18. ⓐ Αφού f γν. μονότονη

$\Rightarrow f\uparrow$ ή $f\downarrow$

ΕΝΟΤΗΤΑ 7

$1<3$

$f(1) > f(3)$

$\|\quad\quad\|$

$2\quad\quad -2$

$f\downarrow$

$\Rightarrow f(3)<1 \Rightarrow$ αντιστρ.

ⓑ $f\left(-2+f^{-1}(x+2)\right)=2$

$f\left(-2+f^{-1}(x+2)\right)=f(1)$

$f(3)=1$

$-2+f^{-1}(x+2)=1$

$f^{-1}(x+2)=3$

$f\left(f^{-1}(x+2)\right)=f(3)$

$x+2=-2$

$x=-4$

(1) $f^{-1}\left(f(e^x-1)-4\right)<3$

$f\downarrow$

$f\left(f^{-1}\left(f(e^x-1)-4\right)\right)>f(3)$

$f(e^x-1)-4>-2$

$f(e^x-1)>2$

$f(e^x-1)>f(1)$

$f\downarrow$

$e^x-1>1$

$e^x>2$

$\ln e^x>\ln 2$

$x>\ln 2$

19. $f(x) = x^3 + x$

(α) Συνάρτηση f $\Rightarrow$ f 1-1 $\Rightarrow$ f αντιστρ.

(β) Ψάχνω το $f^{-1}(10)$

$f($ποιο; $) = 10$

$f(2) = 10 \Leftrightarrow f^{-1}(10) = 2$.

(γ) i) $f(1 + f^{-1}(x^2 - 3x)) = 0$

$f(1 + f^{-1}(x^2 - 3x)) = f(0)$

f 1-1

$1 + f^{-1}(x^2 - 3x) = 0$

$f^{-1}(x^2 - 3x) = -1$

$f(f^{-1}(x^2 - 3x)) = f(-1)$

$x^2 - 3x = -2$

$x^2 - 3x + 2 = 0$

$x = 1$

$x = 2$

11). $f^{-1}\left(f(x^2-1)+4f^{-1}(-10)\right)<1$

Αφού $f(-2)=-10$ το $f^{-1}(-10)=-2$

$$f^{-1}\left(f(x^2-1)+4(-2)\right)<1$$

$$f^{-1}\left(f(x^2-1)-8\right)<1$$

επ f

$$f\left(f^{-1}\left(f(x^2-1)-8\right)\right)<f(1)$$

$$f(x^2-1)-8<2$$

$$f(x^2-1)<10$$

$$f(x^2-1)<f(2)$$

επ f

$$x^2-1<2$$

$$x^2-3<0$$

$x\in(-\sqrt{3},\sqrt{3})$.

22. $f\downarrow$

(α) Αφού $f\downarrow \Rightarrow f$ 1-1 $\Rightarrow f$ αντιστρ.

(β) $f^{-1}(x_1) < f^{-1}(x_2)$

$f\downarrow$

$f(f^{-1}(x_1)) > f(f^{-1}(x_2))$

$x_1 > x_2$ $\quad f^{-1}\downarrow$.

Β' τρόπος

$x_1 < x_2 \Rightarrow f(f^{-1}(x_1)) < f(f^{-1}(x_2))$

$f\downarrow$

$f^{-1}(x_1) > f^{-1}(x_2)$

$f^{-1}\downarrow$

① Αν $f(1)=0$

$$f(x)+f^{-1}(x-1)<1.$$

$$\underbrace{f(x)+f^{-1}(x-1)-1<0}_{g(x)}$$

$$g(x)<g(1)$$

$g\downarrow$

$x>1.$

Μονοτονία $g(x)$

- $x_1<x_2 \Rightarrow f(x_1)>f(x_2)$
- $x_1<x_2 \Rightarrow f^{-1}(x_1-1)>f^{-1}(x_2-1)$ } ⊕

$g\downarrow$

$$g(1)=f(1)+f^{-1}(0)-1 = 0+1-1=0$$

αφού $f(1)=0 \Leftrightarrow f^{-1}(0)=1$

26. $f(x) = 1 - \ln x$ $\quad D_f = (0, +\infty)$.

$g(x) = \ln \frac{e}{x}$ $\quad D_g = (0, +\infty)$

$h(x) = \sqrt{x}$ $\quad D_h = [0, +\infty)$

(1) $D_f = D_g = (0, +\infty)$.

$g(x) = \ln \frac{e}{x} = \ln e - \ln x = 1 - \ln x = f(x)$

Εξαρτάται, διότι έχουν ίδια απεικόνιση.

(3) $f_1(x) = \frac{1}{f(x)} = \frac{1}{1 - \ln x}$

$x \in D_f$ και $f(x) \neq 0$

$x > 0$ $\quad 1 - \ln x \neq 0$

$\ln x \neq 1$

$e^{\ln x} \neq e^1$

$x \neq e$.

$D_{f^{-1}} = (0, e) \cup (e, +\infty)$.

① $\varphi(x) = (h \circ f)(x) = h(f(x)) = \sqrt{1-\ln x}$

$x \in D_f$ con $f(x) \in D_h$

$x > 0$

$1 - \ln x \geq 0$

$1 \geq \ln x$

$e^1 \geq e^{\ln x}$

$e \geq x$

$D_\varphi = (0, e]$

② $\varphi(x) > 1$

$\sqrt{1-\ln x} > 1$

$1 - \ln x > 1$

$0 > \ln x$

$e^0 > e^{\ln x}$

$1 > x > 0$

28. $f(x) = e^x + \ln(x+1) - 1$

$D_f = (-1, +\infty)$

(01) Εύκολα $f \nearrow$

(3). $e^{x^2} + \ln(x^2+1) > 1$

$e^{x^2} + \ln(x^2+1) - 1 > 0$

$f(x^2) > 0$

$f(x^2) > f(0)$

$f \nearrow$

$x^2 > 0$

$x \in \mathbb{R}^*$

(8) $e^{x^2} - e^{x+2} > \ln \frac{x+3}{x^2+1}$

$e^{x^2} - e^{x+2} > \ln(x+3) - \ln(x^2+1)$

$e^{x^2} + \ln(x^2+1) - 1 > e^{x+2} + \ln(x+3) - 1$

$f(x^2) > f(x+2)$

$f \nearrow$

$x^2 > x+2$

$x^2 - x - 2 > 0$

x		-1		2
x^2-x-2	$+$	$\vert$ $-$	$\vert$	$+$

$x \in (-\infty, -1) \cup (2, +\infty)$.

(8) $e^{x^3+x-1} + \ln(x^3+x) - 1 - e = \ln\frac{2}{e}$

$e^{x^3+x-1} + \ln(x^3+x) - 1 = \ln 2 - \ln e + e$

$f(x^3+x-1) = \ln 2 + e - 1$

$f(x^3+x-1) = f(1)$

f 1-1

$x^3+x-1 = 1$

$x^3+x-2 = 0$

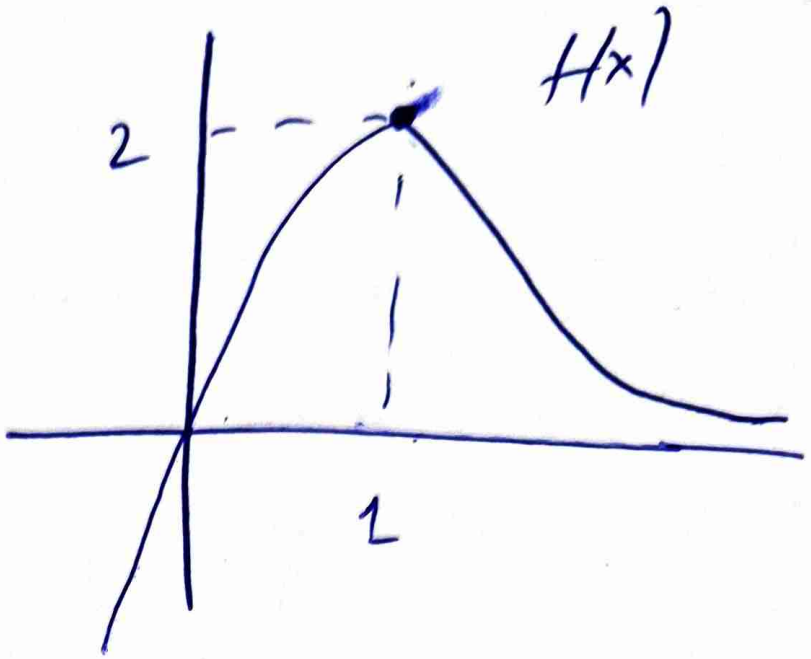
1	0	1	-2	(1)
↓	1	1	2	
1	1	2	0	

$x = 1$

$(x-1)(x^2+x+2) = 0$

$\Delta < 0$

27.



(a) $f\left(\frac{1}{e}\right)$ ve $f\left(\frac{1}{\pi}\right)$

$\frac{1}{\pi} < \frac{1}{e}$

$f \nearrow$ artan

$f\left(\frac{1}{\pi}\right) < f\left(\frac{1}{e}\right)$

(b) $f(2x^2+1) > f(x^2+2)$

$\left.\begin{array}{l} \cdot\ 2x^2+1 > 1 \\ \cdot\ x^2+2 > 1 \end{array}\right\}$ $f \nearrow$

$2x^2+1 < x^2+2$

$x^2 - 1 < 0$

x		-1		1
x^2-1	$+$	0	$-$	0

$x \in (-1, 1)$.

(γ) εξίσωση $f(x) = (x-1)^4 + 2$

$$0 = (x-1)^4 + 2 - f(x)$$

(+)
- $(x-1)^4 \geq 0$ Το "=" για $x=1$
- $f(x) \leq 2 \Rightarrow 2 - f(x) \geq 0$

Το "=" $x=1$.

$$(x-1)^4 + (2-f(x)) \geq 0$$

Το "=" για $x=1$

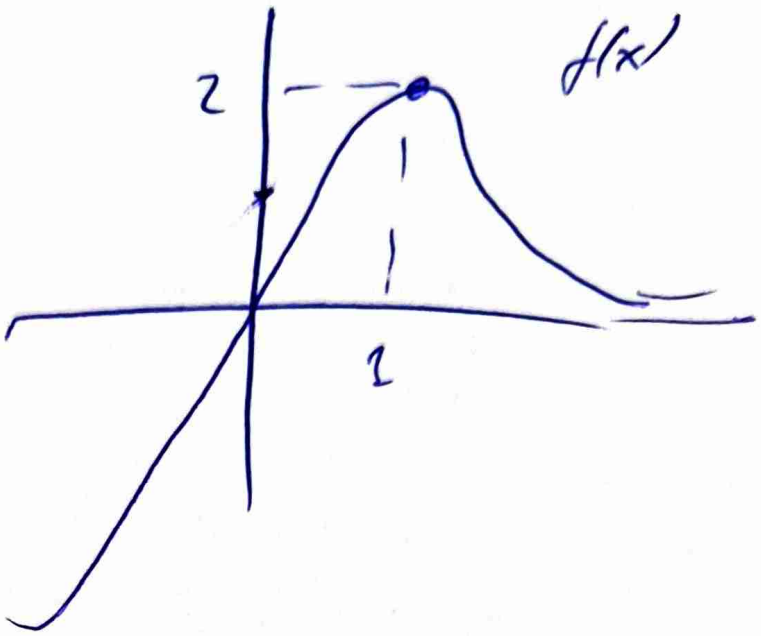
$$x = 1$$

(δ) $e^{-\alpha} \cdot f(x) = 1$ Πλήθος ριζών

$$f(x) = \frac{1}{e^{-\alpha}}$$

$$f(x) = e^{-\alpha} = \lambda$$

$$\underline{\underline{f(x) = \lambda}}$$



- Αν $\lambda > 2$ δηλαδή $e^{-\alpha} > 2 \Rightarrow$

$$\ln e^{-\alpha} > \ln 2$$

$$-\alpha > \ln 2$$

$$\alpha < \ln 2$$

τότε ρίζες.

- Αν $\lambda = 2$ δηλαδή

$\alpha = \ln 2$ 1 ρίζα

- Αν $0 < \lambda < 2 \Rightarrow 0 < e^{-\alpha} < 2$

$\alpha > \ln 2$ ✓

- Αν $\lambda \leq 0 \Rightarrow e^{-\alpha} \leq 0$ Άτοπο! 2 ρίζες

29.

$f(x) = \dfrac{ax-1}{x-2}$ $\quad f(3)=2$

$g(x) = e^x + 1 \quad D_g = \mathbb{R}.$

(a) $f(3) = 2 \Rightarrow \dfrac{3a-1}{3-2} = 2 \Rightarrow 3a = 3$

$\underline{a = 1}$

$f(x) = \dfrac{x-1}{x-2}$

$D_f = \mathbb{R} - \{2\}$

$$(f \circ g)(x) = f(g(x)) = \frac{e^x+1-1}{e^x+1-2} = \frac{e^x}{e^x-1}$$

$x \in D_g$ και $g(x) \in D_f$

$x \in \mathbb{R}$ $\quad e^x + 1 \neq 2$

$e^x \neq 1$

$x \neq 0.$

$D_{f \circ g} = \mathbb{R}^*$

(β) $h(x) = \dfrac{e^x}{e^x - 1}$ $\quad D_h = \mathbb{R}^*$

$h(x_1) = h(x_2)$

$$\frac{e^{x_1}}{e^{x_1} - 1} = \frac{e^{x_2}}{e^{x_2} - 1}$$

$$e^{x_1}(e^{x_2} - 1) = e^{x_2}(e^{x_1} - 1)$$

$$e^{x_1} e^{x_2} - e^{x_1} = e^{x_2} e^{x_1} - e^{x_2}$$

$$e^{x_1} = e^{x_2}$$

$$x_1 = x_2 \quad \text{δ1-1}.$$

(γ) Άρα δ1-1 $\Rightarrow$ h αντιστρ.

$$y = \frac{e^x}{e^x - 1}$$

$$y(e^x - 1) = e^x$$

$$ye^x - y = e^x$$

$$ye^x - e^x = y$$

$$e^x(y-1) = y$$

$$e^x = \frac{y}{y-1} \qquad y \neq 1$$

$$x = \ln \frac{y}{y-1}$$

$$h^{-1}(x) = \ln \frac{x}{x-1}$$

$$D_{h^{-1}} = (-\infty, 0) \cup (1, +\infty$$

$\longrightarrow \quad x \neq 0$

$$\ln \frac{y}{y-1} \neq 0$$

$$\frac{y}{y-1} \neq 1$$

$$y \neq y - 1$$

$$0 \neq -1$$

מתקיים תמיד

$$\frac{y}{y-1} > 0$$

		0		1
y	$-$	$\oplus$	$+$	$+$
y	$-$	$\oplus$	$+$	$+$
$y-1$	$-$	$-$	$\oplus$	$+$
$\frac{1}{y-1}$	$+$	$-$		$+$

$$y \in (-\infty, 0) \cup (1, +\infty)$$

(8) $h^{-1}(x) > \ln 2$

$\ln \frac{x}{x-1} > \ln 2$

$\frac{x}{x-1} > 2$

$\frac{x}{x-1} - 2 > 0$

$\frac{x - 2(x-1)}{x-1} > 0 \Rightarrow \frac{-x+2}{x-1} > 0$

x		1	2	
$2-x$	$+$	$+$	0	$-$
$x-1$	$-$	0	$+$	$+$
$\sim$	$-$	$+$	0	$-$

$x \in (1, 2)$.

31. $f(x) = x^5 + 2x - 2$

(α) Είναι $f \nearrow$ $\Rightarrow$ αντιστρ.

$$f^{-1}(x) = x$$

$$f(x) = x$$

$$x^5 + 2x - 2 = x$$

$$\underbrace{x^5 + x - 2 = 0}_{g(x)} \Rightarrow g(x) = 0$$

$$g(x) = g(1)$$

$g \nearrow$

$x = 1$

(β) $f^{-1}(x) = 0$

$$f(f^{-1}(x)) = f(0)$$

$x = -2$.

x	$-\infty$	-2	$+\infty$
f^{-1}	$-$	0	$+$

- $x < -2 \Rightarrow f^{-1}(x) < f^{-1}(-2)$

  $f^{-1}(x) < 0$

- $x > -2 \Rightarrow f^{-1}(x) > 0$.

Σχόλιο

Η διχοτωμ/

1ης - 3ης

γωνίας είναι

$y = x$

(a) $[f^{-1}(x)]^5 + 2f^{-1}(x) = 0$

$(f^{-1}(x))^5 + 2f^{-1}(x) - 2 = -2$

$f(f^{-1}(x)) = f(0)$

$f \circ 1 = -1$

$f^{-1}(x) = 0$

$x = -2$.

(b) $f^{-1}(x-2) + x^5 \leq -2x$

$f^{-1}(x-2) + x^5 + 2x - 2 \leq -2$

$f^{-1}(x-2) + f(x) + 2 \leq 0$

$\varphi(x)$.

• $x_1 < x_2 \Rightarrow x_1 - 2 < x_2 - 2 \Rightarrow f^{-1}(x_1 - 2) < f^{-1}(x_2 - 2)$

(+)

• $x_1 < x_2 \Rightarrow f(x_1) + 2 < f(x_2) + 2$

$\varphi \nearrow$

Αν είναι $\varphi(x) < 0$

$\varphi(x) < \varphi(0)$

$\varphi \nearrow$

$x < 0$

$f(0) = -2$

$f^{-1}(-2) = 0$

$\varphi(2) = f^{-1}(0) + f(2) + 2$ $\quad$ ΟΧΙ

$\varphi(0) = f^{-1}(-2) + f(0) + 2 =$

$= 0 \quad -2 + 2 = 0$ ✓