Ενοτητα 23

30. $f: [0,+\infty) \to \mathbb{R}$

f συν+παρ

f' συν+παρ

f''

$f(0) = f'(0) = 0$

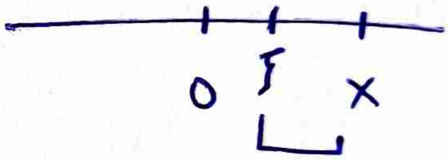
$f''(x) > 0$

$$g(x) = \begin{cases} \dfrac{f(x)}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

Νδο $g \uparrow$

$$g'(x) = \frac{f'(x)x - f(x)}{x^2}$$

Αρκει νδο $f'(x)x - f(x) > 0$



$$f'(\xi) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

$\xi < x$

$f''(x) > 0$

$f' \uparrow$

$f'(\xi) < f'(x)$

$\frac{f(x)}{x} < f'(x)$

$f(x) < x f'(x)$

$x f'(x) - f(x) > 0$

Ενοτητα 26

28. ⓐ $f(x) = e^x + \frac{x^3}{6} - \frac{x^2}{2} - \eta\mu x$

$f'(x) = e^x + \frac{1}{6}\, 3x^2 - \frac{1}{2}\, 2x - \sigma\upsilon\nu x$

$f'(x) = e^x + \frac{1}{2} x^2 - x - \sigma\upsilon\nu x$ $\qquad f'(0) = 0$

$f''(x) = e^x + x - 1 + \eta\mu x$ $\qquad f''(0) = 0$

$f'''(x) = e^x + \underbrace{1 + \sigma\upsilon\nu x}_{+} > 0$

$-1 \leq \sigma\upsilon\nu x \leq 1$

$\boxed{0 \leq \sigma\upsilon\nu x + 1 \leq 2}$

x		0	
f'''	$+$		$+$
f''	↗ $-$		$+$ ↗
f'	↘ $+$	0	$+$ ↗
f	↗		↗

$x < 0 \Rightarrow f''(x) < f''(0) \Rightarrow f''(x) < 0$

$x > 0 \Rightarrow f''(x) > f''(0) \Rightarrow f''(x) > 0$

$f'(x) \geq f'(0)$

$f'(x) \geq 0$

(B) $f(x) = \dfrac{\ln x}{x-2}$, $x \in (0,2) \cup (2,+\infty)$.

$$f'(x) = \frac{\frac{1}{x}(x-2) - \ln x}{(x-2)^2} = \frac{x-2-x\ln x}{x(x-2)^2}$$

$$\boxed{\varphi(x) = x-2-x\ln x}$$

$\varphi'(x) = 1 - \ln x - 1 = -\ln x$

$\rightarrow \varphi'(x)=0 \Rightarrow -\ln x = 0$ $\quad \underline{\underline{x=1}}$

x	0		1		2		$+\infty$
φ'		$+$	0	$-$	$\|\|$	$-$	
φ		$\nearrow$		$\searrow$	$\|\|$	$\searrow$	
f'		$-$		$-$	$\|\|$	$-$	
f		$\searrow$		$\searrow$	$\|\|$	$\searrow$	

$\varphi(x) \leq \varphi(1)$

$\varphi(x) \leq -1$

29. (a) $f(x) = \begin{cases} \dfrac{\ln x}{x-1}, & 0 < x \neq 1 \\ 1, & x = 1 \end{cases}$

$$f'(x) = \frac{\frac{1}{x}(x-1) - \ln x}{(x-1)^2}$$

$$\boxed{\varphi(x) = \frac{x-1}{x} - \ln x} \qquad \varphi(1) = 0$$

$$\varphi'(x) = \frac{x - x + 1}{x^2} - \frac{1}{x} = \frac{1}{x^2} - \frac{1}{x} = \frac{1-x}{x^2}$$

x	0	1	
φ'	$+$		$-$
φ	$\nearrow -$		$- \searrow$
f'	$-$		$-$
f	$\searrow$		$\searrow$

$\varphi(x) \leq \varphi(1) \Rightarrow \varphi(x) \leq 0$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x}}{} = 1$$

$f(1) = 1$ Συνεχής στο 1.

(B) $$f(x) = \begin{cases} x + \dfrac{1-συνx}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0} f(x) = 0 \\ f(0) = 0 \end{array}\right\} \text{Συνεχής στο } 0.$$

$$f'(x) = 1 + \frac{ημx \cdot x - (1-συνx)}{x^2} = \frac{x^2 + xημx - 1 + συνx}{x^2}$$

$$\boxed{φ(x) = x^2 + xημx - 1 + συνx} \qquad φ(0) = 0$$

$$φ'(x) = 2x + \cancel{ημx} + xσυνx - \cancel{ημx}$$

$$φ'(x) = x\underbrace{(2 + συνx)}_{(+)}$$

x		0	
$φ'$	$-$		$+$
$φ$	↘ $+$		$+$ ↗
f'	$+$		$+$
f	↗		↗

$φ(x) \geq φ(0)$

$φ(x) \geq 0$

33.

$f(1) = 2$

$f'(1) = 0$

$f''(x) > 0$

Ⓐ νδο $f(x) \geq 2$

x		1	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$\underline{f(x) \geq 2}$

Ⓑ Nδο η $\dfrac{f(\alpha)-2}{x-2} + \dfrac{f(\beta)-2}{x} = 0$ $(0,2)$ μον. λύση

$g(x) = x\,(f(\alpha)-2) + (x-2)(f(\beta)-2)$

$g(0) = -2\,(f(\beta)-2) < 0$ $\qquad f(\alpha) > 2$

$g(2) = 2\,(f(\alpha)-2) > 0$ $\qquad f(\alpha)-2 > 0$

$g(0)\,g(2) < 0$ Bolzano $\exists\,\xi \in (0,2)$ $\qquad f(\beta) > 2$

τ.ω $g(\xi) = 0$. $\qquad f(\beta)-2 > 0$

$$H \quad g(x) = x\underbrace{\left(f(\alpha)-2\right)}_{\text{νουμερο}} + (x-2)\underbrace{\left(f(\beta)-2\right)}_{\text{νουμερα}}$$

Ειναι πολυωνυμο 1ου βαθμου

αρα εχει το πολυ 1 λυση.

Εχω ηδη βρει 1 με

το Bolzano η οποια

ειναι μοναδικη.

31. $f(x) = 2e^{x-2} - x^2 + 2x - 2$

(α) $f'(x) = 2e^{x-2} - 2x + 2$

$f'(x) = 2\left(\underset{(+)}{e^{x-2} - x + 1}\right) \geq 0 \quad f \nearrow$

- $e^x \geq x + 1$

$e^{x-2} \geq x - 2 + 1$

$e^{x-2} \geq x - 1$

$e^{x-2} - x + 1 \geq 0$

(Β) $g(x) = 2e^{x-2}$

$h(x) = x^2 - 2x + 2$

Να εχ λυση
$$\begin{cases} g(x) = h(x) \Rightarrow 2e^{x-2} = x^2 - 2x + 2 \\ g'(x) = h'(x) \Rightarrow 2e^{x-2} = 2x - 2 \end{cases}$$

$$\begin{cases} 2e^{x-2} - x^2 + 2x - 2 = 0 \\ 2e^{x-2} - 2x + 2 = 0 \end{cases} \Rightarrow \begin{cases} f(x) = 0 \quad x = 2 \\ f'(x) = 0 \quad x = 2 \end{cases}$$

14. $f(x) = \ln x$

$g(x) = \sigma\upsilon\nu\, x$

Ενότητα 27

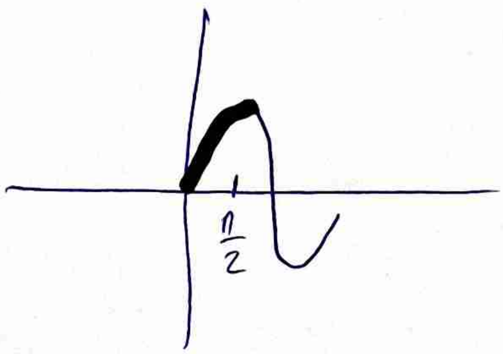
$f(x) = g(x) \qquad x_0 \in (0, \frac{\pi}{2})$

$\ln x = \sigma\upsilon\nu\, x$

$\underbrace{\ln x - \sigma\upsilon\nu\, x}_{h(x)} = 0$

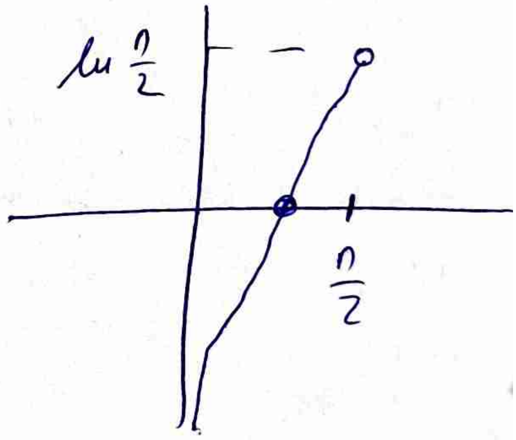
$h'(x) = \frac{1}{x} + \eta\mu\, x > 0$

$\quad (+) \quad (+)$



$h \nearrow$

$h \nearrow$



$\lim_{x \to 0^+} h(x) = -\infty$

$\lim_{x \to \frac{\pi}{2}^-} h(x) = \ln\frac{\pi}{2} > 0$

$\Sigma T_h = (-\infty, \ln\frac{\pi}{2}) \ni 0$

- h συνεχής
- $h \nearrow$
- $\Sigma T_h = (-\infty, \ln\frac{\pi}{2})$

Το $0 \in \Sigma T_h$ οπότε

$\exists!\, \xi$ τ.ω. $h(\xi) = 0$

15. $f(x) = x\ln x + x^2 - 3x + 2$

$$f(x) = f(e)$$

$f'(x) = \ln x + x\,\frac{1}{x} + 2x - 3$

$f'(x) = \ln x + 2x - 2 \qquad f'(1) = 0$

$f''(x) = \frac{1}{x} + 2 > 0$

x	0		1		$+\infty$
f''		$+$		$+$	
f'		$\nearrow$ $-$	0	$+$ $\nearrow$	
f		$\searrow$		$\nearrow$	

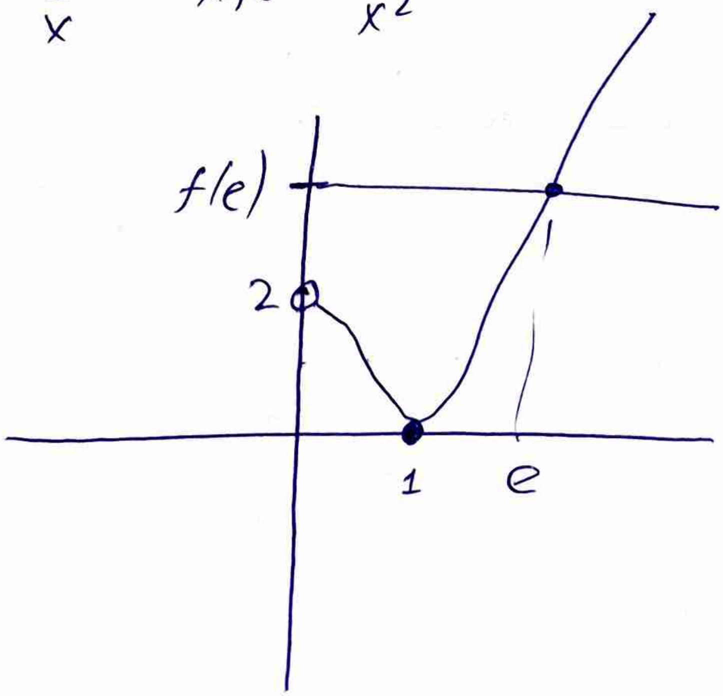
$f(x) \geq f(1)$

$f(x) \geq 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cancel{x\ln x}^{\,0} + x^2 - 3x + 2 = 2$

$\to \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$

$\lim_{x\to +\infty} f(x) = +\infty$



$f(e) = e\ln e + e^2 - 3e + 2$

$f(e) = e + e^2 - 3e + 2 = e^2 - 2e + 2$

$f(e) = e^2 - 2e + 2$

Έστω $f(e) \leq 2$

$e^2 - 2e + \cancel{2} \leq \cancel{2}$

$e^2 - 2e \leq 0$

$e(e-2) \leq 0$

$+ \quad +$

Άτοπο.

Άρα $f(e) > 2$.

Αν $x<1$

$\Sigma T_f = [0, 2)$

το $f(e) > 2$

άρα η εξίσωση

$f(x) = f(e)$

αδύνατη.

$x > 1$

$f(x) = f(e)$

f 1-1

$x = e$

22. $f(x) = x^4 - 6x^2 + 4x$

$\varepsilon: y = 8x$

$y - f(x_0) = f'(x_0)(x - x_0) \parallel y = 8x$

$f'(x_0) = 8$

πόσες λύσεις έχει η εξίσωση αυτή;

$f'(x) = 4x^3 - 12x + 4$

Άρα $f'(x) = 8 \Rightarrow 4x^3 - 12x + 4 = 8$

$4x^3 - 12x - 4 = 0$

$x^3 - 3x - 1 = 0$

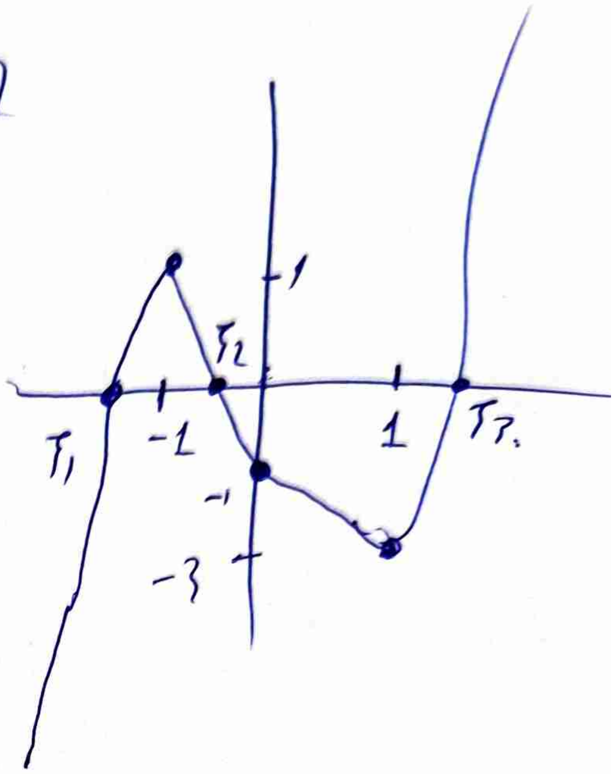
$\varphi(x) = 0$

πλήθος ριζών.

$\varphi(x) = x^3 - 3x - 1$

$\varphi'(x) = 3x^2 - 3 = 3(x^2-1)$

x		-1		1	
φ'	$+$		$-$		$+$
φ	$-\infty \nearrow$		$\searrow$		$\nearrow$



$\lim_{x\to-\infty} \varphi(x) = -\infty$

$\varphi(-1) = 1$

$\varphi(1) = -3$

$\lim_{x\to+\infty} \varphi(x) = +\infty$

$x < -1$

- φ συνεχής
- $\varphi \nearrow$
- $\Sigma T\varphi = (-\infty, 1)$

Το $0 \in \Sigma T\varphi$ άρα

$\exists!\ \xi_1$ τ.ω $\varphi(\xi_1)=0$

$-1 \leq x \leq 1$

- φ συνεχής
- $\varphi \searrow$
- $\Sigma T\varphi = [-3, 1]$

Το $0 \in \Sigma T\varphi$

άρα $\exists\ \xi_2$ τ.ω

$\varphi(\xi_2) = 0$

$x > 1$

- φ συνεχής
- $\varphi \nearrow$
- $\Sigma T\varphi = (-3, +\infty)$

Το $0 \in \Sigma T\varphi$

άρα

$\exists!\ \xi_3$ τ.ω

$\varphi(\xi_3) = 0$

25. $f(x) = (x+1)\ln(x+1)$

$$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(0, -1)$$

$$-1 - f(x_0) = f'(x_0)(0 - x_0)$$

$$-1 - f(x) = -x f'(x)$$

$$1 + f(x) = x f'(x)$$

$$1 + (x+1)\ln(x+1) = x\left(\ln(x+1) + 1\right)$$

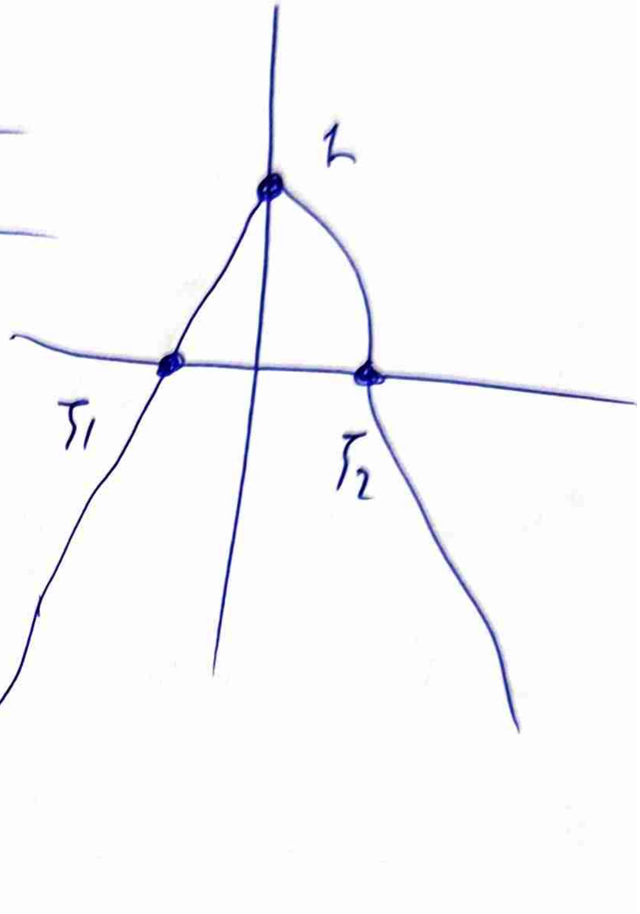
$$1 + \cancel{x\ln(x+1)} + \ln(x+1) = \cancel{x\ln(x+1)} + x$$

$$1 + \ln(x+1) = x$$

$$\underbrace{1 + \ln(x+1) - x}_{\varphi(x)} = 0.$$

$$\varphi'(x) = \frac{1}{x+1} - 1 = \frac{1-x-1}{x+1} = \frac{-x}{x+1}$$

x	-1		0	
φ'		$+$		$-$
φ	$-\infty$	$\nearrow$	1	$\searrow$ $-\infty$



$$\lim_{x \to -1^+} \varphi(x) = -\infty$$

$$\varphi(0) = 1$$

$$\lim_{x \to +\infty} \varphi(x) = \lim_{x \to +\infty} 1 + \ln(x+1) - x =$$

$$= \lim_{x \to +\infty} x\left(\cancel{\frac{1}{x}} + \cancel{\frac{\ln(x+1)}{x}} - 1\right) = -\infty.$$

$$\to \lim_{x \to +\infty} \frac{\ln(x+1)}{x} = \lim_{x \to +\infty} \frac{1}{x+1} = 0$$

$x < 0$

- φ συνεχής
- $\varphi \nearrow$
- $\Sigma T \varphi = (-\infty, 1]$

Το $0 \in \Sigma T \varphi$

άρα $\exists\, \xi_1$ τ.ω

$\varphi(\xi_1) = 0$

$x \geq 0$

- φ συνεχής
- $\varphi \searrow$
- $\Sigma T \varphi = (-\infty, 1]$

Το $0 \in \Sigma T \varphi$

άρα $\exists\, \xi_2$ τ.ω

$\varphi(\xi_2) = 0$

26. $f(x) = e^x$ $\quad f'(x) = e^x$

$g(x) = 2\sqrt{x}$ $\quad g'(x) = \frac{1}{\sqrt{x}}$

$$\begin{cases} f'(\alpha) = g'(\beta) \\ f(\alpha) - \alpha f'(\alpha) = g(\beta) - \beta g'(\beta) \end{cases}$$

$$\begin{cases} e^{\alpha} = \frac{1}{\sqrt{\beta}} \\ e^{\alpha} - \alpha e^{\alpha} = 2\sqrt{\beta} - \beta \cdot \frac{1}{\sqrt{\beta}} \end{cases} \Rightarrow \sqrt{\beta} = e^{-\alpha} \Rightarrow \beta = e^{-2\alpha}$$

$$e^{\alpha}(1-\alpha) = 2e^{-\alpha} - e^{-2\alpha} e^{\alpha}$$

$$e^{\alpha}(1-\alpha) = 2e^{-\alpha} - e^{-\alpha}$$

$$e^{\alpha}(1-\alpha) = e^{-\alpha}$$

$$e^{2\alpha}(1-\alpha) = 1$$

$$e^{2\alpha}(1-\alpha) - 1 = 0$$

$$\boxed{\varphi(x) = e^{2x}(1-x) - 1}$$

$$\varphi'(x) = 2e^{2x}(1-x) - e^{2x}$$

$$\varphi'(x) = e^{2x}(2-2x-1)$$

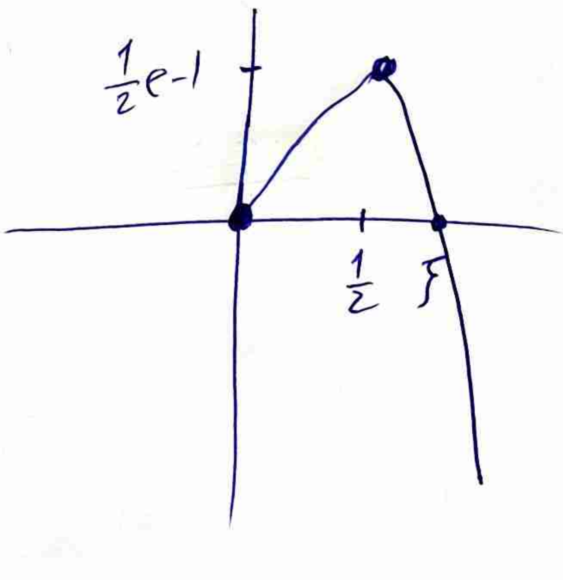
$$\varphi'(x) = e^{2x}(1-2x)$$

x	0		$1/2$	
φ'		$+$		$-$
φ	0	$\nearrow$	θ	$\searrow$ $-\infty$

$\varphi(0) = 1$

$\varphi\left(\frac{1}{2}\right) = \frac{1}{2}e - 1 > 0$

$\lim\limits_{x \to +\infty} \varphi(x) = -\infty$



Αν $0 \leq \alpha \leq \frac{1}{2}$

$\varphi(\alpha) = 0$

$\varphi(\alpha) = \varphi(0)$

φ 1-1

$\alpha = 0$

$\alpha > \frac{1}{2}$

- φ συνεχής
- $\varphi \downarrow$
- $\Sigma T\varphi = (-\infty, \frac{1}{2}c - 1)$.

Το $0 \in \Sigma T\varphi$

άρα $\exists!\, \alpha_1$

τ.ω. $\varphi(\alpha_1) = 0$

2. (a) $f(x) = \dfrac{x-1}{x-2}, \quad x \neq 2$

Ενοτητα 32

$$f'(x) = \frac{x-2-(x-1)}{(x-2)^2} = \frac{-1}{(x-2)^2} < 0$$

$f \downarrow$

x		2	
$f'(x)$	$\searrow$		$\searrow$

$$f''(x) = \frac{-(-1)\,2(x-2)}{(x-2)^4} = \frac{2}{(x-2)^3}$$

x		2	
f''	$-$		$+$
f	$\cap$		$\cup$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-1}{x-2} = -\infty$$

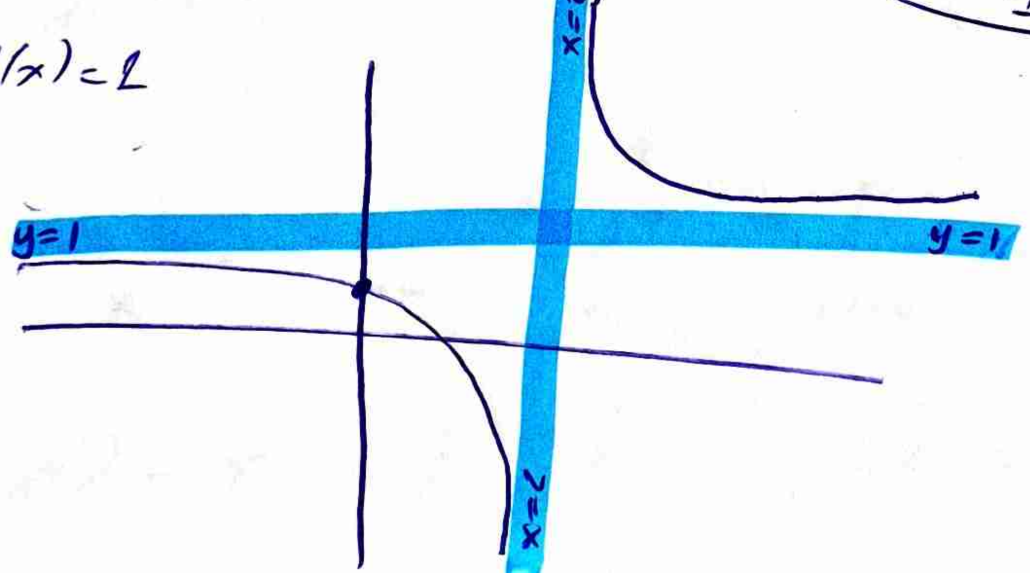
$\varepsilon_1 : x = 2$
κατακόρυφη

$$\lim_{x\to 2^+} f(x) = +\infty$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x-1}{x-2} = 1$$

$\varepsilon_2 : y = 1$
οριζόντια
$\pm\infty$.

$$\lim_{x\to +\infty} f(x) = 1$$



(γ) $f(x) = x - \frac{1}{x-1}$, $x \neq 1$.

$$f'(x) = 1 - \frac{-1}{(x-1)^2} = 1 + \frac{1}{(x-1)^2} > 0 \quad f \nearrow$$

$$f''(x) = \frac{-2(x-1)}{(x-1)^4} = \frac{-2}{(x-1)^3}$$

x		1	
f''	$+$	‖	$-$
f	∪	‖	∩

$$\lim_{x \to 1^-} f(x) = +\infty$$

$$\lim_{x \to 1^+} f(x) = -\infty$$

Ε.Α. $x=1$ κατακόρυφη

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

Δεν έχω οριζόντια

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x - \frac{1}{x-1}}{x} = \lim_{x \to +\infty} 1 + \frac{1}{(x-1)^2} = 1$$

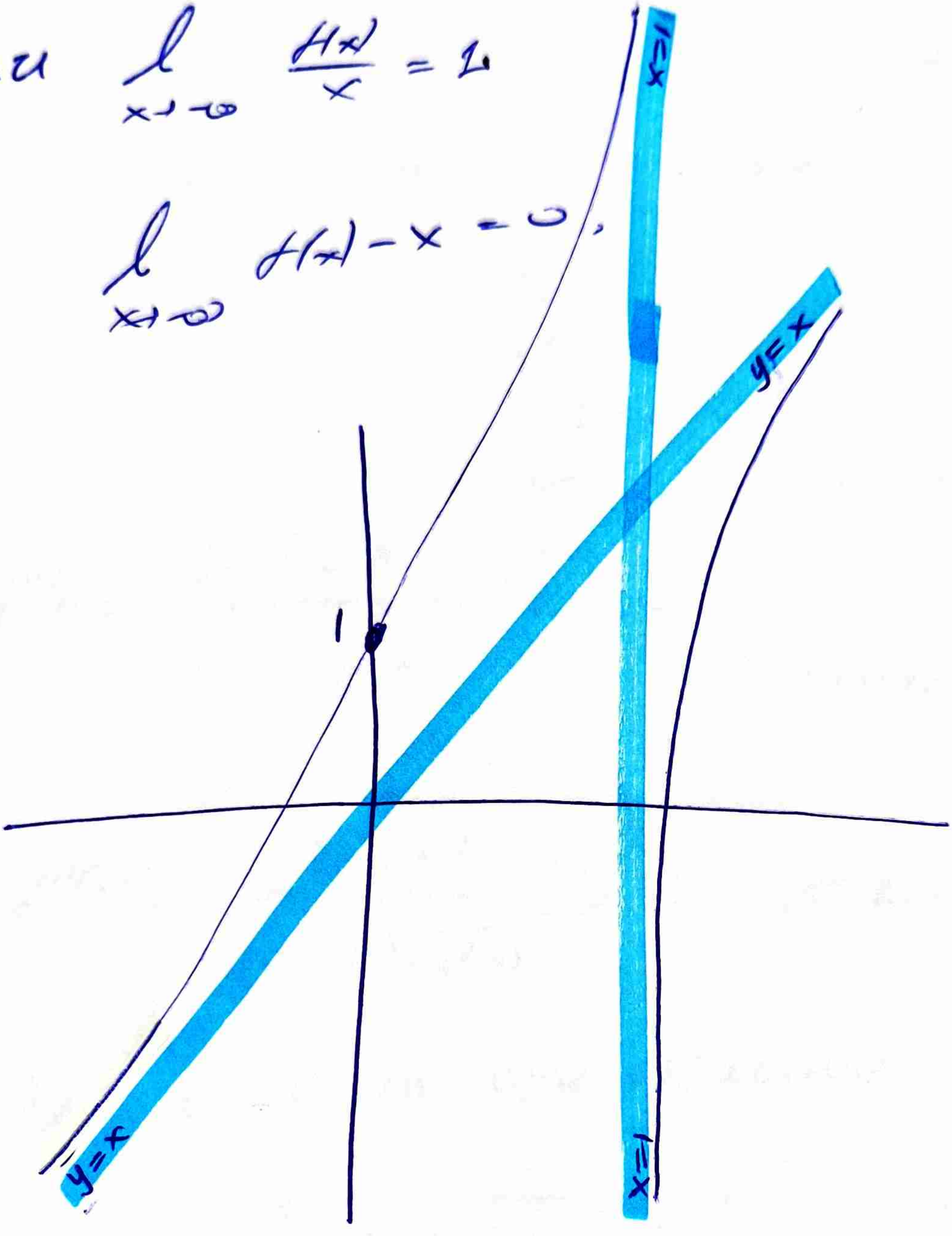
$$\lim_{x \to +\infty} f(x) - x = \lim_{x \to +\infty} \cancel{x} - \frac{1}{x-1} - \cancel{x} = 0$$

$y = x$

$+\infty$
$-\infty$

yoki $\lim\limits_{x \to -\infty} \frac{f(x)}{x} = 1$

$\lim\limits_{x \to -\infty} f(x) - x = 0$,



(ε) $f(x)=\sqrt{x^2+1}-x \quad , x\in\mathbb{R}$

$$f'(x)=\frac{2x}{2\sqrt{x^2+1}}-1=\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}<0$$

$f \downarrow$

- $\sqrt{x^2+1}-x>0 \Rightarrow \sqrt{x^2+1}>x$

αν $x>0$

$\cancel{x^2}+1>\cancel{x^2}$

$1>0$ ✓

αν $x\leq 0$

✓

$$f''(x)=\frac{\sqrt{x^2+1}-x\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x^2+1}=\frac{x^2+1-x}{(x^2+1)\sqrt{x^2+1}}$$

$\Delta<0$

$$f''(x)=\frac{x^2-x+1}{(x^2+1)\sqrt{x^2+1}}>0$$ f κυρτή.

Αφού $D_f=\mathbb{R}$ δεν έχει κατακόρυφη.

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\sqrt{x^2+1}-x=\lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x}=0$$

$\varepsilon_1: y=0$ $+\infty$ οριζόντια.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{x^2+1} - x = +\infty$$

Δεν εχω οριζοντια

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+1} - x}{x} =$$

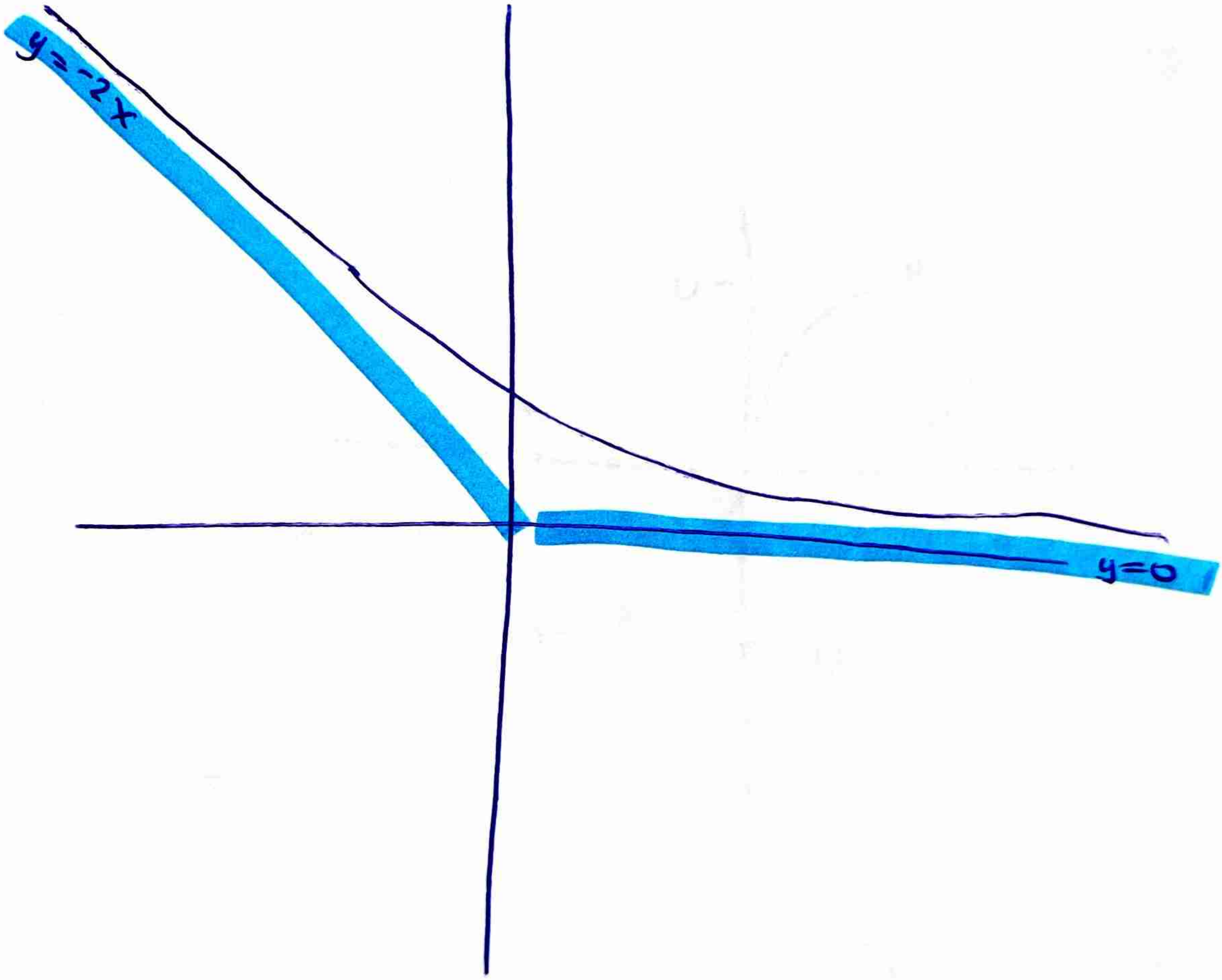
$$= \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{1}{x^2}} - x}{x} = -2$$

$$\lim_{x\to-\infty} f(x)+2x = \lim_{x\to-\infty} \sqrt{x^2+1} - x + 2x$$

$$= \lim_{x\to-\infty} \sqrt{x^2+1} + x = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1} - x} = 0$$

$$y = -2x$$

Πλαγια $-\infty$



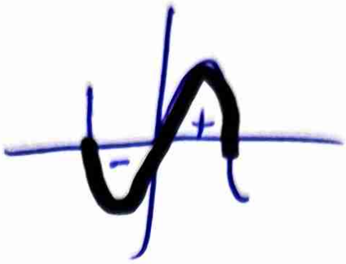
y = -2x
y=0

62) $f(x) = x + ημx \quad , \quad x \in [-π, π]$

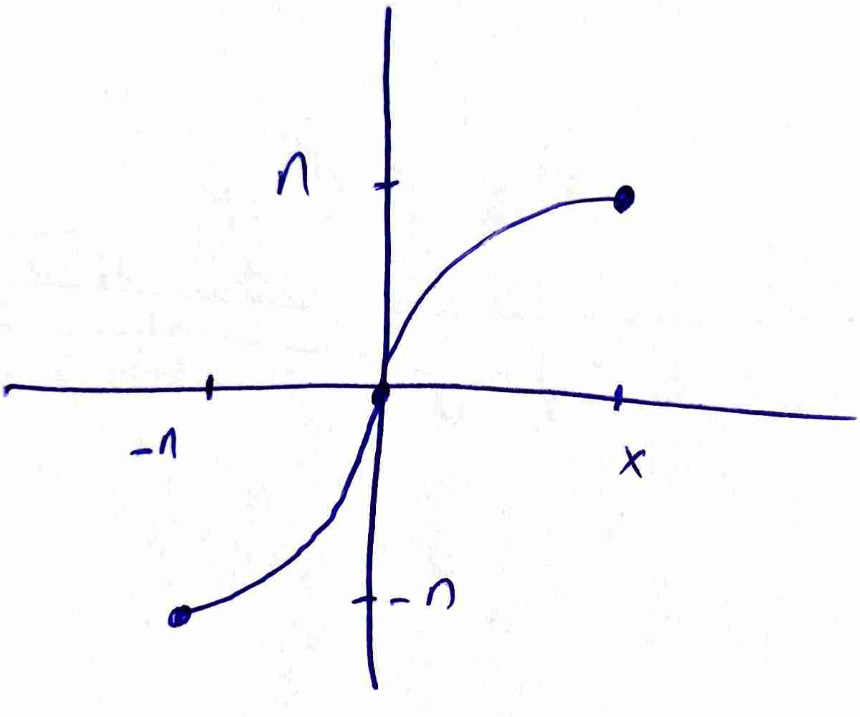
$f'(x) = 1 + συνx \geq 0 \quad f \uparrow$

Άρα $x \in [-π, π]$ δεν έχει ασύμπτωτη.

$f''(x) = -ημx$



x	$-π$	0	$π$
f''	$+$		$-$
f	$\cup$		$\cap$



Επομενο Μαθημα

24

69

70

71.