34. $f(x) = \frac{x^4}{4} + \frac{x^2}{2} - 1$

Νδο $\exists!\ x_0 \in (0,1)$ τ.ω η εφαπτομενη

στο x_0 $\perp$ $\varepsilon: x+y-3=0$.

$$y - f(x_0) = f'(x_0)(x-x_0) \perp y = -x+3$$

Πρεπει $f'(x_0) \cdot (-1) = -1$

$$f'(x_0) = 1$$

$$\boxed{\text{Αρκει νδο } \exists!\ x_0 \in (0,1) \text{ τ.ω } f'(x_0)=1}$$

$$f'(x) = 1 \Rightarrow \frac{1}{4}4x^3 + \frac{1}{2}2x = 1$$

$$\underbrace{x^3+x-1}_{g(x)} = 0$$

Η $g(x)$ είναι συνεχής ως π.δ.σ στο $[0,1]$

$$\left.\begin{aligned} g(0) &= -1 \\ g(1) &= 1 \end{aligned}\right\} g(0)\,g(1) < 0$$

Bolzano $\exists!\, x_0 \in (0,1)$ τ.ω.

$g(x_0) = 0$

γνη $g \nearrow$

33.

$$f(x) = \ln x - \frac{1}{x} + 3$$

$$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow (0,0)$$

$$0 - f(x_0) = f'(x_0)(0 - x_0)$$

$$-f(x) = -x f'(x)$$

$$f(x) = x f'(x)$$

$$\ln x - \frac{1}{x} + 3 = x\left(\frac{1}{x} + \frac{1}{x^2}\right)$$

$$\ln x - \frac{1}{x} + 3 = 1 + \frac{1}{x}$$

$$\underbrace{\ln x - \frac{2}{x} + 2}_{\varphi(x)} = 0 \Rightarrow \varphi(x) = 0$$

$$\varphi(x) = \varphi(1)$$

$$\varphi \text{ 1-1}$$

$$\underline{\underline{x = 1}}$$

- $x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2$

(+)

- $x_1 < x_2 \Rightarrow \frac{1}{x_1} > \frac{1}{x_2} \Rightarrow -\frac{2}{x_1} < -\frac{2}{x_2}$

$$y - f(1) = f'(1)(x - 1)$$

$\varphi \nearrow$

32. $f(x) = x\ln x + \frac{x^3}{3} - 1$

Ψάχνω εφαπτομένη $\perp$ $\varepsilon: x+2y+1=0$.

$y - f(x_0) = f'(x_0)(x - x_0)$ $\perp$ $y = -\frac{1}{2}x - \frac{1}{2}$

Πρέπει $f'(x_0) \cdot \left(-\frac{1}{2}\right) = -1$

$f'(x_0) = 2$

$f'(x) = 2$,

$\ln x + x \cdot \frac{1}{x} + \frac{1}{3} 3x^2 = 2$

$\ln x + 1 + x^2 = 2$

$\underbrace{\ln x + x^2 - 1}_{\varphi(x)} = 0$. $\Rightarrow \varphi(x) = 0$

$\varphi(x) = \varphi(1)$

φ 1-1

$x = 1$

• $x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2$

• $x_1 < x_2 \Rightarrow x_1^2 < x_2^2$ (+)

$\varphi \nearrow$

$y - f(1) = f'(1)(x-1)$

28. $f(x) = x(x-2)$ $\qquad$ $g(x) = -x^2-5$

$f(x) = x^2-2x$ $\qquad$ $g'(x) = -2x$

$f'(x) = 2x-2$

$$\begin{cases} f'(a) = g'(\beta) \\ f(a) - a f'(a) = g(\beta) - \beta g'(\beta) \end{cases}$$

$$\begin{cases} 2a-2 = -2\beta \\ a^2-2a - a(2a-2) = -\beta^2-5 - \beta(-2\beta) \end{cases}$$

$$\begin{cases} a-1 = -\beta \rightarrow a = 1-\beta \\ a^2 - \cancel{2a} - 2a^2 + \cancel{2a} = -\beta^2-5+2\beta^2 \end{cases}$$

$-a^2 = \beta^2-5$

$-(1-\beta)^2 = \beta^2-5$ $\qquad$ $a=-1$ $\qquad$ $a=2$

$1-2\beta+\beta^2 = 5-\beta^2$ $\qquad$ $\beta=2$ $\qquad$ $\beta=-1$

$2\beta^2-2\beta-4=0 \rightarrow \beta^2-\beta-2=0$

25. $f(x) = kx^2 - 2\lambda x + k$

$x_0 = 1$

$g(x) = x^3 - x + \lambda$

$$\begin{cases} f'(1) = g'(1) \\ f(1) = g(1) \end{cases} \Rightarrow \begin{cases} 2k - 2\lambda = 2 \\ k - 2\lambda + k = \lambda \end{cases}$$

$f'(x) = 2kx - 2\lambda$

$g'(x) = 3x^2 - 1$

$$\begin{cases} k - \lambda = 1 \Rightarrow \lambda = k - 1 \\ 2k - 3\lambda = 0 \end{cases}$$

$2k - 3(k-1) = 0$

$2k - 3k + 3 = 0$

$-k + 3 = 0$

$\lambda = 2$

$k = 3$

22. $y = ax+1$ $\qquad x_0 = -1$

$$f(x) = Bx^2 - ax + 2$$

$$\begin{cases} f'(-1) = a \\ f(-1) = -a+1 \end{cases} \qquad f'(x) = 2Bx - a$$

$$\begin{cases} -2B - a = a \\ B + \cancel{a} + 2 = \cancel{-a} + 1 \end{cases}$$

$a = -B$

$B + 2a = -1$

$-a + 2a = -1$

$a = -1$

$B = 1$

21. $f(x) = x^2+x+2$

$$y = \lambda x + 1$$

$$\begin{cases} f'(x) = \lambda \\ f(x) = \lambda x + 1 \end{cases} \Rightarrow \begin{cases} 2x+1 = \lambda \\ x^2+x+2 = \lambda x + 1 \end{cases}$$

$$\begin{cases} 2x = \lambda - 1 \\ x^2+x+1 = \lambda x \end{cases} \Rightarrow \lambda = 2x+1$$

$$x^2+x+1 = (2x+1)x$$

$$x^2 + \cancel{x} + 1 = 2x^2 + \cancel{x}$$

$$x^2 = 1$$

$x = 1$ $\qquad\qquad$ $x = -1$

$\lambda = 3$ $\qquad\qquad$ $\lambda = -1$

17 εντυπωτα

19. $f(x) = x^2 - x + 2$

Ndo η $y = x+1$ εφαπτεται την C_f.

$$\begin{cases} f'(x) = 1 \\ f(x) = x+1 \end{cases} \Rightarrow \begin{cases} 2x - 1 = 1 \Rightarrow 2x = 2 \quad x = 1 \\ x^2 - x + 2 = x + 1 \Rightarrow x^2 - 2x + 1 = 0 \end{cases}$$

Η $y = x+1$ εφαπτεται την C_f

στο $M(1, f(1))$.

29.

$$f(x) = x^2 - 3x + 3$$

$$f'(x) = 2x - 3$$

$$g(x) = 2x^2 - x + 2$$

$$g'(x) = 4x - 1$$

$$\begin{cases} f'(a) = g'(\beta) \\ f(a) - a f'(a) = g(\beta) - \beta g'(\beta) \end{cases}$$

$$\begin{cases} 2\alpha - 3 = 4\beta - 1 \\ a^2 - 3a + 3 - \alpha(2a - 3) = (2\beta^2 - \beta + 2) - \beta(4\beta - 1) \end{cases}$$

$$\begin{cases} 2a - 4\beta = 2 \\ a^2 \cancel{-3a} + 3 - 2a^2 \cancel{+3a} = 2\beta^2 \cancel{-\beta} + 2 - 4\beta^2 \cancel{+\beta} \end{cases}$$

$$\begin{cases} \alpha - 2\beta = 1 \\ -a^2 + 3 = -2\beta^2 + 2 \end{cases}$$

$$\begin{cases} a = 2B + 1 \\ 2B^2 - a^2 = -1 \end{cases}$$

$$2B^2 - (2B+1)^2 = -1$$

$$2B^2 - (4B^2 + 4B + 1) = -1$$

$$2B^2 - 4B^2 - 4B - 1 = -1$$

$$-2B^2 - 4B = 0$$

$$B^2 + 2B = 0$$

$$B(B+2) = 0$$

$B = 0$ $\qquad$ $B = -2$

$a = 1$ $\qquad$ $a = -3$

35. $f(x) = (x-2)\ln x$

Νδο υπαρχει μοναδικη εφαπτομενη η οποια διερχεται απο αρχη αξονων

$$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow (0,0)$$

$$0 - f(x_0) = f'(x_0)(0 - x_0)$$

$$-f(x_0) = -x_0 f'(x_0)$$

$$f(x) = x f'(x)$$

$$(x-2)\ln x = x\left[\ln x + \frac{x-2}{x}\right]$$

$$\cancel{x\ln x} - 2\ln x = \cancel{x\ln x} + x\,\frac{x-2}{x}$$

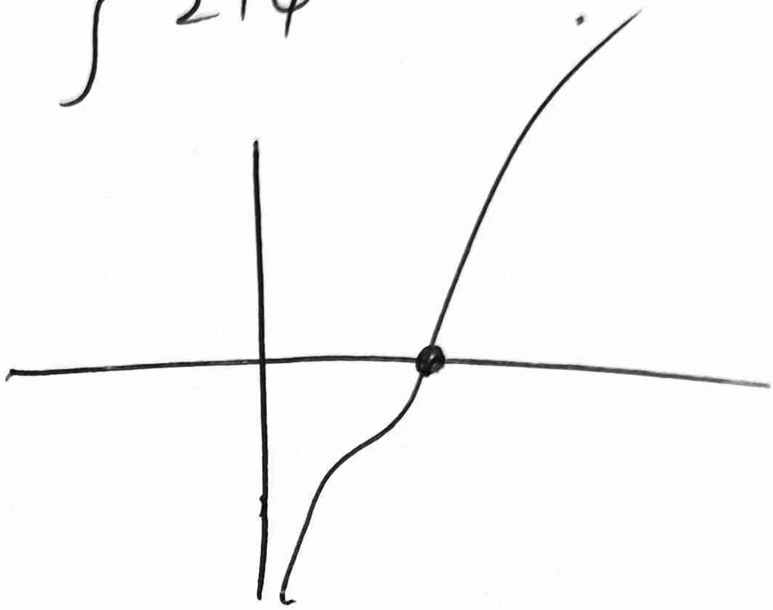
$$-2\ln x = x - 2$$

$$\underbrace{x + 2\ln x - 2}_{\varphi(x)} = 0$$

$\varphi(x) = x + 2\ln x - 2 \quad , x > 0$

$\varphi \nearrow$

$\left.\begin{array}{l} \lim\limits_{x\to 0^+} \varphi(x) = -\infty \\ \lim\limits_{x\to +\infty} \varphi(x) = +\infty \end{array}\right\} \Sigma T\varphi = \mathbb{R}$.



- φ συνεχής
- $\varphi \nearrow$
- $\Sigma T\varphi = \mathbb{R}$.

το $0 \in \Sigma T\varphi$ άρα

$\exists!\ \xi$ τ.ω $\varphi(\xi) = 0$.

Επαναληπτικο Μαθημα στις εφαπτομενες

1. Να βρεθει η εφαπτομενη της $f(x) = x^2$ στο σημειο της $M(2, f(2))$

$\varepsilon : y - f(2) = f'(2)(x-2)$

- $f(2) = 4$
- $f'(2) = 4$

$\boxed{f'(x) = 2x}$

$y - 4 = 4(x-2)$

$y - 4 = 4x - 8$

$\boxed{y = 4x - 4}$

1 extra

Να Βρεθεί η εφαπτομένη της

$f(x) = x^3$

ⓐ στο σημείο της με τεταγμένη $y_0 = -8$

$f(x) = x^3$

$-8 = x^3$

$\underline{\underline{x = -2}}$

$M(-2, -8)$

$f'(x) = 3x^2$

$f'(-2) = 12$

$\varepsilon: y - f(-2) = f'(-2)(x+2)$

$y - (-8) = 12(x+2)$

$y + 8 = 12x + 24$

$y = 12x + 16$

ⓑ εκεί που τέμνει τον $x'x$.

$f(x) = x^3$

όπου $y_0 = 0$

$0 = x^3$

$x = 0$

$M(0,0)$

$f'(x) = 3x^2$

$f'(0) = 0$

$y - f(0) = f'(0)(x - 0)$

$y - 0 = 0(x - 0)$

$y = 0$

$\underline{\underline{x'x}}$

2. Να βρεθει η εφαπτομενη της $f(x)=x^2$

$f'(x)=2x$

η οποια

(α) εχει συντελεστη διευθυνσης 4

$\varepsilon: y-f(x_0)=f'(x_0)(x-x_0)$

Γνωριζω οτι $f'(x_0)=4$

$2x_0=4$

$\underline{\underline{x_0=2}}$

$\boxed{\varepsilon: y-f(2)=f'(2)(x-2)}$

$y-4=4(x-2) \longrightarrow y=4x-4$

(β) ειναι παραλληλη στον $x'x$

$\varepsilon: y-f(x_0)=f'(x_0)(x-x_0)$

Γνωριζω $f'(x_0)=0$

$2x_0=0$

$\underline{x_0=0}$

$y-f(0)=f'(0)(x-0)$

$y-0=0(x-0)$

$y=0$

$x'x$

(γ) είναι παράλληλη στην $\eta: 2x-y+4=0$

$$\varepsilon: y-f(x_0)=f'(x_0)(x-x_0) \parallel y=2x+4$$

Γνωρίζω $f'(x_0)=2$

$$2x_0=2$$

$$\underline{\underline{x_0=1}}$$

$$y-f(1)=f'(1)(x-1)$$

$$y-1=2(x-1)$$

$$\boxed{y=2x-1}$$

(δ) είναι κάθετη $\eta: 2x-y+4=0$.

$$\varepsilon: y-f(x_0)=f'(x_0)(x-x_0) \perp y=2x+4.$$

Γνωρίζω

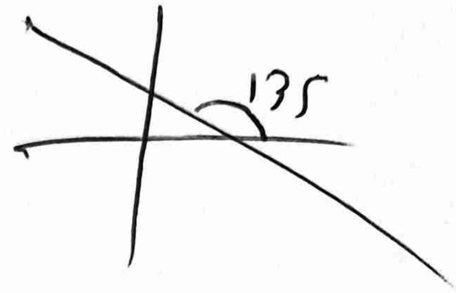
$$f'(x_0)\cdot 2=-1$$

$$2x_0=-\frac{1}{2}$$

$$x_0=-\frac{1}{4}$$

$$\varepsilon: y-f\left(-\frac{1}{4}\right)=f'\left(-\frac{1}{4}\right)\left(x+\frac{1}{4}\right)$$

(ε) σχηματίζει 135° με τον $x'x$



$$y - f(x_0) = f'(x_0)(x - x_0)$$

Γνωρίζω ότι $f'(x_0) = εφ135$

$$2x_0 = -1$$

$$x_0 = -\frac{1}{2}$$

$$y - f\left(-\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)\left(x + \frac{1}{2}\right).$$

3. Να βρεθει η εφαπτομενη της

$f(x)=\sqrt{x}$ η οποια διερχεται $A(0,1)$

$$y-f(x_0)=f'(x_0)(x-x_0) \longrightarrow A(0,1)$$

$$1-f(x_0)=f'(x_0)(0-x_0)$$

$$1-\sqrt{x_0}=\frac{1}{2\sqrt{x_0}}(-x_0)$$

$$1-\sqrt{x}=\frac{-x}{2\sqrt{x}}$$

$$2\sqrt{x}-\cancel{2\sqrt{x}}^{\,x}=-x$$

$$2\sqrt{x}=x$$

$$4x=x^2 \Rightarrow x^2-4x=0$$

$$x(x-4)=0$$

$$\cancel{x=0} \qquad x=4$$

$$y-f(4)=f'(4)(x-4)$$

4. (a) Να εξετασαι αν η $y=x+1$ εφαπτεται την $f(x)=x^2$

$$\begin{cases} f'(x)=1 \\ f(x)=x+1 \end{cases} \Rightarrow \begin{cases} 2x=1 \Rightarrow x=\frac{1}{2} \\ x^2=x+1 \end{cases}$$

$$\frac{1}{4}=\frac{1}{2}+1$$

$$1=2+4$$

$$1=6 \text{ Ατοπο!}$$

Το συστημα δεν εχει λυση!

Αρα η $y=x+1$ δεν ειναι εφαπτομενη

την $f(x)$

(B). Η εξ. $y=10x-9$ εφάπτεται της

$f(x)= x^3+ax^2+\beta x+3$. στο $x_0=2$.

$$\begin{cases} f'(2)=10 \\ f(2)=10\cdot 2-9 \end{cases} \Rightarrow \begin{cases} 12+4a+\beta=10 \\ 8+4a+2\beta+3=11 \end{cases}$$

$$\boxed{f'(x)=3x^2+2ax+\beta.}$$

$$\begin{cases} 4a+\beta=-2 \\ 4a+2\beta=0 \end{cases} \ominus \quad -\beta=-2$$

$$\beta=2$$

$4a+2=-2$

$4a=-4$

$$a=-1$$

(γ) Νδο η ε: $y=-4x-1$ εφάπτεται

την $f(x)=x^2-2x$.

$$\begin{cases} f'(x)=-4 \\ f(x)=-4x-1 \end{cases} \Rightarrow \begin{cases} 2x-2=-4 \\ x^2-2x=-4x-1 \end{cases}$$

$$\begin{cases} 2x=-2 \\ x^2+2x+1=0 \end{cases}$$

$x=-1$

$1-2+1=0$

$0=0$ ✓

Συμπέρασμα

Η $y=-4x-1$ εφάπτεται την

C_f στο -1.

5. $f(x) = a\ln x + \beta x^2$ $f'(x) = \frac{a}{x} + 2\beta x$

$g(x) = x^2 + 2\beta x + a$ $g'(x) = 2x + 2\beta$

Εχουν κοινη εφαπτομενη στο

κοινο τους σημειο $x_0 = 1$

$$\begin{cases} f'(1) = g'(1) \\ f(1) = g(1) \end{cases} \Rightarrow \begin{cases} a + \cancel{2\beta} = 2 + \cancel{2\beta} \\ a = 2 \\ \beta = 1 + 2\beta + a \\ 0 = 1 + \beta + 2 \\ \beta = -3. \end{cases}$$

κοινη εφαπτομενη

κοινο σημειο.

6.

$$f(x) = x^2 - 3x + 4$$
$$g(x) = x^2 + x + 4$$

Εχουν κοινη εφαπτομενη στο κοινο τους σημειο;

$$\begin{cases} f'(x) = g'(x) \\ f(x) = g(x) \end{cases} \Rightarrow \begin{cases} \cancel{2x} - 3 = \cancel{2x} + 1 \\ x^2 - 3x + 4 = x^2 + x + 4 \end{cases}$$

$$\begin{cases} -3 = 1 \\ ; \end{cases} \text{ Ατοπο!}$$

Αρα δεν εχουν κοινη εφαπτομενη σε κοινο σημειο.

Σε μη κοινα σημεια;

Υπαρχει κοινη εφαπτομενη

$$\begin{cases} f'(a) = g'(B) \\ f(a) - a f'(a) = g(B) - B g'(B) \end{cases}$$

$$\begin{cases} 2a - 3 = 2B + 1 \\ a^2 - 3a + 4 - a(2a-3) = B^2 + B + 4 - B(2B+1) \end{cases}$$

$$\begin{cases} 2a - 2B = 4 \\ a^2 - \cancel{3a} + \cancel{4} - 2a^2 + \cancel{3a} = B^2 + \cancel{B} + \cancel{4} - 2B^2 - \cancel{B} \end{cases}$$

$$\begin{cases} a - B = 2 \quad \Rightarrow \underline{\underline{a = B + 2}} \\ -a^2 = -B^2 \end{cases}$$

$$(B+2)^2 = B^2$$

$$\cancel{B^2} + 4B + 4 = \cancel{B^2}$$

$$4B = -4$$

$$B = -1 \qquad a = 1$$

$$y - f(1) = f'(1)(x-1)$$

$$y - 2 = -(x-1)$$

$$y = -x + 3$$