# Ενότητα 1.5

**6.** $x+2 \le f(x) \le x^2+x+2$

(α) Νδο $f(0)=2$

Για $x=0$ έχω $0+2 \le f(0) \le 0^2+0+2$

$2 \le f(0) \le 2$

$\Rightarrow \underline{\underline{f(0)=2}}$

(β) Νδο $f'(0)=1$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)-2}{x}$$

$$x+2 \le f(x) \le x^2+x+2$$

$$x+2-2 \le f(x)-2 \le x^2+x+2-2$$

$$x \le f(x)-2 \le x^2+x$$

• Για $x>0$

$$\frac{x}{x} \leq \frac{f(x)-2}{x} \leq \frac{x^2+x}{x}$$

$$\boxed{1 \leq \frac{f(x)-2}{x} \leq x+1}$$

$$\left.\begin{aligned} &\lim_{x\to 0^+} 1 = 1 \\ &\lim_{x\to 0^+} x+1 = 1 \end{aligned}\right\} \text{Από Κ.Π}$$

$$\lim_{x\to 0^+} \frac{f(x)-2}{x} = 1$$

Για $x<0$

$$1 \geq \frac{f(x)-2}{x} \geq x+1$$

όμοια $\lim_{x\to 0^-} \frac{f(x)-2}{x} = 1$

$$\lim_{x\to 0} \frac{f(x)-2}{x} = 1 \Rightarrow \underline{f'(0)=1}$$

**7.** $2x\,\eta\mu x \le f(x) \le x^2 + \eta\mu^2 x$

Ψάχνω την εφαπτομένη στο $x_0=0$

$$y - f(0) = f'(0)(x-0)$$

<u>Για $x=0$</u>

$0 \le f(0) \le 0 \rightarrow \underline{\underline{f(0)=0}}$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)-0}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}$$

<u>Για $x>0$</u>

$$2\,\eta\mu x \le \frac{f(x)}{x} \le x + \frac{\eta\mu^2 x}{x}$$

- $\lim_{x\to 0^+} 2\,\eta\mu x = 0$
- $\lim_{x\to 0^+} x+\frac{\eta\mu^2 x}{x} = \lim_{x\to 0^+} x + \frac{\eta\mu x}{x}\cdot \eta\mu x = 0+1\cdot 0 = 0$

Από Κ.Π $\lim_{x\to 0^+}\frac{f(x)}{x} = 0$

Για $x<0$

$$2\eta\mu x \geq \frac{f(x)}{x} \geq x + \frac{\eta\mu^2 x}{x}$$

$$\lim_{x\to 0^-} 2\eta\mu x = 0$$

$$\lim_{x\to 0^-} x + \frac{\eta\mu^2 x}{x} = 0$$

Από Κ.Π. $\lim_{x\to 0^-} \frac{f(x)}{x} = 0$

Άρα $\lim_{x\to 0} \frac{f(x)}{x} = 0 \Rightarrow f'(0) = 0.$

$$\varepsilon: y - 0 = 0 \cdot (x - 0)$$

$y = 0$ $\quad x'x$

# Θεωρια

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

- $$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \underset{\substack{h\to 0\\ x\to x_0}}{\overset{x_0+h=x}{=\!=\!=}} \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$ ✓

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x + 1} = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

Για να είναι η $f$ παραγωγίσιμη

στο $x_0$ πρέπει $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ να

υπάρχει και να είναι πραγμ. αριθμός.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \underset{\substack{x \to x_0 \\ k \to x_0 - 5}}{\overset{x = 5 + k}{=\!=}} \lim_{k \to x_0 - 5} \frac{f(5+k) - f(x_0)}{5 + k - x_0}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \underset{\substack{x \to x_0 \\ h \to 0}}{\overset{x = x_0 + h}{=\!=}} \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

**9.** $f(2+h) = 1+2h+3h^2 \quad \forall h \in \mathbb{R}$.

(α) Nδο $f(2) = 1$

$\underline{h=0}$

$f(2+0) = 1+2\cdot 0+3\cdot 0^2$

$\underline{\underline{f(2) = 1}}$

(β) Nδο $f'(2) = 2$

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{1}+2h+3h^2-\cancel{1}}{h} = \lim_{h\to 0} \frac{2h+3h^2}{h}$$

$$= \lim_{h\to 0} 2+3h = 2$$

$\underline{\underline{f'(2) = 2}}$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

**15.** Η ευθεία $\varepsilon: y=2x$ εφάπτεται της $C_f$ στο $M(1, f(1))$.

$f'(1)=2$

$f(1)=2\cdot 1$

$f(1)=2$

(α) $\lim\limits_{x\to 1} \dfrac{f(x)-2x^2}{x^2-x}$

Η πληροφορία $f'(1)=2$ γίνεται όριο.

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=2$$

$$\boxed{\lim_{x\to 1}\frac{f(x)-2}{x-1}=2}$$

Θεωρώ βοηθητική $g(x)=\dfrac{f(x)-2}{x-1}$

$f(x)=g(x)(x-1)+2$

$$\lim_{x\to 1}\frac{g(x)(x-1)+2-2x^2}{x^2-x}=\lim_{x\to 1}\frac{g(x)(x-1)}{x(x-1)}-2\frac{x^2-1}{x(x-1)}$$

$$= \lim_{x\to 1} \frac{g(x)}{x} - 2\,\frac{\cancel{(x-1)}(x+1)}{x\,\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{g(x)}{x} - 2\,\frac{x+1}{x}$$

$$= \frac{2}{1} - 2\,\frac{2}{1} = -2$$

(B) $$\lim_{x\to 1} \frac{x\,H(x) - 2}{x^2-1} = \lim_{x\to 1} \frac{x\left(g(x)(x-1)+2\right) - 2}{x^2-1}$$

$$= \lim_{x\to 1} \frac{x\,g(x)(x-1)}{x^2-1} + \frac{2x-2}{x^2-1} =$$

$$= \lim_{x\to 1} \frac{x\,g(x)\,\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} + \frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{1\cdot 2}{2} + \frac{2}{2}$$

2.

**18.** Η $f$ παρ/μη στο $\alpha$

$$f'(\alpha)=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{x-\alpha}.$$

ⓐ Νδο $\lim_{x\to\alpha}\frac{xf(x)-\alpha f(\alpha)}{x-\alpha}=f(\alpha)+\alpha f'(\alpha).$

• $\lim_{x\to\alpha}\frac{xf(x)-\alpha f(\alpha)}{x-\alpha}=\lim_{x\to\alpha}\frac{xf(x)-xf(\alpha)+xf(\alpha)-\alpha f(\alpha)}{x-\alpha}$

$$\lim_{x\to\alpha}\frac{x\left(f(x)-f(\alpha)\right)}{x-\alpha}+\frac{\cancel{(x-\alpha)}f(\alpha)}{\cancel{x-\alpha}}$$

$$=\alpha\cdot f'(\alpha)+f(\alpha)$$

(3) NSO $\lim_{x\to a} \frac{af(x)-xf(a)}{x^2-ax} = f'(a) - \frac{f(a)}{a}$

$\cdot \lim_{x\to a} \frac{af(x)-xf(a)}{x^2-ax} = \lim_{x\to a} \frac{af(x)-af(a)+af(a)-xf(a)}{x^2-ax}$

$= \lim_{x\to a} \frac{a(f(x)-f(a))}{x(x-a)} - \frac{f(a)\cancel{(x-a)}}{x\cancel{(x-a)}}$

$= f'(a) - \frac{f(a)}{a}$ ✓

# 19.

$f(2) = 0 \qquad f'(2) = -2$

$$\lim_{x\to 1} \frac{f(3x-1)}{x-1} \xlongequal[\substack{3x = t+1 \\ x = \frac{t+1}{3} \\ x\to 1 \\ t \to 2}]{3x-1=t} \lim_{t\to 2} \frac{f(t)}{\frac{t+1}{3} - 1} =$$

$$= \lim_{t\to 2} \frac{3f(t)}{t+1-3} = \lim_{t\to 2} \frac{3f(t)}{t-2} =$$

$$= 3 \lim_{x\to 2} \frac{f(x)}{x-2} =$$

$$= 3 \lim_{x\to 2} \frac{f(x)-0}{x-2} =$$

$$= 3 \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = 3 \cdot f'(2)$$

$$= 3 \cdot (-2) = -6.$$

**20.** Η $f$ παρ/μη στο 2 !

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\ell\in\mathbb{R}.$$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\ell.$$

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ⓐ ΝΔΟ $\displaystyle\lim_{h\to 0}\frac{f(2+3h)-f(2)}{h}=3f'(2)$

$$\lim_{h\to 0}\frac{f(2+3h)-f(2)}{h} \ \underset{\substack{h\to 0\\ x\to 2\\ 3h=x-2\\ h=\frac{x-2}{3}}}{\overset{2+3h=x}{=\!=\!=\!=}} \ \lim_{x\to 2}\frac{f(x)-f(2)}{\frac{x-2}{3}}$$

$$=\lim_{x\to 2}\frac{3\left(f(x)-f(2)\right)}{x-2}=3f'(2)$$

(B) $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2-h)}{h} = 2f'(2)$.

• $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2) + f(2) - f(2-h)}{h} =$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} - \frac{f(2-h) + f(2)}{h} \quad (*)$$

$$\rightarrow \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \overset{2+h=x}{\underset{\substack{h=x-2 \\ h \to 0 \\ x \to 2}}{=\!=}} \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = f'(2)$$

$$\rightarrow \lim_{h \to 0} \frac{f(2-h) - f(2)}{h} \overset{2-h=x}{\underset{\substack{h=2-x \\ h \to 0 \\ x \to 2}}{=\!=}} \lim_{x \to 2} - \frac{f(x) - f(2)}{x-2}$$

$$= -f'(2)$$

$$\overset{*}{=\!=} f'(2) - (-f'(2))$$

$$= 2f'(2)$$

1.
$$\lim_{h\to+\infty} \frac{h^3+h}{h^2+1} \cdot \left(f\left(2+\frac{1}{h}\right)-f(2)\right) = f'(2)$$

$$\frac{h\cancel{(h^2+1)}}{\cancel{h^2+1}}$$

$$2+\frac{1}{h} = x \Rightarrow \frac{1}{h} = x-2$$

$$h = \frac{1}{x-2}$$

$h \to +\infty$

$x \to 2$

$$\lim_{x\to 2} \frac{1}{x-2} \cdot \left(f(x)-f(2)\right) =$$

$$= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = f'(2)$$

**24.** Η $f$ συνεχής στο 1.

$$\lim_{x\to 1}\frac{f(x)-2}{x-1}=3.$$

ⓐ Βρείτε το $f(1)=2$.

$$g(x)=\frac{f(x)-2}{x-1} \Leftrightarrow \boxed{f(x)=g(x)(x-1)+2}$$

$$\lim_{x\to 1} f(x)=\lim_{x\to 1} g(x)(x-1)+2.$$

$$\boxed{f(1)=2}$$

ⓑ Ν.δ.ο. $f'(1)=3$

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{f(x)-2}{x-1}=3$$

$$\underline{f'(1)=3}$$

**25.** $\lim\limits_{x\to 0} \dfrac{f(x)+2\eta\mu x}{x} = 5$

Νδο $f(0)=0$ και $f'(0)=3$.

$g(x) = \dfrac{f(x)+2\eta\mu x}{x}$ $\qquad \lim\limits_{x\to 0} g(x)=5$

$\boxed{f(x)= x\,g(x) - 2\eta\mu x}$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x\,g(x) - 2\eta\mu x$

$\lim\limits_{x\to 0} f(x) = 0$.

$f(0)=0$

$f'(0)=\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{f(x)}{x}$

$= \lim\limits_{x\to 0} \dfrac{x\,g(x)-2\eta\mu x}{x} = \lim\limits_{x\to 0} \dfrac{\cancel{x}\,g(x)}{\cancel{x}} - \dfrac{2\eta\mu x}{x} = 5-2=3.$

26. (a) $\lim_{x \to +\infty} x f\left(\frac{2x+1}{x}\right) = 3$.

$\frac{2x+1}{x} = u$. $\Rightarrow 2x+1 = xu$

$2x - xu = -1$

$x(2-u) = -1$

$x \to +\infty$

$x = -\frac{1}{2-u}$

$\lim_{x \to +\infty} \frac{2x+1}{x} = \lim_{x \to +\infty} \frac{2\cancel{x}}{\cancel{x}} = 2$.

$u \to 2$

$$\lim_{u \to 2} \frac{f(u)}{u-2} = 3$$

$$\lim_{x \to 2} \frac{f(x)}{x-2} = 3$$.

$$\lim_{x \to 2} \frac{f(x)}{x-2} = 3$$

$$g(x) = \frac{f(x)}{x-2}$$

$$\boxed{g(x)(x-2) = f(x)}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x)(x-2) = 0$$

$$f(2) = 0$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2} \frac{f(x)}{x-2}$$

$$f'(2) = 3$$

**28.** Η $f$ παρ/μη στο 1 $\Rightarrow \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1} = \ell$

$$h(x) = \begin{cases} f(x^2), & x \le 1 \\ f(2x-1), & x > 1 \end{cases}$$

$$\lim_{x\to 1^-} \frac{h(x)-h(1)}{x-1} = \lim_{x\to 1^-} \frac{f(x^2)-f(1)}{x-1} \;\; \frac{x^2 = t}{x = \sqrt{t}}$$

$$= \lim_{t\to 1} \frac{f(t)-f(1)}{\sqrt{t}-1} = \lim_{t\to 1} \frac{f(t)-f(1)}{t-1}\left(\sqrt{t}+1\right)$$

$$= \ell \cdot 2$$

$$\lim_{x\to 1^+} \frac{h(x)-h(1)}{x-1} = \lim_{x\to 1^+} \frac{f(2x-1)-f(1)}{x-1} \;\; \frac{2x-1=t}{\substack{2x = t+1 \\ x = \frac{t+1}{2}}}$$

$$= \lim_{t\to 1} \frac{f(t)-f(1)}{\frac{t+1}{2}-1} = \lim_{t\to 1} \frac{f(t)-f(1)}{\frac{t-1}{2}} =$$

$$= 2\lim_{t\to 1} \frac{f(t)-f(1)}{t-1} = 2\cdot\ell$$