3. $f(x) = \ln x - x - \frac{1}{x} + 2,\ x>0$

Εκφώνηση 3ο

ⓐ Ανίσωση $f'(e^x) > 1$

$$f'(x) = \frac{1}{x} - 1 + \frac{1}{x^2}$$

$$f''(x) = -\frac{1}{x^2} + \frac{-2x}{x^4}$$

$$f''(x) = -\frac{1}{x^2} - \frac{2}{x^3} < 0 \quad f \text{ κοίλη}$$

$f' \downarrow$

$f'(e^x) > 1$

$f'(e^x) > f'(1)$

$f' \downarrow$

$e^x < 1$

$\boxed{x < 0}$

ⓑ i) Νδο $f(x) \le x-1 \quad \forall x>0$

$y - f(1) = f'(1)(x-1)$

$y - 0 = 1 \cdot (x-1)$

$y = x - 1$

Αφού f κοίλη

$f(x) \le x - 1$

Το "=" $x = 1$

11). Νδο $e^{f(\ln x)} \leq \frac{x}{e}$ $\forall x > 1$

$f(\ln x) \leq \ln \frac{x}{e}$

$f(\ln x) \leq \ln x - \ln e$

$f(\ln x) \leq \ln x - 1$

Αν $x > 1 \Rightarrow \ln x > \ln 1 \Rightarrow \ln x > 0$

Γνωρίζω ότι $f(x) \leq x - 1$ $\forall x > 0$

$f(\ln x) \leq \ln x - 1$ ✓

γ) Ανίσωση $x^x < e^{2x^2 - 3x + 1}$ $(0, +\infty)$

$\ln x^x < \ln e^{2x^2 - 3x + 1}$

$x \ln x < 2x^2 - 3x + 1$

$\ln x < 2x - 3 + \frac{1}{x}$

$\ln x - \frac{1}{x} - x + 2 < -x + 2 + 2x - 3$

$f(x) < x - 1$ $\quad x \in (0, 1) \cup (1, +\infty)$

13. (γ) $f(x) = e^x - \ln(x+1)$, $x > -1$

$$f'(x) = e^x - \frac{1}{x+1} \qquad f'(0) = 0$$

$$f''(x) = e^x + \frac{1}{(x+1)^2} > 0 \qquad f \text{ κυρτή}$$

x	-1	0	
f''	$+$		$+$
f'	$\nearrow \; -$	0	$+ \; \nearrow$
f	$\searrow$		$\nearrow$

$$f(x) \geq f(0)$$

$$f(x) \geq 1$$

$\varepsilon_1: x = -1$ κατακόρυφη

$$\lim_{x \to -1^+} f(x) = +\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^x - \ln(x+1) =$$

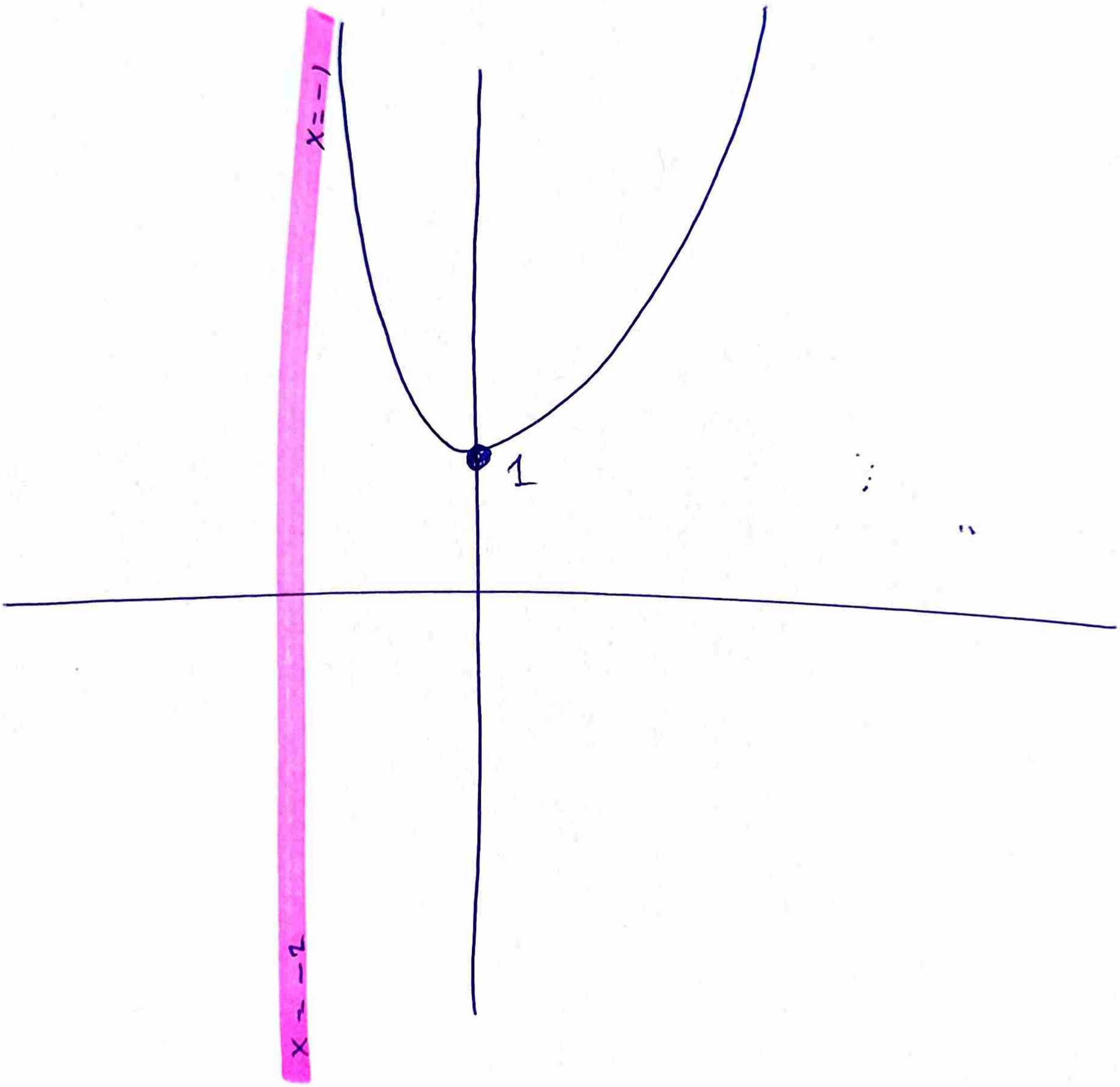
$$= \lim_{x \to +\infty} e^x \left(1 - \frac{\ln(x+1)}{e^x}\right) = +\infty$$

$$\to \lim_{x \to +\infty} \frac{\ln(x+1)}{e^x} = \lim_{x \to +\infty} \frac{1}{(x+1)e^x} = 0$$

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{e^x-\ln(x+1)}{x}=$$

$$=\lim_{x\to+\infty}\left(e^x-\frac{1}{x+1}\right)=+\infty$$

δεν εχ
οριζοντια
ουτε πλαγια,



4. $f: \mathbb{R} \to \mathbb{R}$ $\quad f \nearrow$ $\quad f(1)=2$

$f(\mathbb{R})=\mathbb{R}$ $\quad f$ κοίλη $\quad f'(1)=2$

(α) Ν.δ.ο. $f'(e^x - x) \le 2$

$f'(e^x - x) \le f'(1)$

$f' \downarrow$

$e^x - x \geq 1$

$e^x \geq x+1$ ✓

(β) $y - f(1) = f'(1)(x-1)$

$y - 2 = 2(x-1)$

$\boxed{y=2x}$

Αφού f κοίλη

$f(x) \le 2x$

Το "=" $x=1$

Αφού f κοίλη

$\Rightarrow f' \downarrow$

(γ) Εξίσωση $f(x) - |x-1| = 2x$

$f(x) - 2x - |x-1| = 0$

Προφανής ρίζα $\underline{\underline{x=1}}$

$$\left.\begin{array}{l} \bullet\ f(x) \le 2x \Rightarrow f(x) - 2x \le 0 \\ \bullet\ |x-1| \geq 0 \Rightarrow -|x-1| \le 0 \end{array}\right\} \stackrel{(+)}{\Rightarrow} f(x) - 2x - |x-1| \le 0$$

(ϛ) $\lim_{x\to 1} \frac{1}{f(x)-2x} = -\infty$

Γνωρίζω ότι $f(x) \leq 2x \Rightarrow f(x)-2x \leq 0$

(ε) Αφού f ↓ αντιστρέψιμη

Ν.δ.ο. $f^{-1}(x) \geq \frac{x}{2}$

$f(f^{-1}(x)) \geq f(\frac{x}{2})$

$x \geq f(\frac{x}{2})$

Γνωρίζω ότι $f(x) \leq 2x \quad \forall x \in R$

$f(\frac{x}{2}) \leq 2\frac{x}{2}$

$f(\frac{x}{2}) \leq x$ ✓

12. f κυρτή $\Rightarrow f' \nearrow$

$f(0)=1$

$$f(x) \geq e^x + x - \ln(x^2+1)$$

Εξίσωση $f(x) - 2x = 1$

$$f(x) = 2x - 1$$

$$f(x) \geq e^x + x - \ln(x^2+1)$$

$$\underbrace{f(x) - e^x - x + \ln(x^2+1)} \geq 0$$

$$g(x) \geq 0$$

$$g(x) \geq g(0)$$

Ακρότατο στο 0

Fermat $g'(0) = 0$

$$g'(x) = f'(x) - e^x - 1 + \frac{2x}{x^2+1}$$

$$g'(0) = f'(0) - 1 - 1 = 0$$

$$\underline{\underline{f'(0) = 2}}$$

$$y - f(0) = f'(0)(x-0)$$

$$y - 1 = 2x$$

$$\underline{\underline{y = 2x+1}}$$

Αφού f κυρτή

$$f(x) \geq 2x+1$$

Το "=" $\underline{\underline{x=0}}$

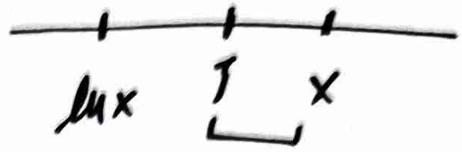
f κυρτη $\Rightarrow f' \nearrow$

15. (i) Νδο $f(\ln x) - f(x) > (\ln x - x) f'(x)$

$\forall x > 0$

$\cdot \ \ln x \leq x - 1$

$\ln x - x \leq -1$



$\ln x - x < 0$

$$f'(\xi) = \frac{f(x) - f(\ln x)}{x - \ln x}$$

$\ln x < x$

$\xi < x$

$f' \nearrow$

$f'(\xi) < f'(x)$

$$\frac{f(x) - f(\ln x)}{x - \ln x} < f'(x)$$

$f(x) - f(\ln x) < (x - \ln x) f'(x)$

$f(\ln x) - f(x) > (\ln x - x) f'(x)$ ✓

14. f δυο φορές παραγωγίσιμη $\quad$ f συν+παρ

$f''(x) \neq 0 \quad \forall x \in \mathbb{R}$ $\quad$ f' συν+παρ

• $f(1) - f'(1) < f(0)$ $\quad$ f'' συνεχής

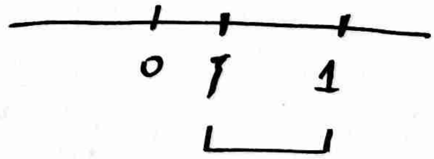
Νδο f κυρτή

Αφού $f''(x) \neq 0$ και συνεχής

$\Rightarrow f''(x) > 0$ ή $f''(x) < 0$

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0)$$

$$f''(\rho) = \frac{f'(1) - f'(\xi)}{1 - \xi}$$



$$f''(\rho) = \frac{\overset{\oplus}{f'(1) - (f(1) - f(0))}}{\underset{\oplus}{1 - \xi}} > 0$$

$f(1) - f'(1) < f(0)$

$f(1) - f(0) < f'(1)$

$f''(\rho) > 0 \Rightarrow f''(x) > 0$ $\quad$ f κυρτή.

7. $f(x) = \dfrac{x+2}{e^x}, \quad x \in \mathbb{R}$

ⓐ $f'(x) = \dfrac{e^x - (x+2)e^x}{e^{2x}} = \dfrac{1-x-2}{e^x} = \dfrac{-1-x}{e^x}$

$f''(x) = \dfrac{-e^x - (-1-x)e^x}{e^{2x}} = \dfrac{-1+1+x}{e^x} = \dfrac{x}{e^x}$

x		0	
f''	$-$		$+$
f	∩		∪

$y - f(0) = f'(0)(x-0)$

$y - 2 = -1(x-0)$

$\boxed{y = 2 - x}$

$\forall x<0$ η f κοίλη $f(x) \leq 2-x$

$\forall x>0$ η f κυρτή $f(x) \geq 2-x$

Το "=" $x = 0$.

(β) Ανίσωση $f'(x^2) < f'(x^4)$

x		0	
f''	$-$		$+$
f'	$\searrow$		$\nearrow$

$f'(x) \geq f'(0)$

$f'(x) \geq -1$

$\cdot\ x^2 \geq 0$
$\cdot\ x^4 \geq 0$ $\}$ $\forall x > 0$ $f' \nearrow$

$x^2 < x^4$

$x^2 - x^4 < 0$

$x^2(1-x^2) < 0$

x		-1	0	1	
x^2	$+$	$+$		$+$	$+$
$1-x^2$	$-$	$+$		$+$	$-$
$p(x)$	$-$	$+$		$+$	$-$

$x \in (-\infty,-1) \cup (1,+\infty)$

(γ) Ν.δ.ο $f(x) \leq 2-x$ $\forall x \leq 0$

Το έδειξα πριν!

(δ) i) Εξίσωση $x + f(x) = 2$

$f(x) = 2 - x$ Προφανής ρίζα $x=0$

$\forall x<0$ $f(x)$ κοίλη $f(x) < 2-x$

$\forall x>0$ $f(x)$ κυρτή $f(x) > 2-x$

ii) Ανισωση $2-f(x) > x$

$$f(x) < 2-x$$

$$x \in (-\infty, 0)$$

(c) ΝΔο $f(e^{x-1}-x)+e^{x-1} \geq x+2 \quad \forall x \in \mathbb{R}$

- $e^{x} \geq x+1$

$e^{x-1} \geq x-1+1$

$e^{x-1} \geq x$

$e^{x-1} - x \geq 0$

$\forall x > 0 \quad f(x) > 2-x$

$$f(e^{x-1}-x) > 2-(e^{x-1}-x)$$

$$f(e^{x-1}-x) > 2-e^{x-1}+x$$

$$f(e^{x-1}-x) - x + e^{x-1} > 2$$

(5c) NVO $\quad f(e^{-x}-1)+e^{-x} \leq 3 \quad \forall x \geq 0$

$$x \geq 0 \Rightarrow -x \leq 0 \Rightarrow e^{-x} \leq e^{0}$$

$$e^{-x} \leq 1$$

$$e^{-x}-1 \leq 0$$

$$\forall x \leq 0 \qquad f(x) \leq 2-x$$

$$f(e^{-x}-1) \leq 2-(e^{-x}-1)$$

$$f(e^{-x}-1) \leq 2-e^{-x}+1$$

$$f(e^{-x}-1)+e^{-3} \leq 3.$$

Ενοτητα 32

2. (α) $f(x) = \frac{x-1}{x-2}, \quad x \neq 2$

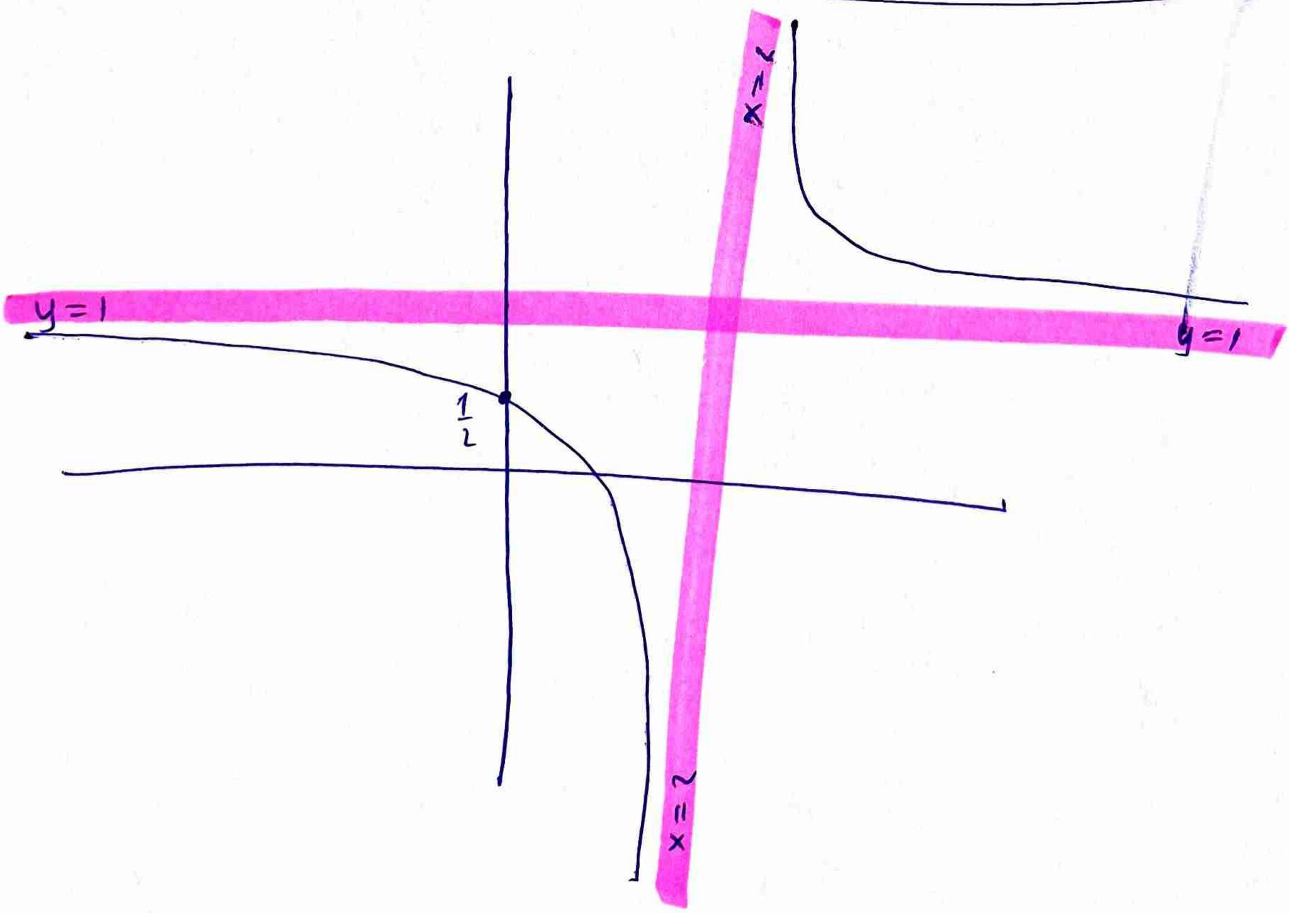
$f'(x) = \frac{x-2-x+1}{(x-2)^2} = -\frac{1}{(x-2)^2} < 0$ $\quad$ $f \downarrow$ $(-\infty, 2)$ και $(2, +\infty)$

$f''(x) = -\frac{-2(x-2)}{(x-2)^4} = 2\frac{1}{(x-2)^3}$

x		2	
f''	$-$	‖	$+$
f	$\cap$	‖	$\cup$

$\lim_{x \to \pm\infty} f(x) = 1$ $\quad$ $\varepsilon_1: y = 1$ $\pm\infty$ οριζοντια.

$\lim_{x \to 2^-} f(x) = -\infty$ $\quad$ $\varepsilon_2: x = 2$ κατακορυφη



40. $f(x) = \dfrac{x}{x^2+1}, \quad x \in R$

(α1) $f'(x) = \dfrac{x^2+1 - x \cdot 2x}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

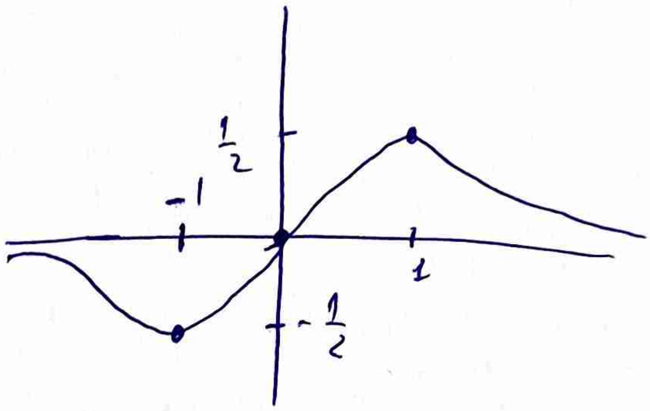
x		-1		1	
f'	$-$		$+$		$-$
f	$0 \searrow$	$-\frac{1}{2}$	$\nearrow$	$\frac{1}{2}$	$\searrow 0$

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{\cancel{x}}{x^{\cancel{2}}} = 0$

$f(-1) = \dfrac{-1}{2}$

$f(1) = \dfrac{1}{2}$

$\lim\limits_{x \to +\infty} f(x) = 0$



$ΣΤ_f = [-\frac{1}{2}, \frac{1}{2}]$

(B) $(x^2+1)^2 = x$

$x^2+1 = \dfrac{x}{x^2+1}$

$f(x) = x^2+1$

Αφού $ΣΤ_f = [-\frac{1}{2}, \frac{1}{2}]$

και $x^2+1 > 1$

τότε αδύνατη!

Ενότητα 27

21.

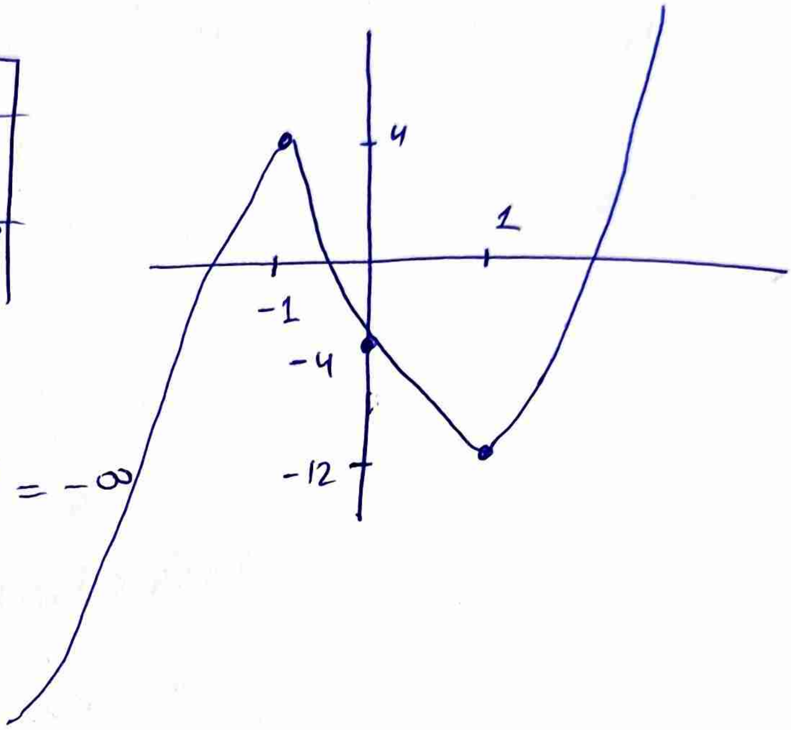
$f(x) = x^4 - 6x^2 - 4x - 1$

$f'(x) = 4x^3 - 12x - 4$

$f''(x) = 12x^2 - 12 = 12(x^2 - 1)$

$\longrightarrow f''(x) = 0 \Rightarrow \underline{\underline{x = \pm 1}}$

x		-1		1	
f''	$+$	0	$-$	0	$+$
f'	$-\infty \nearrow$	4	$\searrow$	-12	$\nearrow +\infty$



$\lim\limits_{x \to -\infty} f'(x) = \lim\limits_{x \to -\infty} 4x^3 = -\infty$

$\lim\limits_{x \to +\infty} f'(x) = +\infty$

$f'(-1) = -4 + 12 - 4 = 4$

$f'(1) = -12$

$x < -1$

- f' συνεχής
- $f' \uparrow$
- $\Sigma T_{f'} = (-\infty, 4)$

Το $0 \in \Sigma T_{f'}$

άρα $\exists!\ \xi_1 < 0$

τ.ω $f'(\xi_1) = 0$

$-1 \le x \le 1$

- f' συνεχής
- $f' \downarrow$
- $\Sigma T_{f'} = [-12, 4]$

Το $0 \in \Sigma T_{f'}$

άρα $\exists!\ \xi_2$

τ.ω

$f'(\xi_2) = 0$

$x > 1$

- f' συνεχής
- $f' \uparrow$
- $\Sigma T_{f'} = (-12, +\infty)$

Το $0 \in \Sigma T_{f'}$

άρα $\exists!\ \xi_3 > 0$

τ.ω $f'(\xi_3) = 0$

Το $\xi_2 < 0$

$f \downarrow$

$f(\xi_2) > f(0)$

$0 > -2$

✓

35. $f(x) = \begin{cases} x e^{x}, & x \leq 0 \\ \sqrt{x} \ln x, & x > 0 \end{cases}$

(a) $\lim\limits_{x \to 0^-} f(x) = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sqrt{x} \ln x = \lim\limits_{x \to 0^+} \dfrac{\sqrt{x}}{x} \, x \ln x$

$\rightarrow \lim\limits_{x \to 0^+} x \ln x = \lim\limits_{x \to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x \to 0^+} -\dfrac{x^2}{x} = 0$

$\rightarrow \lim\limits_{x \to 0^+} \dfrac{\sqrt{x}}{x} = \lim\limits_{x \to 0^+} \dfrac{1}{2\sqrt{x}} = +\infty$ Δεν βγαίνει έτσι.

$\rightarrow \lim\limits_{x \to 0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}} = \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{\frac{-\frac{1}{2\sqrt{x}}}{x}} = \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2x\sqrt{x}}} =$

$= \lim\limits_{x \to 0^+} -\dfrac{2x\sqrt{x}}{x} = 0$

Άρα $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0) = 0$

Συνεχής στο 0.

$\underline{x < 0}$

$f_1(x) = xe^x$

$f_1'(x) = e^x + xe^x = e^x(x+1)$

$\to f_1'(x) = 0 \iff \underline{\underline{x = -1}}$

$\underline{x > 0}$

$f_2(x) = \sqrt{x}\ln x$

$f_2'(x) = \frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x}$

$f_2'(x) = \frac{x \ln x + 2x}{2x\sqrt{x}}$

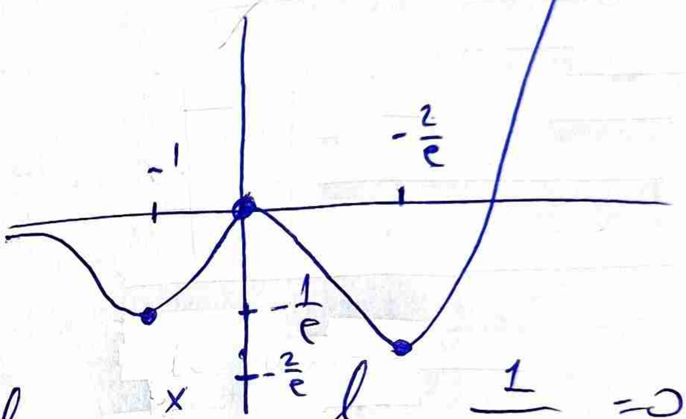
$f_2'(x) = \frac{\ln x + 2}{\sqrt{x}}$

$\to \ln x + 2 = 0$

$\ln x = -2$

$x = e^{-2}$

x		-1		0		$1/e^2$	
f_1'	$-$	0	$+$		////		////
f_2'	////		////		$-$	0	$+$
f'	$-$		$+$		$-$		$+$
f	$0 \searrow$	$-\frac{1}{e}$	$\nearrow$	0	$\searrow$	$-\frac{2}{e}$	$\nearrow +\infty$



$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$

$f(-1) = -\frac{1}{e}$

$\Sigma \,?_f = [-\frac{2}{e}, +\infty[$

$f(0) = 0$

$f(\frac{1}{e^2}) = \sqrt{\frac{1}{e^2}} \ln \frac{1}{e^2} = \frac{1}{e} \ln e^{-2} = -\frac{2}{e}$

41. $f(x) = \ln(x+1) - e^x + 2 \quad , x > -1$

α) $f'(x) = \frac{1}{x+1} - e^x \qquad f'(0) = 0$

$f''(x) = -\frac{1}{(x+1)^2} - e^x < 0$

x	-1		0	
f''		$-$		$-$
f'		$+ \searrow$	0	$\searrow -$
f	$-\infty$	$\nearrow$	1	$\searrow -\infty$

$f(x) \le f(0)$

$f(x) \le 1$

$\lim_{x\to -1^+} f(x) = -\infty$

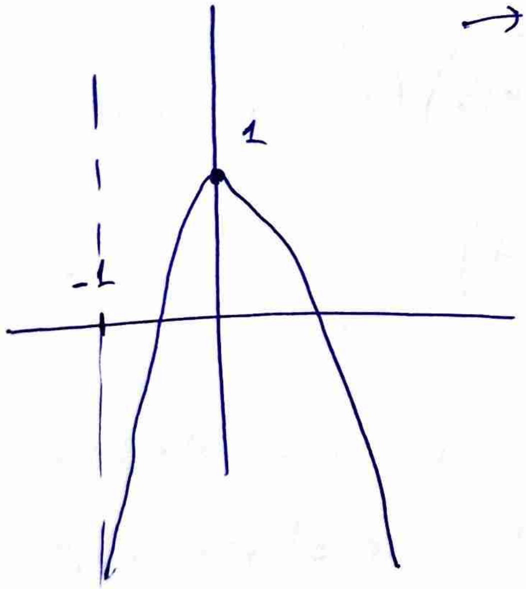
$f(0) = 1$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln(x+1) - e^x + 2$

$= \lim_{x\to+\infty} e^x\left(\frac{\ln(x+1)}{e^x} - 1 + \frac{2}{e^x}\right)$

$\to \lim_{x\to+\infty} \frac{\ln(x+1)}{e^x} = \lim_{x\to+\infty} \frac{1}{(x+1)e^x} = 0$

$= -\infty$,



$[T_f = (-\infty, 1]$,

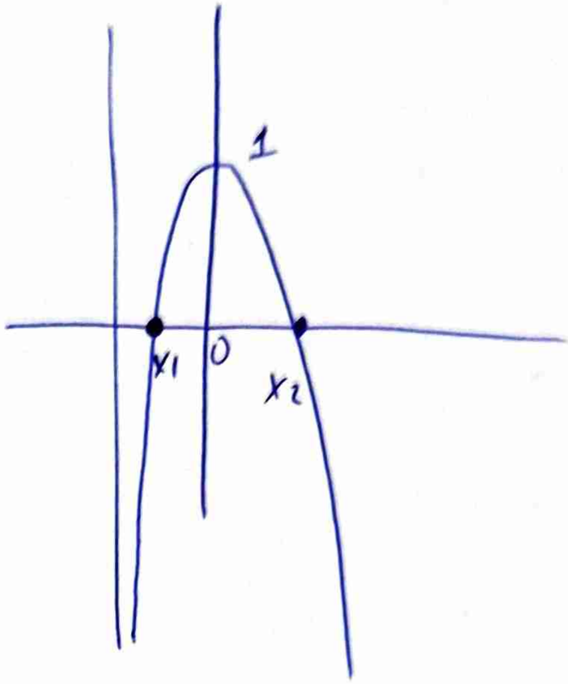
(Β) $x<0$

- f συνεχ
- $f \uparrow$
- $\sigma T_f = (-\infty, 1]$

Το $0 \in \sigma T_f$

άρα $\exists\, \xi_1 < 0$

τ.ω $f(\xi_1)=0$



$x \geq 0$

- f συνεχ
- $f \downarrow$
- $\sigma T_f = (-\infty, 1]$

Το $0 \in \sigma T_f$

άρα $\exists!\, \xi_2 > 0$

τ.ω $f(\xi_2)=0$.

(Γ) $$\frac{f(\alpha)-1}{x-x_1} + \frac{f(\beta)-1}{x-x_2} = 0$$

$$g(x) = (x-x_2)(f(\alpha)-1) + (x-x_1)(f(\beta)-1)$$

$$g(x_1) = \overset{\ominus}{(x_1-x_2)}\,\overset{\ominus}{(f(\alpha)-1)} > 0$$

$$g(x_2) = \underset{\oplus}{(x_2-x_1)}\,\underset{\ominus}{(f(\beta)-1)} < 0$$

Αφού $f(x) \leq 1$ $\forall x$

$f(\alpha)<1$

$f(\beta)<1$

$g(x_1)g(x_2)<0$ Bolzano $\exists \rho \in (x_1, x_2)$

τ.ω $g(\rho)=0$. Αφού $g(x)$ πολυώνυμο 1ου βαθμού έχει το πολύ 1 ρίζα.

Επόμενο Μάθημα

3ο

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15 α β