# Ασκηση 1

Εστω $f: R \to R$ συνεχ.

• $f^2(x) - 2f(x) = x^2 - 2x \quad \forall x \in R.$

Να βρεθουν οι πιθανοι τυποι $f(x)$

$$f^2(x) - 2f(x) + 1 = x^2 - 2x + 1.$$

$$(f(x)-1)^2 = (x-1)^2$$

$$|f(x)-1| = |x-1|$$

$$|g(x)| = |x-1|$$

**Ριζα $g(x)$**

$g(x) = 0$

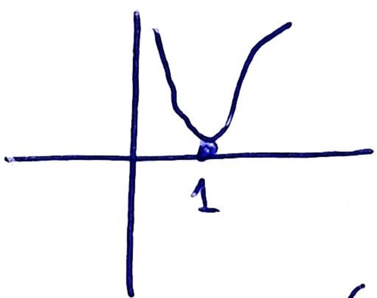
$|g(x)| = 0$

$|x-1| = 0$

$x-1=0$

$\boxed{x=1}$

| $x$ | | 1 | |
|---|---|---|---|
| $g(x)$ | ; | 0 | ; |

1. 

+

$|g(x)| = |x-1|$

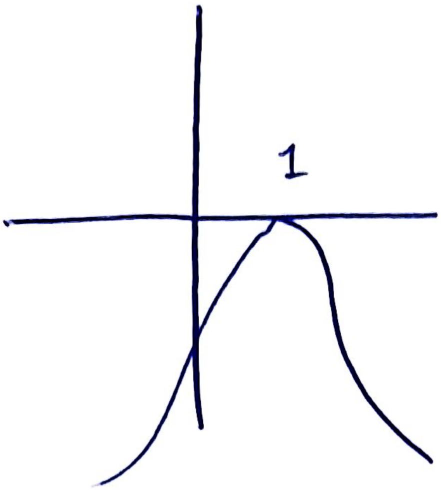
$g(x) = |x-1|$

$$g(x) = \begin{cases} 1-x, & x<1 \\ x-1, & x \geq 1 \end{cases}$$

$$f(x)-1 = \begin{cases} 1-x, & x<1 \\ x-1, & x \geq 1 \end{cases}$$

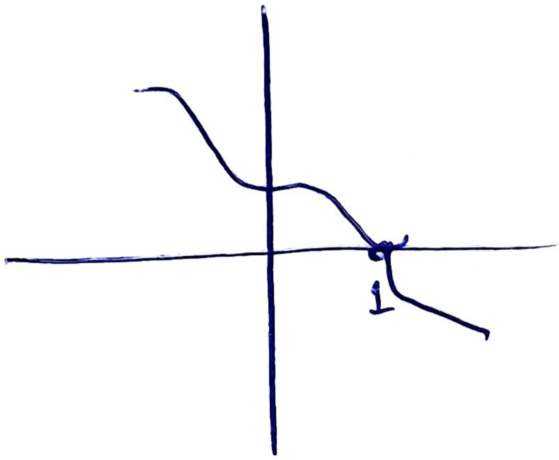
$$f(x) = \begin{cases} 2-x, & x<1 \\ x-2, & x \geq 1 \end{cases}$$

2.




$$f(x)=\begin{cases} x-2, & x<1 \\ 2-x, & x\geq 1 \end{cases}$$

3.




• Αν $x<1$ τοτε

$$|g(x)|=|x-1|$$

$$g(x)=1-x$$

$$f(x)-1=1-x$$

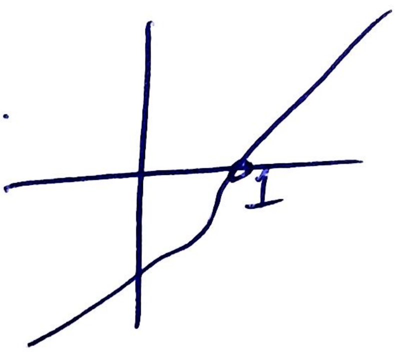
$$f(x)=2-x$$

• Αν $x\geq 1$ τοτε $|g(x)|=|x-1|$

$$g(x)=1-x$$

$$f(x)=2-x$$

4.




Αν $x<1$ τοτε $|g(x)|=|x-1|$

$$g(x)=x-1$$

$$f(x)-1=x-1$$

$$f(x)=x$$

Αν $x\geq 1 \Rightarrow g(x)=x-1$

$f(x)-1=x-1$ $f(x)=x$

Σελ 47

**2.** $f: (0,+\infty) \to \mathbb{R}$ παρ/μη.

$$f'(x) = \frac{1 - xf(x)}{x^2} \quad \forall x > 0.$$

(α) Νδο $g(x) = xf(x) - \ln x$, $x > 0$ σταθερή.

$$g'(x) = f(x) + xf'(x) - \frac{1}{x} = f(x) + x \cdot \frac{1 - xf(x)}{x^2} - \frac{1}{x}$$

$$g'(x) = f(x) + \frac{1 - xf(x)}{x} - \frac{1}{x}$$

$$g'(x) = f(x) + \frac{1}{x} - f(x) - \frac{1}{x} = 0.$$

$g'(x) = 0$. άρα $g(x) = c$.

(β). Αν $f(1) = 0$. Βρείτε τον $f(x)$

$g(x) = c$

$xf(x) - \ln x = c \longrightarrow xf(x) - \ln x = 0$

$$f(x) = \frac{\ln x}{x}$$

$\xrightarrow{x=1}$

$f(1) - 0 = c \Rightarrow \underline{0 = c}$

**3.** (a) $g(x) = \dfrac{1}{f(x)} - x^2$.

$\boxed{f'(x) = -2x f^2(x)}$

$$g'(x) = \frac{-f'(x)}{f^2(x)} - 2x$$

$$g'(x) = -\frac{-2x\cancel{f^2(x)}}{\cancel{f^2(x)}} - 2x = 2x - 2x = 0$$

$$g'(x) = 0 \Rightarrow g(x) = c.$$

(β). Αν $f(0) = 1$.

$$g(x) = c$$

$$\frac{1}{f(x)} - x^2 = c$$

$$\frac{1}{f(0)} - 0 = c$$

$$1 - 0 = c \qquad \underline{\underline{c = 1}}$$

$$\frac{1}{f(x)} - x^2 = 1 \Rightarrow \frac{1}{f(x)} = x^2 + 1$$

$$f(x) = \frac{1}{x^2+1}$$

5. $f: [0,\pi] \to \mathbb{R}$ $\quad f(0)=0.$

$$f'(x) = x\,\eta\mu x$$

Λύσ $\quad f(x) = \eta\mu x - x\,\sigma\upsilon\nu x$

Θεωρώ $\boxed{g(x) = f(x) - \eta\mu x + x\,\sigma\upsilon\nu x.}$

$$g'(x) = f'(x) - \cancel{\sigma\upsilon\nu x} + \cancel{\sigma\upsilon\nu x} + x(-\eta\mu x)$$

$$g'(x) = f'(x) - x\,\eta\mu x$$

$$g'(x) = x\,\eta\mu x - x\,\eta\mu x = 0$$

$$g(x) = c.$$

$$f(x) - \eta\mu x + x\,\sigma\upsilon\nu x = c$$

$x=0$

$$f(0) - \eta\mu 0 + 0\,\sigma\upsilon\nu 0 = c$$

$$0 = c$$

$$f(x) = \eta\mu x - x\,\sigma\upsilon\nu x$$

**6.** $f(0)=0$

$$x\left(2f(x)+xf'(x)\right)=1-f'(x).$$

ΝΔΟ $f(x)=\dfrac{x}{x^2+1}$.

$$\boxed{g(x)=(x^2+1)f(x)-x}$$

$$g'(x)=\underbrace{2xf(x)+(x^2+1)f'(x)}-1$$

$$g'(x)=1-1=0$$

$$2xf(x)+x^2f'(x)+f'(x)=1$$

$$2xf(x)+(x^2+1)f'(x)=1.$$

Άρα $g(x)=c$

$$(x^2+1)f(x)-x=c \Rightarrow (x^2+1)f(x)=x$$

$$\xrightarrow{x=0}$$

$$f(0)=c$$

$$0=c$$

$$f(x)=\frac{x}{x^2+1}$$

8. $f(0)=1$.

$$x\left(f(x)+x f'(x)\right) = -f'(x)$$

Νδο $f(x) = \dfrac{1}{\sqrt{x^2+1}}$

$$x f(x) + x^2 f'(x) + f'(x) = 0$$

$$\boxed{f'(x)(x^2+1) + x f(x) = 0.}$$

$$\boxed{\text{Θεωρώ } g(x) = \sqrt{x^2+1}\, f(x) - 1.}$$

$$g'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} f(x) + \sqrt{x^2+1}\, f'(x)$$

$$g'(x) = \frac{x f(x) + (x^2+1) f'(x)}{\sqrt{x^2+1}} = 0 \Rightarrow g(x) = c$$

$\sqrt{x^2+1}\, f(x) - 1 = c$

$x=0$

$f(0) - 1 = c$

$1 - 1 = c$

$c = 0$

$$f(x) = \frac{1}{\sqrt{x^2+1}}$$

**10.** $f'(x) + f(x)\,\varepsilon\varphi x = 0$

$$f'(x) + f(x)\,\frac{\eta\mu x}{\sigma\upsilon\nu x} = 0$$

$$\boxed{f'(x) + \frac{\eta\mu x}{\sigma\upsilon\nu x}\,f(x) = 0}$$

Μορφή

$f'(x) + g(x) f(x) = 0$

$$g(x) = \frac{\eta\mu x}{\sigma\upsilon\nu x}.$$

$$G(x) = -\ln(\sigma\upsilon\nu x)$$

$$e^{G(x)} = e^{-\ln(\sigma\upsilon\nu x)} = \frac{1}{e^{\ln\sigma\upsilon\nu x}} = \frac{1}{\sigma\upsilon\nu x}$$

$$\frac{1}{\sigma\upsilon\nu x}\,f'(x) + \frac{\eta\mu x}{\sigma\upsilon\nu^2 x}\,f(x) = 0.$$

$$\left(\frac{1}{\sigma\upsilon\nu x}\,f(x)\right)' = 0$$

$$\frac{f(x)}{\sigma\upsilon\nu x} = c \qquad \frac{x=0}{\frac{f(0)}{1} = c} \qquad c = 1$$

$f(x) = \sigma\upsilon\nu x$

**11.** $f'(x) - f(x) = \dfrac{e^x}{x}$

- $g(x) = -1$
- $G(x) = -x$
- $e^{G(x)} = e^{-x}$

$f(1) = 0$

$$e^{-x} f'(x) - e^{-x} f(x) = e^{-x} \frac{e^x}{x}$$

$$\left(e^{-x} f(x)\right)' = \frac{e^0}{x}$$

$$\left(e^{-x} f(x)\right)' = \frac{1}{x}$$

$$\left(e^{-x} f(x)\right)' = (\ln x)'$$

$$e^{-x} f(x) = \ln x + c, \quad \Rightarrow e^{-x} f(x) = \ln x$$

$\underline{x=1}$

$e^{-1} f(1) = \ln 1 + c$

$\underline{\underline{c=0}}$

$$\frac{f(x)}{e^x} = \ln x$$

$$f(x) = e^x \ln x$$

**13.**

$f(0) = \ln 2$

$f'(x) = e^{x - f(x)}$

$$f'(x) = \frac{e^x}{e^{f(x)}}$$

$$e^{f(x)} f'(x) = e^x$$

$$\left(e^{f(x)}\right)' = \left(e^x\right)'$$

$$e^{f(x)} = e^x + c \longrightarrow e^{f(x)} = e^x + 1$$

$\underline{x = 0}$

$e^{f(0)} = e^0 + c$

$e^{\ln 2} = 1 + c$

$2 = 1 + c$

$\underline{c = 1}$

$$f(x) = \ln\left(e^x + 1\right)$$

**15.** $f''(x) = g''(x)$

$f'(x) = g'(x) + c$

$\underline{\underline{x=1}}$

$f'(1) = g'(1) + c$

$g'(1) = g'(1) + c$

$\underline{\underline{c = 0}}$

$f'(x) = g'(x)$

$f(x) = g(x) + c$

$\underline{x = 0}$

$f(0) = g(0) + c$

$f(0) - g(0) = c$

$3 = c$

$f(x) = g(x) + 3$

$f(1) = g(1) + 3$ ✓

**17.** $f'(x) = f(x)(1-\sigma\varphi x) \qquad x \in (0,\pi)$

$$f'(x) - (1-\sigma\varphi x) f(x) = 0$$

$$f'(x) - \left(1 - \frac{\sigma\upsilon\nu x}{\eta\mu x}\right) f(x) = 0$$

$$\boxed{g(x) = -1 + \frac{\sigma\upsilon\nu x}{\eta\mu x}}$$

$$G(x) = \ln(\eta\mu x) - x$$

$$e^{G(x)} = e^{\ln \eta\mu x - x} = \frac{e^{\ln \eta\mu x}}{e^{x}} = \frac{\eta\mu x}{e^{x}}$$

$$\frac{\eta\mu x}{e^x} f'(x) - \frac{\eta\mu x}{e^x}\left(1 - \frac{\sigma\upsilon\nu x}{\eta\mu x}\right) f(x) = 0$$

$$\left(\frac{\eta\mu x}{e^x} f(x)\right)' = 0$$

$$\frac{\eta\mu x}{e^x} f(x) = c$$

$$\frac{1}{e^{\pi/2}} \cancel{f(\pi/2)} = c$$

$$c = 1$$

$$f(x) = \frac{e^x}{\eta\mu x}$$

## Β' τρόπος

$$f'(x) = f(x)(1 - σφx)$$

(α) $g(x) = f(x)\, ημx$

$$g'(x) = f'(x)\, ημx + f(x)\, συνx$$

$$g'(x) = f(x)(1-σφx)\, ημx + f(x)\, συνx$$

$$g'(x) = f(x)\left((1-σφx)\, ημx + συνx\right)$$

$$g'(x) = f(x)\left(ημx - \frac{συνx}{\cancel{ημx}}\cancel{ημx} + συνx\right)$$

$$g'(x) = f(x)\, ημx$$

$$g'(x) = g(x)$$

$$\boxed{\begin{array}{l} h(x) = -1 \\ H(x) = -x \\ e^{H(x)} = e^{-x} \end{array}}$$

$$g'(x) - g(x) = 0 \Rightarrow e^{-x} g'(x) - e^{-x} g(x) = 0$$

$$\left(e^{-x} g(x)\right)' = 0$$

$$e^{-x} g(x) = c$$

$$\frac{g(x)}{e^{x}} = c \qquad \underline{g(x) = ce^{x}}$$

(B) $g(x) = ce^x$

$$f(x) \cdot \sin x = ce^x$$

$$x = \frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) \cdot 1 = ce^{\pi/2}$$

$$e^{\frac{\pi}{2}} = ce^{\pi/2}$$

$$c = 1$$

$$f(x) = \frac{e^x}{\sin x}$$

**29.** (B) $f(x) = \begin{cases} \dfrac{x\ln x}{x-1} & , 0<x \neq 1 . \\ 1 & , x=1 . \end{cases}$

$$f'(x) = \frac{(x\ln x)'(x-1) - x\ln x\,(x-1)'}{(x-1)^2}$$

$$f'(x) = \frac{(\ln x+1)(x-1) - x\ln x}{(x-1)^2}$$

$$f'(x) = \frac{\cancel{x\ln x} - \ln x + x - 1 - \cancel{x\ln x}}{(x-1)^2}$$

$$f'(x) = \frac{x-\ln x-1}{(x-1)^2} \geqslant 0 \qquad f \nearrow .$$

. $\ln x \leq x-1$

$0 \leq x - \ln x - 1$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x\ln x}{x-1} = \lim_{x\to 1} \frac{\ln x+1}{1} = 1$$

$f(1)=1$ ✓

**29.** (a) $f(x) = \begin{cases} \dfrac{xe^x}{e^x-1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$f'(x) = \frac{(x\cdot e^x)'\cdot(e^x-1) - x\cdot e^x\cdot(e^x-1)'}{(e^x-1)^2} = \frac{(e^x+e^x\cdot x)(e^x-1) - x\cdot e^x\cdot e^x}{(e^x-1)^2}$$

$$= \frac{[e^x(x+1)(e^x-1) - xe^{2x}]}{(e^x-1)^2} = e^x\,\frac{(x+1)(e^x-1) - xe^x}{(e^x-1)^2}$$

$$f'(x) = e^x\,\frac{\cancel{xe^x} + 1 + e^x - 1 - \cancel{xe^x}}{(e^x-1)^2} = \frac{e^{2x}}{(e^x-1)^2} > 0$$

$f \nearrow$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{xe^x}{e^x-1} = \lim_{x\to 0}\frac{e^x+xe^x}{e^x} =$$

$$= \lim_{x\to 0}\frac{1+x}{1} = 1.$$

$f(0) = 1$ Συνεχής στο 0.

**26.** (α) $f(x) = \sqrt{x^2+1} + x$

ΠΕΔ $x^2+1 \geq 0$ που ισχύει!

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}} + 1 = \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} > 0$$

$\longrightarrow x+\sqrt{x^2+1} = 0$ $\qquad f \nearrow$

$\sqrt{x^2+1} = -x$

1. Αν $x \geq 0$ τότε αδύνατη

2. Αν $x < 0$ τότε $x^2+1 = x^2$

$1 = 0$ Αδύνατη.

Άρα $\underline{x+\sqrt{x^2+1} > 0}$ ή $x+\sqrt{x^2+1} < 0$

**27.** a) $f(x) = \dfrac{1}{x^2} - \dfrac{1}{x-4}$

$x^2 \neq 0 \Leftrightarrow \boxed{x \neq 0}$

$x - 4 \neq 0 \Leftrightarrow x \neq 4$

$D_f = (-\infty, 0) \cup (0, 4) \cup (4, +\infty)$

$$f'(x) = \frac{(1)' \cdot x^2 - 1 \cdot (x^2)'}{x^4} - \frac{(1)' (x-4) - 1 \cdot (x-4)'}{(x-4)^2} = \frac{-2x}{x^4} - \frac{-1}{(x-4)^2}$$

$$= \frac{-2x(x-4)^2 + x^4}{x^4 \cdot (x-4)^2} = \frac{x^4 - 2x \cdot (x^2 - 8x + 16)}{x^4 (x-4)^2} = \frac{x^4 - 2x^3 + 16x^2 - 32x}{x^4 (x-4)^2}$$

$$= \frac{x(x^3 - 2x^2 + 16x - 32)}{x^4 (x-4)^2} = \frac{x^2(x-2) + 16(x-2)}{x^3 (x-4)^2}$$

$$= \frac{(x-2)(x^2+16)}{x^3 (x-4)^2}$$

$\oplus$ $\oplus$

| $x$ | | 0 | | 2 | | 4 | |
|---|---|---|---|---|---|---|---|
| $x-2$ | $-$ | ‖ | $-$ | | $+$ | ‖ | $+$ |
| $x^3$ | $-$ | ‖ | $+$ | | $+$ | ‖ | $+$ |
| $f'$ | $+$ | ‖ | $-$ | | $+$ | ‖ | $+$ |
| $f$ | $\nearrow$ | ‖ | $\searrow$ | | $\nearrow$ | ‖ | $\nearrow$ |

**26.** (γ) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

πρέπει $x^2+1 \geq 0$ που ισχύει
και
$\sqrt{x^2+1} \neq 0$
$x^2+1 \neq 0$ $x \in \mathbb{R}$

$$f'(x) = \frac{(x)'(\sqrt{x^2+1}) - x(\sqrt{x^2+1})'}{(\sqrt{x^2+1})^2}$$

$$f'(x) = \frac{\sqrt{x^2+1} - x\left(\frac{1}{2\sqrt{x^2+1}}\right)(x^2+1)}{(\sqrt{x^2+1})^2}$$

$\cancel{f'(x) =}$

$$f'(x) = \frac{\sqrt{x^2+1} - x\left(\frac{x}{\sqrt{x^2+1}}\right)}{(\sqrt{x^2+1})^2}$$

$$f'(x) = \frac{(\sqrt{x^2+1})(\sqrt{x^2+1}) - x^2}{(\sqrt{x^2+1})^2}$$

$$f'(x) = \frac{\sqrt{x^2+1}^2 - x^2}{(\sqrt{x^2+1})^2} = \frac{1}{\sqrt{x^2+1}^2} = \frac{1}{x^2+1} > 0$$

(+)

$f'(x) = 0$
$-x^2 = 0$
$x = 0$

$\cancel{f'(x)}$

| $x$ | $-\infty$ | $0$ | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | $+$ | | $-$ |
| $f(x)$ | $\nearrow$ | | $\searrow$ |

$f \nearrow$

**26.** (B) $f(x) = x\sqrt{1-x^2}$

$1-x^2 \geq 0$

$x^2 \leq 1$

$-1 \leq x \leq 1$

αρα AF $[-1,1]$

$f'(x) = (x)' \cdot \sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} (1-x^2)'$

$f'(x) = \sqrt{1-x^2} + \frac{x}{2\sqrt{1-x^2}} \cdot (-2x)$

$f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$

$f'(x) = \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$

$1-2x^2 = 0$

$x^2 = \frac{1}{2}$ $\quad x = \pm \frac{\sqrt{2}}{2}$

| $x$ | $-1$ | | $-\frac{\sqrt{2}}{2}$ | | $\frac{\sqrt{2}}{2}$ | | $-1$ |
|---|---|---|---|---|---|---|---|
| $f'$ | | $-$ | 0 | $+$ | 0 | $-$ | |
| $f$ | | ↘ | | ↗ | | ↘ | |

**27.** (γ) $f(x) = \ln(\sqrt{x^2+1} - x)$.

πρέπει $x^2+1 \geq 0$ που ισχύει

και $\sqrt{x^2+1} - x > 0$

$\sqrt{x^2+1} > x$

$D_f = \mathbb{R}$.

αν $x \geq 0$ : $\cancel{x^2} + 1 > \cancel{x^2} \Rightarrow 1 > 0$ που ισχύει

αν $x < 0$ : $\sqrt{x^2+1} > x$ ισχύει.

$$f'(x) = \frac{1}{\sqrt{x^2+1} - x}(\sqrt{x^2+1} - x)' = \frac{1}{\sqrt{x^2+1} - x} \cdot \left(\frac{2x}{2\sqrt{x^2+1}} - 1\right)$$

$$= \frac{1}{\sqrt{x^2+1} - x} \cdot \frac{2x - 2\sqrt{x^2+1}}{2\sqrt{x^2+1}}$$

$$= \cancel{\frac{1}{\sqrt{x^2+1} - x}} \cdot \frac{\cancel{x - \sqrt{x^2+1}}}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{x^2+1}} < 0$$

$f \searrow$

**27.** $f(x) = \ln\left(\dfrac{x}{1-x}\right)$

πρπ $1-x \neq 0$ και $\dfrac{x}{1-x} > 0$

$x \neq 1$

| $x$ | | $0$ | | $1$ | |
|---|---|---|---|---|---|
| $x$ | $-$ | $0$ | $+$ | | $+$ |
| $1-x$ | $+$ | | $+$ | $0$ | $-$ |
| $\dfrac{x}{1-x}$ | $-$ | | $+$ | | $-$ |

$x \in (0, 1)$

$$f'(x) = \frac{1}{\frac{x}{1-x}}\left(\frac{x}{1-x}\right)' = \frac{\cancel{1-x}}{x} \cdot \frac{1-x+x}{(1-x)^{\cancel{2}}}$$

$$f'(x) = \frac{1}{x(1-x)} > 0$$

$+$ $+$

$f \nearrow$

# Επομενο Μαθημα

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