)=e^x+2x-2$

$f''(x)=e^x+2>0$

1η απόδειξη

x	$-\infty$	$+\infty$
f''	$+$	
f'	$\nearrow$	
f		

$\Sigma T_{f'}$

$$\left.\begin{aligned} &\lim_{x\to-\infty} f'(x)=-\infty \\ &\lim_{x\to+\infty} f'(x)=+\infty \end{aligned}\right\} \Sigma T_{f'}\ \mathbb{R}.$$

2η απόδειξη

$$\left.\begin{aligned} &\bullet\ f' \text{ συνεχής} \\ &\bullet\ f' \nearrow \\ &\cdot\ \Sigma T_{f'}=\mathbb{R} \end{aligned}\right\} \begin{aligned} &\text{Το } 0\in\Sigma T_{f'} \\ &\text{άρα } \exists!\, x_0 \\ &\text{τ.ω } f'(x_0)=0 \end{aligned}$$

x		x_0	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

$x<x_0 \Rightarrow f'(x)<f'(x_0) \Rightarrow f'(x)<0$

Επόμενο Μάθημα

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