$f(0) = f'(0) = 0$

24. $f(x) - 2f'(x) + f''(x) = 2e^x$

(α) νδο $g(x) = 2x - \dfrac{f'(x) - f(x)}{e^x}$ σταθερη.

$$g'(x) = 2 - \frac{(f''(x) - f'(x))e^x - (f'(x) - f(x))e^x}{(e^x)^2}$$

$$g'(x) = 2 - \frac{f''(x) - f'(x) - f'(x) + f(x)}{e^x}$$

$$g'(x) = 2 - \frac{f''(x) - 2f'(x) + f(x)}{e^x} = 2 - \frac{2\cancel{e^x}}{\cancel{e^x}}$$

$$g'(x) = 2 - 2 = 0$$

$$g'(x) = 0 \Rightarrow \underline{\underline{g(x) = c}}$$

(β) $2x - \dfrac{f'(x) - f(x)}{e^x} = c.$

$\underline{x = 0}$

$$0 - \frac{f'(0) - f(0)}{1} = c \iff \underline{\underline{c = 0}}$$

$$2x - \frac{f'(x) - f(x)}{e^x} = 0,$$

$$\frac{f'(x) - f(x)}{e^x} = 2x$$

$$\frac{e^x f'(x) - e^x f(x)}{e^{2x}} = (x^2)'$$

$$\left(\frac{f(x)}{e^x}\right)' = (x^2)'$$

$$\frac{f(x)}{e^x} = x^2 + c$$

$$\underline{x=0}$$

$$c = 0$$

$$f(x) = x^2 e^x$$

26. $f(0)=g(0)=1$.

$f'(x)=g(x)$

$g'(x)=1-f(x)$

νδο $\quad f(x)=1+\eta\mu x \quad$ και $\quad g(x)=\sigma\upsilon\nu x$

$\varphi(x)=f(x)-1-\eta\mu x$

$\varphi'(x)=f'(x)-\sigma\upsilon\nu x = g(x)-\sigma\upsilon\nu x$

$\varphi'(x)=g(x)-\sigma\upsilon\nu x$

$\varphi''(x)=g'(x)+\eta\mu x$.

$\varphi''(x)=1-f(x)+\eta\mu x$.

$\varphi''(x)=\varphi(x)$

$\varphi''(x)+\varphi'(x)=\varphi'(x)+\varphi(x)$

$e^x\varphi''(x)+e^x\varphi'(x)=e^x\varphi'(x)+e^x\varphi(x)$

$\left[e^x\varphi'(x)\right]'=\left[e^x\varphi(x)\right]'$

$$e^x \varphi'(x) = e^x \varphi(x) + C$$

$$e^0 \varphi'(0) = e^0 \varphi(0) + C$$

$$\cancel{g(0)} - \cancel{1} = \cancel{f(0)} - \cancel{1} + C$$

$$C = 0$$

$$\cancel{e^x} \varphi'(x) = \cancel{e^x} \varphi(x)$$

$$\varphi'(x) = \varphi(x)$$

$$\varphi(x) = ce^x.$$

$$\underline{x=0}$$

$$\varphi(0) = ce^0$$

$$f(0) - 1 = C.$$

$$c = 0.$$

$$\varphi(x) = 0.$$

$$f(x) - 1 - \eta\mu x = 0$$

$$\boxed{f(x) = \eta\mu x + 1}.$$

Αφού

$$g'(x) = 1 - f(x)$$

$$g'(x) = 1 - \eta\mu x - 1$$

$$g'(x) = -\eta\mu x$$

$$g'(x) = (\sigma\upsilon\nu x)'$$

$$g(x) = \sigma\upsilon\nu x + C$$

$$\underline{x=0}$$

$$g(0) = 1 + C.$$

$$1 = 1 + C \quad C = 0$$

$$g(x) = \sigma\upsilon\nu x$$

29.

ⓐ $f(x) = \begin{cases} \dfrac{xe^x}{e^x - 1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{xe^x}{e^x-1}\right) = \lim_{x\to 0}\frac{e^x + xe^x}{e^x} = 1.$$

$f(0) = 1$.

$$\boxed{\text{Άρα } f(0) = \lim_{x\to 0} f(x) \text{ η } f \text{ συνεχής στο } 0.}$$

Η f συνεχής στο $(-\infty, 0)$ και $(0, +\infty)$ ως π.δ.σ

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\frac{xe^x}{e^x-1} - 1}{x} = \lim_{x\to 0}\frac{xe^x - e^x + 1}{x(e^x-1)}$$

$$= \lim_{x\to 0}\frac{\cancel{e^x} + xe^x - \cancel{e^x}}{e^x - 1 + xe^x} = \lim_{x\to 0}\frac{e^x + xe^x}{e^x + e^x + xe^x}$$

$$= \frac{1+0}{1+1+0} = \frac{1}{2}$$

$f'(0) = \frac{1}{2}$.

$$f'(x) = \frac{(xe^x)'(e^x-1) - xe^x(e^x-1)'}{(e^x-1)^2}$$

$$f'(x) = \frac{(e^x+xe^x)(e^x-1) - xe^x e^x}{(e^x-1)^2}$$

$$f'(x) = \frac{e^{2x} - e^x + \cancel{xe^x e^x} - xe^x - \cancel{xe^x e^x}}{(e^x-1)^2}$$

$$f'(x) = \frac{\overset{+}{e^x}\,\overset{+}{(e^x-1-x)}}{\underset{+}{(e^x-1)^2}} \geq 0 \qquad f \nearrow$$

- $e^x \geq x+1 \Rightarrow e^x - x - 1 \geq 0$

37. (B) $f(x) = e^x - \frac{1}{2}x^2 - \ln(x^2+1)$.

$$f'(x) = (e^x)' - \frac{1}{2}(x^2)' - \frac{1}{x^2+1}\cdot(x^2+1)'$$

$$f'(x) = e^x - \frac{\cancel{2}x}{\cancel{2}} - \frac{2x}{x^2+1}$$

$$f'(x) = e^x - x - \frac{2x}{x^2+1}$$

$$f'(x) = \underbrace{e^x - x - 1}_{+} + 1 - \frac{2x}{x^2+1}$$

$$f'(x) = e^x - x - 1 + \frac{x^2+1-2x}{x^2+1} = \underbrace{e^x - x - 1}_{+} + \frac{(x-1)^2}{x^2+1} \quad +$$

15. 01 $f(x) = \dfrac{n\mu x}{1+\sigma w x}$ $x \in (-n, n)$.

$$F'(x) = \frac{(n\mu x)'(1+\sigma u v x) - n\mu x(1+\sigma u v x)'}{(1+\sigma u v x)^2}$$

$-1 \leq \sigma w x \leq 1$

$\sigma w x + 1 \geq 0$

$$F'(x) = \frac{\sigma u v x(1+\sigma u v x) - n\mu x(-n\mu x)}{(1+\sigma u v x)^2}$$

$$F'(x) = \frac{\sigma u v x + \sigma u v^2 x + n\mu^2 x}{(1+\sigma u v^2 x)^2} = \frac{\sigma u \cancel{v x + 1}}{(1+\sigma u v x)^{\cancel{2}}} = \frac{1}{1+\sigma u v x} > 0$$

[redacted]

37. (α) $f(x)=\frac{1}{2}x^2 - x(\ln x - 1) - \etaμ x\ ;\ x>0$

$$f'(x) = \frac{1}{2}(2x) - [(x)'(\ln x - 1) + x(\ln x - 1)'] - συν x$$

$$f'(x) = x - [(\ln x - 1) + x\left(\frac{1}{x}\right)] - συν x$$

$$f'(x) = x - [\ln x - 1 + 1] - συν x$$

$$\boxed{f'(x) = x - \ln x - συν x}$$

• $\ln x \leq x - 1$

$\underbrace{συν x \leq 1 \leq}\ x - \ln x$

$συν x \leq x - \ln x$

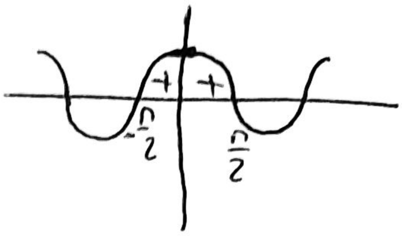
$0 \leq x - \ln x - συν x$

$f'(x) \geq 0$

$f \nearrow$

15. (B) $f(x) = \eta\mu x - \varepsilon\varphi x, \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f'(x) = \sigma\upsilon\nu x - \frac{1}{\sigma\upsilon\nu^2 x} \le 0$$



$\cancel{f'(x) = \sigma\upsilon\nu x(\sigma\upsilon\nu^2 x) - 1}$

$-1 \le \sigma\upsilon\nu x \le 1 \quad \Rightarrow \quad |\sigma\upsilon\nu x| \le 1$

$-1 \ge \frac{1}{\sigma\upsilon\nu x} \ge 1 \qquad \sigma\upsilon\nu^2 x \le 1$

$-1 \ge \frac{1}{\sigma\upsilon\nu^2 x} \ge 1 \qquad \frac{1}{\sigma\upsilon\nu^2 x} \geq 1$

$$-\frac{1}{\sigma\upsilon\nu^2 x} \le -1.$$

$$-1 \le \sigma\upsilon\nu x \le 1$$

$$\sigma\upsilon\nu x - \frac{1}{\sigma\upsilon\nu^2 x} \le 0$$

$f \downarrow$

34. (B) $f(x) = e^{x} - e^{-x} + 2\sigma\upsilon\nu\, x$.

$$f'(x) = e^{x} + e^{-x} - 2\eta\mu x$$

Αρκει νδο $e^{x} + e^{-x} - 2\eta\mu x > 0$.

$$-1 \leq \eta\mu x \leq 1$$

$$2 \geq -2\eta\mu x \geq -2$$

$$e^{x} + e^{-x} + 2 \geq e^{x} + e^{-x} - 2\eta\mu x \geq e^{x} + e^{-x} - 2$$

$$f'(x) \geq e^{x} + e^{-x} - 2$$

$$f'(x) \geq e^{x} + \frac{1}{e^{x}} - 2$$

$$f'(x) \geq \frac{e^{2x} - 2e^{x} + 1}{e^{x}}$$

$$f'(x) \geq \frac{(e^{x} - 1)^2}{e^{x}} \geq 0.$$

$f \nearrow$.

34. ⓞ1 $f(x) = \frac{1}{2}x^2 + \ln x - 2\sin x$

$$f'(x) = x + \frac{1}{x} - 2\cos x$$

$$-1 \leq \cos x \leq 1$$

$$2 \geq -2\cos x \geq -2$$

$$x + \frac{1}{x} + 2 \geq x + \frac{1}{x} - 2\cos x \geq x + \frac{1}{x} - 2$$

$$f'(x) \geq x + \frac{1}{x} - 2$$

$$f'(x) \geq \frac{x^2 + 1 - 2x}{x}$$

$$f'(x) \geq \frac{(x-1)^2}{x} \geq 0$$

f ↗

30. $f: [0, +\infty) \to \mathbb{R}$ 2 φορές παρ/μη.

$f(0) = f'(0) = 0$

$f''(x) > 0 \quad \forall x > 0$.

Ν.δ.ο. $g(x) = \begin{cases} \dfrac{f(x)}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$ αύξουσα.

$g(x) = \dfrac{f(x)}{x}$

$g'(x) = \dfrac{f(x)' \cdot x + f(x)}{x^2}$

$h(x) = f'(x) \cdot x + f(x)$

$h'(x) = f''(x)x + f'(x) + f'(x)$

$h'(x) = f''(x)x + 2f'(x) > 0$

x	0	$+\infty$
h'		+
h	0	+ ↗
g'		+
g		↗

Αφού $f''(x) > 0 \Rightarrow f' \nearrow$

Για $x > 0 \Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 0$

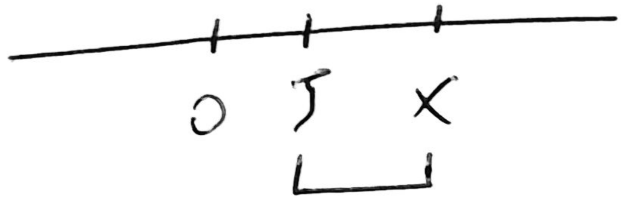
α' τρόπος

β' τρόπ

$$g'(x) = \frac{x f'(x) + f(x)}{x^2}.$$

Αρκει νδο $x f'(x) + f(x) > 0$

$f' \nearrow$



$$f'(\xi) = \frac{f(x) - f(0)}{x - 0}$$

$$f'(\xi) = \frac{f(x)}{x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x)$$

$$\frac{f(x)}{x} < f'(x)$$

$$f(x) < x f'(x)$$

$$0 < x f'(x) - f(x)$$

29. (B) $f(x)=\begin{cases} \dfrac{x\ln x}{x-1} & , \ 0<x\neq 1 \\ 1 & , \ x=1 . \end{cases}$

$\lim\limits_{x\to 1} F(x) = \lim\limits_{x\to 1} \dfrac{x\ln x}{x-1} = \lim\limits_{x\to 1} \dfrac{(x)'\ln x + x(\ln x)'}{1}$

$= \lim\limits_{x\to 1} (\ln x + 1) = 1$ $\qquad f(1)=1.$

Άρα $\lim\limits_{x\to 1} f(x) = f(1)$

Συνεχής στο 1.

$\lim\limits_{x\to 1} \dfrac{F(x)-F(1)}{x-1} =$

$= \lim\limits_{x\to 1} \dfrac{\dfrac{x\ln x}{x-1} - 1}{x-1} = \lim\limits_{x\to 1} \dfrac{x\ln x - x + 1}{x-1}$

$= \lim\limits_{x\to 1} \dfrac{\ln x + x\frac{1}{x} - 1}{1} = \lim\limits_{x\to 1} \ln x = 0$

$f'(1)=0$

$F'(x) = \dfrac{x\ln x}{x-1}'$

$F'(x) = \dfrac{(x\ln x)'\cdot(x-1) - (x\ln x)\cdot(x-1)'}{(x-1)^2}$

$f'(x) = \dfrac{\left(\ln x + x\cdot\frac{1}{x}\right)(x-1) - x\ln x}{(x-1)^2} = \dfrac{(\ln x+1)(x-1) - x\ln x}{(x-1)^2}$

$$f'(x) = \frac{\cancel{x\ln x} - \ln x + x - 1 - \cancel{x\ln x}}{(x-1)^2}$$

$$f'(x) = \frac{\overset{(+)}{x - \ln x - 1}}{(x-1)^2} \geq 0$$

$f \nearrow$.

$\ln x \leq x - 1$

$0 \leq x - \ln x - 1$

33. (α) $f(x) = 3x^3 - ax^2 + 9x - 1$. $f \nearrow$

$f'(x) = 3 \cdot 3x^2 - a \cdot 2x + 9$

$\boxed{f'(x) = 9x^2 - 2ax + 9}$.

Πρέπει να βρω το α ώστε $f'(x) > 0$.

Απαιτώ $\underline{\underline{\Delta < 0}}$ $\Delta = 4a^2 - 4 \cdot 9 \cdot 9$

$= 4(a^2 - 81)$

$a^2 - 81 < 0$

$a^2 < 81$

$a^2 < 9^2$

$|a| < |9|$ $|a| < 9$ $\underline{a \in (-9, 9)}$

$f \searrow$

(β) $f(x) = ax^3 + x^2 - x + 1$, $a \neq 0$

$f'(x) = 3x^2 \cdot a + 2x - 1$

$\boxed{f'(x) = 3ax^2 + 2x - 1}$

$\Delta < 0$ και $3a < 0 \Leftrightarrow$

$\boxed{a < 0}$.

$\Delta = 4 - 4 \cdot 3a \cdot (-1)$

$= 4 + 4 \cdot 3a$

$= 4 \cdot (3a + 1)$

$3a + 1 < 0 \Leftrightarrow$

$\boxed{a < -\frac{1}{3}}$ $a \in (-\infty, -\frac{1}{3})$

32. $g(x) = e^x - x + \dfrac{f(x)-1}{x}, \quad x > 0$

$$g'(x) = e^x - 1 + \frac{f'(x)x - (f(x)-1)}{x^2}$$

$$g'(x) = \overset{(+)}{e^x - 1} + \frac{x f'(x) - f(x) + 1}{x^2}$$

$$\boxed{x > 0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1 \Rightarrow e^x - 1 > 0}$$

Αφού $f''(x) > 0$

$f' \nearrow$

$0 \qquad x$

$$f'(\xi) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(\xi) = \frac{f(x) - 1}{x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x) \Rightarrow \frac{f(x)-1}{x} < f'(x)$$

$$\Rightarrow x f'(x) > f(x) - 1$$

$$\Rightarrow \boxed{x f'(x) - f(x) + 1 > 0}$$

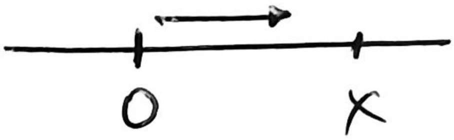
Άρα $g \nearrow$

31. $f: [0,+\infty] \longrightarrow \mathbb{R}$ $\qquad f(0)=0$

$f'(0)=0 \qquad f' \nearrow$

$$g(x)=\begin{cases} \dfrac{f(x)}{x}, & x>0 \\ 0, & x=0 \end{cases}$$

νδο $g(x) \nearrow$.



$$g'(x)=\frac{f'(x)x-f(x)}{x^2}$$

Αρκει νδο $xf'(x)-f(x)>0$

Από ΘΜΤ $\exists \xi \in (0,x)$ τ.ω $\dfrac{f(x)-f(0)}{x-0}=f'(x)$

$\dfrac{f(x)-0}{x-0}=f'(x)$, $\dfrac{f(x)}{x}=f'(x)$

$f'(x)=\dfrac{f(x)}{x}$, $f'(\xi)=\dfrac{f(x)}{x}$, $\xi<x$

$f'(\xi)<f'(x) \Rightarrow \dfrac{f(x)}{x}<f'(x)$

$f(x)<xf'(x)$ οεδ.