Σελ 95

4. $f(x) = x^2 + \ln x - 1$ $\qquad D_f = (0, +\infty)$.

ⓐ $f(x) = 0 \iff x^2 + \ln x - 1 = 0$

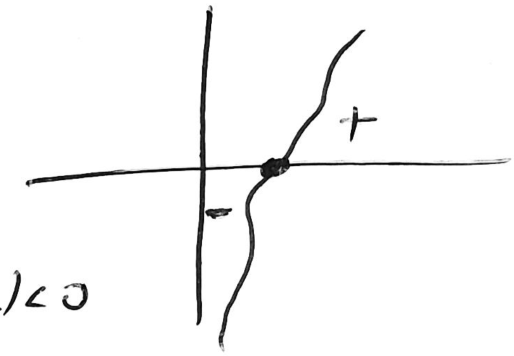
$f(x) = f(1)$

f 1-1

$x = 1$

$$\boxed{f'(x) = 2x + \frac{1}{x} > 0 \qquad f \nearrow \Rightarrow f \text{ 1-1}}$$

x	0	1	
$f(x)$	$\nearrow$ −	0	+ $\nearrow$



$x < 1 \Rightarrow f(x) < f(1) \Rightarrow f(x) < 0$

$x > 1 \Rightarrow f(x) > f(1) \Rightarrow f(x) > 0$

ⓑ $(x^2+1)^2 + \ln(x^2+1) = 1$

$(x^2+1)^2 + \ln(x^2+1) - 1 = 0$

$f(x^2+1) = f(1)$

f 1-1

$x^2 + 1 = 1$

$x^2 = 0$

$\boxed{x = 0}$.

(γ). $xe^{x^2} > e$ στο $(0,+\infty)$.

$\ln xe^{x^2} > \ln e$

$\ln x + \ln e^{x^2} > 1$

$\ln x + x^2 \ln e > 1$

$\ln x + x^2 > 1$.

$x^2 + \ln x - 1 > 0$

$f(x) > 0$

$f(x) > f(1)$

$f \nearrow$

$x > 1$

(δ). $\ln \dfrac{x^2+1}{x+1} > (x+1)^2 - (x^2+1)^2$

$\ln(x^2+1) - \ln(x+1) > (x+1)^2 - (x^2+1)^2$

$(x^2+1)^2 + \ln(x^2+1) - 1 > (x+1)^2 + \ln(x+1) - 1$.

$f(x^2+1) > f(x+1)$

$f \nearrow$

$x^2+1 > x+1 \Rightarrow x^2 - x > 0$

$x(x-1) > 0$

x		0		1	
x^2-x	+	0	−	0	+

$x \in (-\infty, 0) \cup (1,$

$\downarrow$

$x \in (-1, 0) \cup (1,$

5. $f(x) = \dfrac{x^3}{x^2+1}$ $\quad D_f = \mathbb{R}$.

ⓐ $f'(x) = \dfrac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2}$

$f'(x) = \dfrac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \dfrac{x^4+3x^2}{(x^2+1)^2} \geqslant 0$

$f \nearrow$

Ⓑ i) $f(5x^3) > f(8x^2+8)$

$f \nearrow$

$5x^3 > 8x^2+8$.

$5x^3 > 8(x^2+1)$

$\dfrac{x^3}{x^2+1} > \dfrac{8}{5}$

$f(x) > f(2)$

$f \nearrow$

$x > 2$

11). $(x^2+1)f(e^x-1)-x^3=0$.

$$(x^2+1)f(e^x-1)=x^3$$

$$f(e^x-1)=\frac{x^3}{x^2+1}$$

$$f(e^x-1)=f(x)$$

f 1-1

$$e^x-1=x \Rightarrow e^x=x+1$$

$$\underline{\underline{x=0}}$$

- $e^x \geqslant x+1$

$\boxed{x=0}$

To "="

(γ). $$\lim_{x\to 0}\frac{\ln x}{f(2x+1)-f(x+1)}=\lim_{x\to 0^+}\ln x\cdot\frac{1}{\underset{(+)}{f(2x+1)-f(x+1)}}$$

$$=-\infty\cdot(+\infty)=-\infty$$

- $x<2x \Rightarrow x+1<2x+1 \Rightarrow f(x+1)<f(2x+1)$

$$\underline{\underline{0<f(2x+1)-f(x+1)}}$$

6. $f(x) = x - \ln(1+e^x)$ $D_f = \mathbb{R}$.

ⓐ $f'(x) = 1 - \dfrac{e^x}{1+e^x} = \dfrac{1+e^x-e^x}{1+e^x} = \dfrac{1}{1+e^x} > 0$

$f \nearrow$

$f''(x) = \dfrac{-e^x}{(e^x+1)^2} < 0$.

$f' \searrow$

ⓑ i) $2f'(f(x)+\ln 2) < 1$

$f'(f(x)+\ln 2) < \frac{1}{2}$

$f'(f(x)+\ln 2) < f'(0)$

$f' \searrow$

$f(x) + \ln 2 > 0$

$f(x) > -\ln 2$

$f(x) > f(0)$

$f \nearrow$

$x > 0$

ii). $f'\left(\ln\frac{1+e^x}{1+e^{x^2}}\right) = f'(x-x^2)$

f' 1-1

$\ln\frac{1+e^x}{1+e^{x^2}} = x-x^2$

$\ln(1+e^x) - \ln(1+e^{x^2}) = x-x^2$

$x^2 - \ln(1+e^{x^2}) = x - \ln(1+e^x)$

$f(x^2) = f(x)$

f 1-1

$x^2 = x$

$x^2 - x = 0 \Rightarrow x(x-1) = 0$

$x=0$ $x=1$

iii). $f(x) + 2f'(1-e^x) = \ln\frac{e}{2}$

$f(x) + 2f'(1-e^x) = \ln e - \ln 2$

$f(x) + 2f'(1-e^x) - 1 + \ln 2 = 0$

$\varphi(x) = 0$

$\varphi(x) = \varphi(0)$

φ 1-1

$x = 0$

$$\boxed{\varphi(x) = f(x) + 2f'(1-e^x) - 1 + \ln 2}$$

$$\varphi(0) = f(0) + 2f'(1-e^0) - 1 + \ln 2$$

$$\varphi(0) = -\cancel{\ln 2} + 2f'(0) - 1 + \cancel{\ln 2}$$

$$\varphi(0) = \cancel{2}\frac{1}{\cancel{2}} - 1 \qquad = 1 - 1$$

$$\varphi(0) = 0$$

- $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2} \Rightarrow -e^{x_1} > -e^{x_2} \Rightarrow 1 - e^{x_1} > 1 - e^{x_2}$

(+)

$$f'(1-e^{x_1}) < f'(1-e^{x_2})$$

$$2f'(1-e^{x_1}) < 2f'(1-e^{x_2})$$

$$f(x_1) + 2f'(1-e^{x_1}) - 1 + \ln 2 < f(x_2) + 2f'(1-e^{x_2}) - 1 + \ln 2$$

$$\varphi(x_1) < \varphi(x_2)$$

$$\varphi \nearrow \Rightarrow \varphi \text{ 1-1}.$$

γ) $\lim_{x\to 0} \frac{\ln x}{f(x)+\ln 2} = \lim_{x\to 0^+} \ln x \frac{1}{f(x)+\ln 2}$

$\oplus$

$= -\infty \cdot (+\infty) = \underline{\underline{-\infty}}$

Αν $x>0 \Rightarrow f(x)>f(0) \Rightarrow f(x) > -\ln 2$

$f(x)+\ln 2 > 0$

7. $f(x) = \dfrac{e^x}{x}, \ x \geq 1.$

$$f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x (x-1)}{x^2} \geq 0$$

$f \nearrow$

$$f(x) + 1 > e + \ln f(x)$$

$$f(x) - \ln f(x) + 1 - e > 0$$

$\boxed{\varphi(x) = x - \ln x + 1 - e}$

$\varphi'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x} \geq 0$

$$\varphi(f(x)) > \varphi(e)$$

$$\varphi \nearrow$$

$$f(x) > e$$

$$f(x) > f(1)$$

$$f \nearrow$$

$$x > 1$$

$\boxed{x > 1 \Rightarrow f(x) > f(1) \Rightarrow f(x) > e \Rightarrow f(x) > 1}$

8. $f(x) = e^{1-x} + \ln x - 1$ $\qquad D_f = (0, +\infty)$.

ⓐ $f(x) = 0 \Rightarrow f(x) = f(1) \xleftrightarrow{f\ 1-1} x = 1$

$f'(x) = -e^{1-x} + \frac{1}{x}$ $\qquad f'(1) = 0$

$$f'(x) = -\frac{e^1}{e^x} + \frac{1}{x} = \frac{1}{x} - \frac{e}{e^x} = \frac{e^x - ex}{xe^x}$$

$$\boxed{f'(x) = \frac{e^x - e\cdot x}{xe^x}}$$

$\varphi(x) = e^x - e\cdot x$

$\varphi'(x) = e^x - e$

$\rightarrow e^x - e = 0$

$e^x = e$

$\boxed{x = 1}$

x	0		1		$+\infty$
φ'		$-$	0	$+$	
φ		$\searrow$ $+$		$+$ $\nearrow$	
xe^x		$+$		$+$	
f'		$+$		$+$	
f		$\nearrow$		$\nearrow$	

$\varphi(x) \geq \varphi(1)$

$\varphi(x) \geq 0$

x		1	
$f(x)$	$\nearrow$ $-$	0	$+$ $\nearrow$

(B) i) $e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -\ln \varepsilon\varphi x \quad (0, \frac{\pi}{2})$

$$e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -\ln \frac{\eta\mu x}{\sigma\upsilon\nu x}$$

$$e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -(\ln \eta\mu x - \ln \sigma\upsilon\nu x)$$

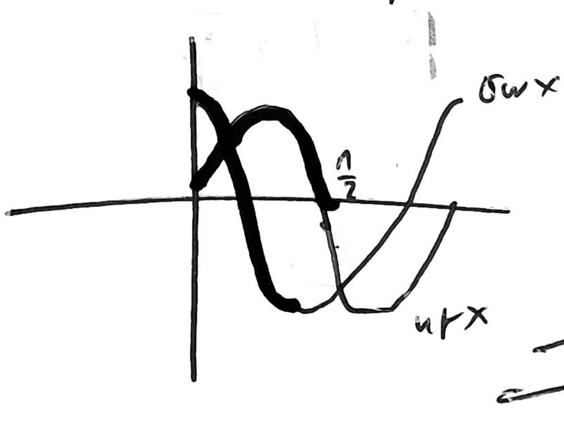
$$e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -\ln \eta\mu x + \ln \sigma\upsilon\nu x$$

$$e^{1-\eta\mu x} + \ln(\eta\mu x) - 1 = e^{1-\sigma\upsilon\nu x} + \ln(\sigma\upsilon\nu x) - 1$$

$$f(\eta\mu x) = f(\sigma\upsilon\nu x)$$

f 1-1

$$\eta\mu x = \sigma\upsilon\nu x \quad (0, \frac{\pi}{2})$$



$$\underline{\underline{x = \frac{\pi}{4}}}$$

φουλ ανάλυση

$$\eta\mu x = \sigma\upsilon\nu x$$

$$\sigma\upsilon\nu\left(\frac{\pi}{2} - x\right) = \sigma\upsilon\nu x.$$

$$\frac{\pi}{2} - x = 2\kappa\pi + x \quad \text{ή} \quad \frac{\pi}{2} - \cancel{x} = 2\kappa\pi - \cancel{x}$$

Αδύνατη.

$$-2x = 2\kappa\pi - \frac{\pi}{2}$$

$$\boxed{x = -\kappa\pi + \frac{\pi}{4}}$$

$$0 < x < \frac{\pi}{2}$$

$$0 < -\kappa\pi + \frac{\pi}{4} < \frac{\pi}{2}$$

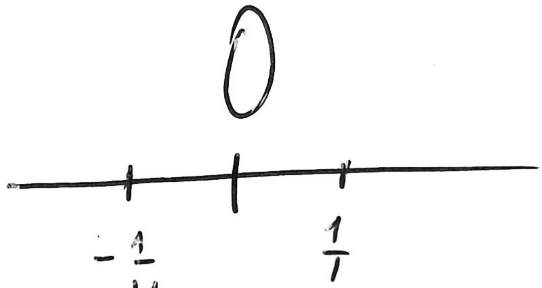
$$0 < -4\kappa\pi + \pi < 2\pi$$

$$-\pi < -4\kappa\pi < \pi$$

$$-1 < -4\kappa < 1$$

$$\frac{1}{4} > \kappa > -\frac{1}{4}$$

$$\kappa \in \mathbb{Z}$$



Για $\kappa = 0$

$$x = \frac{\pi}{4}$$

ii). $e^{1-x} - e^{1-x^2} > \ln x$

$e^{1-x} - e^{1-x^2} > 2\ln x - \ln x$

$$\underbrace{e^{1-x} + \ln x - 1}_{f(x)} > \underbrace{e^{1-x^2} + \ln x^2 - 1}_{f(x^2)}$$

$f \nearrow$

$x > x^2$

$x - x^2 > 0$

$x(1-x) > 0$

x		0		1	
$x-x^2$	$-$	0	$+$	0	$-$

$x \in (0,1)$.

(γ). $\lim\limits_{x \to 1^+} \dfrac{\ln x}{(x-1) f(x)} = \lim\limits_{x \to 1^+} \dfrac{\ln x}{x-1} \cdot \dfrac{1}{f(x)} = 1 \cdot (+\infty)$

$+$

$= +\infty$

$\rightarrow \lim\limits_{x \to 1^+} \dfrac{\ln x}{x-1} = \lim\limits_{x \to 1^+} \dfrac{\frac{1}{x}}{1} = 1$

$x > 1 \Rightarrow f(x) > f(1) \Rightarrow f(x) > 0$

9. α) $e^{x^2}+\ln(x^2+1)=1$

$$\boxed{f(x)=e^x+\ln(x+1)-1 \quad x>-1}$$

$$f'(x)=e^x+\frac{1}{x+1}>0$$

$f \nearrow$

$$e^{x^2}+\ln(x^2+1)-1=0$$

$$f(x^2)=0$$

$$f(x^2)=f(0)$$

f 1-1

$$x^2=0$$

$$\underline{\underline{x=0}}$$

β) $\ln\dfrac{x^2+1}{x+1}>e^x-e^{x^2}$

$$\ln(x^2+1)-\ln(x+1)>e^x-e^{x^2}$$

$$\ln(x^2+1)+e^{x^2}-1>e^x+\ln(x+1)-1$$

$$f(x^2)>f(x)$$

$f \nearrow$

$x^2>x \Rightarrow x^2-x>0$

x		0		1	
x^2-x	+	0	−	0	+

$x\in(-\infty,0)\cup(1,\infty)$

(γ) $e^{x^2-1}+\ln x=e^{x-1}$

$e^{x^2-1}+2\ln x-\ln x=e^{x-1}$

$e^{x^2-1}+\ln x^2-1=e^{x-1}+\ln x-1$

$f(x^2)=f(x)$

f 1-1

$x^2=x$

$x=0$ $x=1$

11. (α) $x \ln x = 1 - x^3$ $\qquad x > 0$

$$\ln x = \frac{1}{x} - x^2$$

$$\underbrace{\ln x - \frac{1}{x} + x^2}_{f(x)} = 0$$

$f(x) = 0$

$f(x) = f(1)$

$x = 1$

$$f'(x) = \frac{1}{x} + \frac{1}{x^2} + 2x > 0$$

$f \nearrow$

(B) $e^{x^3 - 1} \cdot x^x > 1$

$$\ln\left(e^{x^3-1} \cdot x^x\right) > \ln 1$$

$$\ln e^{x^3-1} + \ln x^x > 0$$

$$(x^3 - 1) + x \ln x > 0$$

$$x^3 - 1 + x \ln x > 0$$

$x^2 - \frac{1}{x} + \ln x > 0 \Rightarrow f(x) > 0 \Rightarrow f(x) > f(1)$

$\underline{\underline{x > 1}}$