Έστω $f'(x) = f(x)$

Νδο $f(x) = ce^{x}$

$$f'(x) - f(x) = 0$$

$$e^{-x} f'(x) - e^{-x} f(x) = 0$$

$$\left(e^{-x} f(x)\right)' = 0$$

$$e^{-x} f(x) = c$$

$$\frac{f(x)}{e^{x}} = c$$

$$\underline{\underline{f(x) = ce^{x}}}$$

18. $2f(x) + 4x f'(x) + (x^2+1) f''(x) = 0.$

$f(0) = 0$

$f'(0) = 1$

Νδο $f(x) = \dfrac{x}{x^2+1}$

$$\boxed{g(x) = (x^2+1) f(x) - x}$$

$g'(x) = 2x f(x) + (x^2+1) f'(x) - 1$

$g''(y) = 2 f(x) + 2x f'(x) + 2x f'(x) + (x^2+1) f''(x)$

$$\boxed{g''(x) = 2 f(x) + 4x f'(x) + (x^2+1) f'''(x)}$$

$g''(x) = 0$

$g'(x) = c$

$2x f(x) + (x^2+1) f'(x) - 1 = c \quad \Rightarrow 2x f(x) + (x^2+1) f'(x) - 1 = 0$

$\underline{x = 0}$

$f'(0) - 1 = c$

$1 - 1 = c$

$c = 0$

$2x f(x) + (x^2+1) f'(x) = 1$

$[(x^2+1) f(x)]' = (x)'$

$(x^2+1) f(x) = x + c \qquad \underline{x=0} \quad f(0) = c = 0$

$f(x) = \dfrac{x}{x^2+1}$

19. $|f(x)-f(y)| \le (x-y)^{2020} \quad \forall x,y \in \mathbb{R}$.

$$|f(x)-f(y)| \le |x-y|^{2020}$$

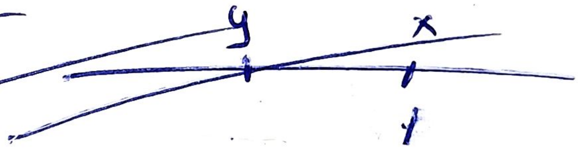
$$\frac{|f(x)-f(y)|}{|x-y|} \le |x-y|^{2019}$$

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^{2019}$$

$$\boxed{-|x-y|^{2019} \le \frac{f(x)-f(y)}{x-y} \le |x-y|^{2019}}$$

~~Έστω $y<x$ τότε~~

$\cancel{f'(f) = \frac{f(x)-f(y)}{x-y}}$



h

x

lim

Θέτω $y = x_0$

$$-|x-x_0|^{2019} \le \frac{f(x)-f(x_0)}{x-x_0} \le |x-x_0|^{2019}$$

$$\left.\begin{array}{l} \lim\limits_{x\to x_0} -|x-x_0|^{2019} = 0 \\ \\ \lim\limits_{x\to x_0} |x-x_0|^{2019} = 0 \end{array}\right\} \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$$

$$f'(x) = 0$$

$$f(x) = c$$

14. Σελ 48

- $f'(x) = g'(x)$ $\forall x \in R$.
- $f(0) = 0$.

$x + g(x) = \dfrac{x\eta\mu x}{x^2+1}$

Αφού $f'(x) = g'(x) \Rightarrow f(x) = g(x) + c$

$$f(x) = \frac{x\eta\mu x}{x^2+1} - x + c$$

$$f(0) = \frac{0\cdot\eta\mu 0}{0+1} - 0 + c = 0$$

$$c = 0$$

$$\boxed{f(x) = \frac{x\eta\mu x}{x^2+1} - x}$$

16. $f'(x) = \frac{(x+1)^2}{x^2+1} f(x) \quad \forall x \in R.$

(α) Αν $g(x) = \frac{f(x)}{x^2+1}$ νδο $g(x) = ce^x$

$$g'(x) = \frac{f'(x)(x^2+1) - f(x)2x}{(x^2+1)^2}$$

$$g'(x) = \frac{\frac{(x+1)^2}{\cancel{x^2+1}} f(x)\,\cancel{(x^2+1)} - 2xf(x)}{(x^2+1)^2}$$

$$g'(x) = \frac{(x+1)^2 f(x) - 2xf(x)}{(x^2+1)^2}$$

$$g'(x) = \frac{f(x)(x^2+\cancel{2x}+1-\cancel{2x})}{(x^2+1)^2} = \frac{f(x)\cancel{(x^2+1)}}{(x^2+1)^{\cancel{2}}} = \frac{f(x)}{x^2+1}$$

$g'(x) = g(x)$

$\underline{\underline{g(x) = ce^x}}$

(Β) Αφού $g(x) = ce^x$

$$\frac{f(x)}{x^2+1} = ce^x$$

$$f(x) = c\, e^x (x^2+1)$$

$f(0) = 1$

$$f(0) = ce^0 (0^2+1)$$

$$1 = c.$$

$$\boxed{f(x) = e^x (x^2+1)}$$

20. $f(x) = e^x - x f'(x) \quad \forall x \in \mathbb{R}$.

(α) $g(x) = e^x - x f(x)$ σταθερή

$$g'(x) = e^x - f(x) - x f'(x) = \underbrace{e^x - x f'(x)} - f(x)$$

$$g'(x) = f(x) - f(x)$$

$$g'(x) = 0 \Rightarrow g(x) = c$$

(β) $g(x) = c$

$e^x - x f(x) = c$.

$\underline{x = 0}$

$1 = c$

$e^x - x f(x) = 1$

$e^x - 1 = x f(x)$

$$f(x) = \frac{e^x - 1}{x} \quad \forall x \neq 0$$

$f(x) = e^x - x f'(x)$

$\underline{x = 0}$

$f(0) = 1$

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

$$f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

21 $f : (0,+\infty) \longrightarrow \mathbb{R}$

$$x\left(f(x) - f'(x)\right) = 1 - x^2 f'(x) \quad \forall x > 0 .$$

(a) $g(x) = (x-1)\, f(x) - \ln x .$

$$g'(x) = f(x) + (x-1)\, f'(x) - \frac{1}{x} .$$

$$g'(x) = \frac{x f(x) + x(x-1) f'(x) - 1}{x}$$

$$g'(x) = \frac{x f(x) + x^2 f'(x) - x f'(x) - 1}{x}$$

$$g'(x) = \frac{0}{x} = 0$$

$\underline{\underline{g(x) = c}}$

$$\boxed{x f(x) - x f'(x) - 1 + x^2 f'(x) = 0}$$

(B) $g(x) = c$.

$(x-1)f(x) - \ln x = c$

$x\left(f(x) - f'(x)\right) = 1 - x^2 f'(x)$

$\underline{x=1}$

$f(1) - \cancel{f'(1)} = 1 - \cancel{f'(1)}$

$$f(1) = 1$$

$x = 1$

$0 - \ln 1 = c$

$c = 0$

$f(x) = \dfrac{\ln x}{x-1}, \; x \neq 1$

$(x-1)f(x) = \ln x$

$f(1) = \lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{\ln x}{x-1} = \lim\limits_{x \to 1} \dfrac{\frac{1}{x}}{1} = 1$

$$f(x) = \begin{cases} \dfrac{\ln x}{x-1}, & x \in (0,1) \cup (1, +\infty) \\ 1, & x = 1. \end{cases}$$

22. $f'(x) = \dfrac{x}{f(x)} \iff \underline{f(x)f'(x) - x = 0}$

ⓐ $g(x) = f^2(x) - x^2$

$g'(x) = 2f(x)f'(x) - 2x = 2\left(f(x)f'(x) - x\right) = 0.$

$g(x) = c.$

ⓑ Αν $f(0) = 1$ Βρείτε την $f(x)$.

$f^2(x) - x^2 = c$

$\xrightarrow{x=0}$

$f^2(0) - 0 = c$

$c = 1$

$f^2(x) = x^2 + 1$

$f^2(x) = \sqrt{x^2+1}^{\,2}$

$|f(x)| = |\overset{+}{\sqrt{x^2+1}}|$

$|\overset{+}{f(x)}| = \sqrt{x^2+1}$

$f(x) = \sqrt{x^2+1}$

<u>Ρίζες $f(x)$</u>

$f(x) = 0$

$|f(x)| = 0$

$\sqrt{x^2+1} = 0$ Άτοπο!

$\left.\begin{array}{l} f(x) \neq 0 \\ \text{συνεχής} \end{array}\right\}$ $f(x) > 0$ ή $f(x) < 0$

$f(0) = 1$

23. $f(0)=1$

$$f'(x)(x+f(x))+f(x)=0.$$

α) νδο $f^2(x)=1-2x\,f(x)$.

$$x f'(x) + f(x) f'(x) + f(x).$$

$$\underbrace{x f'(x) + f(x)} + \underbrace{f(x) f'(x)} = 0.$$

$$\left[f(x)\cdot x + \frac{1}{2} f^2(x)\right]' = 0.$$

$$x f(x) + \frac{1}{2} f^2(x) = c$$

$$\xrightarrow{x=0}$$

$$\frac{1}{2} f^2(1) = c \iff c = \frac{1}{2}.$$

$$x f(x) + \frac{1}{2} f^2(x) = \frac{1}{2}$$

$$2x f(x) + f^2(x) = 1.$$

$$\boxed{f^2(x) = 1-2x\,f(x)}$$

(B). $f^2(x) + 2x f(x) = 1$.

$$f^2(x) + 2x f(x) + x^2 = x^2 + 1$$

$$\left(f(x) + x\right)^2 = \sqrt{x^2+1}^{\,2}$$

$$|f(x) + x| = |\sqrt{x^2+1}|$$

$$|g(x)| = \sqrt{x^2+1}$$

$$g(x) = \sqrt{x^2+1}$$

P,Tl $g(x)$

$\cancel{f(x) + x = \sqrt{x^2+1}}$

$f(x) = \sqrt{x^2+1} - x$

$g(x) = 0$

$|g(x)| = 0$

$\sqrt{x^2+1} = 0$

Άτοπο

$g(x) \neq 0$ συνεχής $\Big\}$ $g(x) > 0$ ή $g(x) < 0$

$g(0) = f(0) + 0 = 1$.

$g(x) > 0$

25. $xf'(x)+(x^2+1)f''(x)=0 \quad \forall x\in\mathbb{R}.$

$f(0)=0 \quad f'(0)=1.$

$f(x)=\ln(x+\sqrt{x^2+1})$

$$\boxed{g(x)=f(x)-\ln(x+\sqrt{x^2+1})}$$

$$g'(x)=f'(x)-\frac{1+\frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$$

$$g'(x)=f'(x)-\frac{\frac{2\sqrt{x^2+1}+2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}=f(x)-\frac{\cancel{2}\cancel{(\sqrt{x^2+1}+x)}}{\cancel{2}\sqrt{x^2+1}\,\cancel{(x+\sqrt{x^2+1})}}$$

$$g'(x)=f'(x)-\frac{1}{\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}-\frac{1}{\sqrt{x^2+1}}=0.$$

$g'(x)=0 \quad \forall x\in\mathbb{R}$

$g(x)=c.$

Επεξήγηση

πίσω →

$$x f'(x) + (x^2+1) f''(x) = 0$$

$$f''(x) + \frac{x}{x^2+1} f'(x) = 0$$

$$h(x) = \frac{x}{x^2+1}$$

$$H(x) = \frac{1}{2} \ln(x^2+1)$$

$$e^{H(x)} = e^{\frac{1}{2}\ln(x^2+1)} = e^{\ln(x^2+1)^{1/2}} = \sqrt{x^2+1}$$

$$\sqrt{x^2+1}\, f''(x) + \frac{x}{x^2+1}\sqrt{x^2+1}\, f'(x) = 0$$

$$\left(\sqrt{x^2+1}\, f'(x)\right)' = 0.$$

$$\sqrt{x^2+1}\, f'(x) = C$$

$$f'(x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{x=0}{f'(0) = 1 = C}$$

$g(x) = c.$

$f(x) - \ln\left(x+\sqrt{x^2+1}\right) = c$

$$\underline{x=0}$$

$f(0) - \ln 1 = c$

$c = 0.$

$$\boxed{f(x) = \ln\left(x+\sqrt{x^2+1}\right)}$$

В'гром

$$-xf'(x) = (x^2+1)f''(x)$$

$$-\frac{x}{x^2+1} = \frac{f''(x)}{f'(x)}$$

$$\left(-\frac{1}{2}\ln(x^2+1)\right)' = \left(\ln(f'(x))\right)'$$

$$-\frac{1}{2}\ln(x^2+1) = \ln f'(x) + C$$

$$\frac{x=0}{C=0}.$$

$$-\frac{1}{2}\ln(x^2+1) = \ln f'(x)$$

$$-\ln\sqrt{x^2+1} = \ln f'(x)$$

$$0 = \ln f'(x)\sqrt{x^2+1}$$

$$1 = f'(x)\sqrt{x^2+1} \qquad f'(x) = \frac{1}{\sqrt{x^2+1}}$$

β' τρόπος

$$g'(x) = f'(x) - \frac{1}{\sqrt{x^2+1}}$$

$$g'(x) = \frac{\sqrt{x^2+1}\, f'(x) - 1}{\sqrt{x^2+1}}$$

$$\varphi(x) = \sqrt{x^2+1}\; f'(x) - 1$$

$$\varphi'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\, f'(x) + \sqrt{x^2+1}\; f''(x)$$

$$\varphi'(x) = \frac{\overbrace{x f'(x) + (x^2+1) f''(x)}^{0}}{\sqrt{x^2+1}} = 0$$

$$\varphi'(x) = 0 \implies \varphi(x) = c.$$

$$\sqrt{x^2+1}\; f'(x) - 1 = 0.$$

Άρα $g'(x) = 0$

$g(x) = c$

...