47. $f(x) = x - \ln(x^2+1)$, $x \in \mathbb{R}$.

(a) $f'(x) = 1 - \dfrac{2x}{x^2+1} = \dfrac{x^2+1-2x}{x^2+1} = \dfrac{(x-1)^2}{x^2+1} \geq 0$

$f \nearrow$

(B) $x - 2 < \ln \dfrac{(2x-1)^2+1}{x^2+2x+2}$

$x - 2 < \ln\left((2x-1)^2+1\right) - \ln\left(x^2+2x+2\right)$

$x - \ln\left((2x-1)^2+1\right) < 2 - \ln\left(x^2+2x+2\right)$

$2x-1 - \ln\left((2x-1)^2+1\right) < x+1 - \ln\left((x+1)^2+1\right)$

$f(2x-1) < f(x+1)$

$f \nearrow$

$2x - 1 < x + 1$

$2x - x < 1 + 1$

$x < 2$

(γ) Νδο $f\left(\ln\left(x^4+1+\frac{1}{x^4+1}\right)\right) < f\left((x^4+1)-x^2\right)$

$f \nearrow$

$$\ln\left(x^4+1+\frac{1}{x^4+1}\right) < x^4+1-x^2$$

$$\ln\left(\frac{(x^4+1)^2+1}{x^4+1}\right) < x^4-x^2+1$$

$$\ln\left((x^4+1)^2+1\right) - \ln(x^4+1) < x^4-x^2+1$$

$$x^2 - \ln\left((x^2)^2+1\right) < x^4+1 - \ln\left((x^4+1)^2+1\right)$$

$$f(x^2) < f(x^4+1)$$

$f \nearrow$

$$x^2 < x^4+1$$

$$0 < x^4-x^2+1$$

✓

(δ) Εξίσωση $(4x^2+1)e^{-2x} = 1.$

$$\frac{4x^2+1}{e^{2x}} = 1$$

$$4x^2+1 = e^{2x}$$

$$\ln(4x^2+1) = 2x$$

$$\ln((2x)^2+1) = 2x$$

$$0 = 2x - \ln((2x)^2+1)$$

$$0 = f(2x)$$

$$f(0) = f(2x)$$

f 1-1

$$2x = 0$$

$$\underline{x=0}$$

7. $f(x) = \dfrac{e^x}{x}, \; x \geq 1$

$$f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2} \geq 0 \; \checkmark$$

$$f'(x) = 0 \Rightarrow \frac{e^x(x-1)}{x^2} = 0 \Leftrightarrow x - 1 = 0$$

$$\underline{\underline{x = 1}}$$

x		1	
f'			$+$
f			$\nearrow$

$f(x) \geq f(1) \Rightarrow \underline{\underline{f(x) \geq e}}$

$$f(x) + 1 > e + \ln f(x)$$

$$e^{f(x)+1} > e^{e + \ln f(x)}$$

$$e^{f(x)} \cdot e^1 > e^e \cdot e^{\ln f(x)}$$

$$e \cdot e^{f(x)} > e^e \cdot f(x)$$

$$\frac{e^{f(x)}}{f(x)} > \frac{e^e}{e} \Leftrightarrow f(f(x)) > f(e)$$

$f \nearrow$

$f(x) > e$

$f(x) > f(1) \quad x \geq 1$

12. $e^x = x^2+1$

α' τρόπος

$$\underbrace{e^x - x^2 - 1}_{f(x)} = 0$$

$$f'(x) = e^x - 2x$$

$$f''(x) = e^x - 2$$

$$\rightarrow \quad e^x - 2 = 0$$

$$e^x = 2$$

$$x = \ln 2$$

x		$\ln 2$
f''	$-$	$+$
f'	$\searrow$ $+$	$+$ $\nearrow$
f	$\nearrow$	$\nearrow$

$$f'(x) \geq f'(\ln 2)$$

$$f'(x) \geq 2 - 2\ln 2$$

$$f'(x) \geq 2(1 - \ln 2)$$

$$f'(x) \geq 2\left(\ln \frac{e}{2}\right)$$

$+$

β' τρόπος

$$\frac{e^x}{x^2+1} = 1$$

$$\underbrace{\frac{e^x}{x^2+1} - 1}_{f(x)} = 0$$

$$f'(x) = \frac{e^x(x^2+1) - 2xe^x}{(x^2+1)^2}$$

$$f'(x) = \frac{e^x(x+1)^2}{(x^2+1)^2} \geq 0$$

$f \nearrow$

(B) Νδο $2e^{\eta\mu\theta} \le e(2-\sigma\upsilon\nu^2\theta)$ $\forall \theta \in \mathbb{R}$.

$$2e^{\eta\mu\theta} \le e\left(2-(1-\eta\mu^2\theta)\right)$$

$$2e^{\eta\mu\theta} \le e(1+\eta\mu^2\theta)$$

$$\frac{e^{\eta\mu\theta}}{1+\eta\mu^2\theta} - 1 \le \frac{e}{2} - 1$$

$$f(\eta\mu\theta) \le f(1)$$

$$f \nearrow$$

$$\eta\mu\theta \le 1$$

που ισχύει.

17. $f:(2,+\infty) \to \mathbb{R}$

$f(3)=0$ $\qquad$ $f'(x)<0$ $\;\forall x>2.$

$(x-2)f(x)+3=x \Rightarrow (x-2)f(x)+3-x=0$

$$\underbrace{f(x)+\frac{3-x}{x-2}}_{g(x)}=0 \Rightarrow g(x)=0$$

$g(x)=g(3)$

g 1-1

$\underline{\underline{x=3}}$

$$g'(x)=f'(x)+\frac{-x+2-(3-x)}{(x-2)^2}$$

$$g'(x)=f'(x)+\frac{-1}{(x-2)^2}<0$$

$g \searrow$

$f'(x)<0$

$\Rightarrow f\downarrow$

18. (a) $f(2)-f(1)<e^2-e$

$$f(2)-e^2<f(1)-e$$

$$\boxed{\varphi(x)=f(x)-e^x}$$

$$\varphi(2)<\varphi(1)$$

$$\varphi\downarrow$$

$$2>1$$ ✓

$\varphi'(x)=f'(x)-e^x<0$

(γ) $f(2)+e^2<f(1)+e^3$

$$e^2-f(1)<e^3-f(2)$$

$$\boxed{\varphi(x)=e^{x+1}-f(x)}$$

$$\varphi(1)<\varphi(2)$$

$$\varphi\uparrow$$

$$1<2$$ ✓

$\varphi'(x)=e^{x+1}-f'(x)>0$

$f'(x)<0$

$f(e)=e$

19. (a) צ״ל $f(\pi) < e + \cos e + 1$

$$f(\pi) < f(e) + \cos e + 1$$

$$f(\pi) - 1 < f(e) + \cos e$$

$$f(\pi) + \cos \pi < f(e) + \cos e$$

$$\boxed{\varphi(x) = f(x) + \cos x}$$

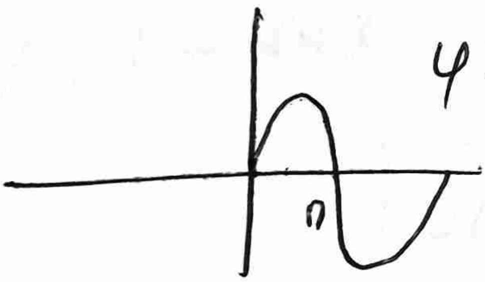
$$\varphi(\pi) < \varphi(e)$$

φ

$\pi > e$

$$\varphi'(x) = \underset{\ominus}{f'(x)} - \underset{\oplus}{\sin x} < 0$$

φ



π

בת (e, π)

לכ $\sin x > 0$

(B) $f(1) > \ln \frac{3^{e+1}}{e+2}$ $\qquad f(e) = e\ln 3$

$f(1) > \ln 3^{e+1} - \ln(e+2)$

$f(1) > (e+1)\ln 3 - \ln(e+2)$

$f(1) > e\ln 3 + \ln 3 - \ln(e+2)$

$f(1) > f(e) + \ln 3 - \ln(e+2)$

$f(1) - \ln 3 > f(e) - \ln(e+2)$

$\boxed{\varphi(x) = f(x) - \ln(x+2)}$

$\varphi'(x) = f'(x) - \frac{1}{x+2}$

$\varphi'(x) < 0$

$\varphi \searrow$

$\varphi(1) > \varphi(e)$

$\varphi \searrow$

$1 < e$ ✓

20. ⓐ $5^x + 12^x = 13^x$

$$\frac{5^x}{13^x} + \frac{12^x}{13^x} = \frac{13^x}{13^x}$$

$$\left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1$$

$$\underbrace{\left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x - 1}_{f(x)} = 0 \Rightarrow f(x) = 0$$

$f(x) = f(2)$

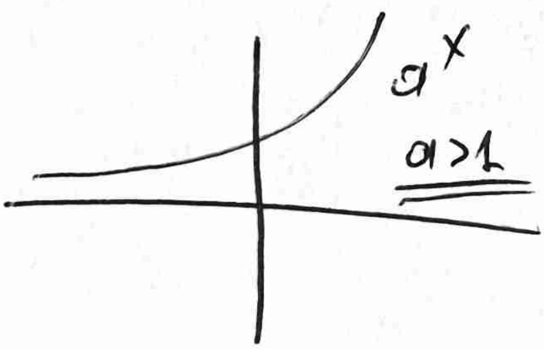
$\underline{\underline{x = 2}}$

• $x_1 < x_2 \Rightarrow \left(\frac{5}{13}\right)^{x_1} > \left(\frac{5}{13}\right)^{x_2}$

• $x_1 < x_2 \Rightarrow \left(\frac{12}{13}\right)^{x_1} > \left(\frac{12}{13}\right)^{x_2}$

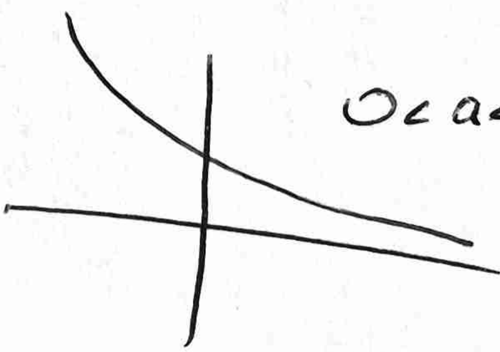
$\Big\}$ ⊕

$f \downarrow$



a^x

$\underline{\underline{a > 1}}$



$0 < a < 1$

21. $2x\,\eta\mu x + 2\sigma\upsilon\nu x = \pi \qquad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

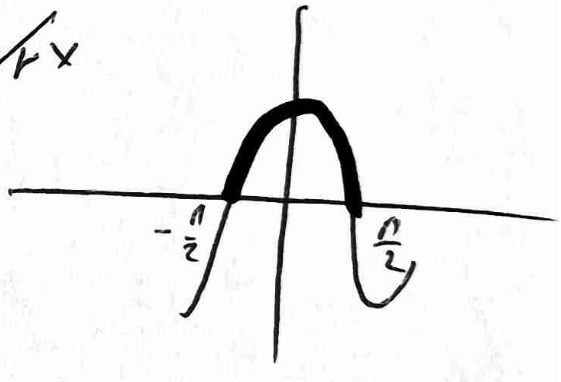
Προφανή ρίζα: $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$

$$\boxed{f(x) = 2x\,\eta\mu x + 2\sigma\upsilon\nu x - \pi}$$

$$f\left(-\frac{\pi}{2}\right) = 2\left(-\frac{\pi}{2}\right)\eta\mu\left(-\frac{\pi}{2}\right) + \cancel{2\sigma\upsilon\nu\left(-\frac{\pi}{2}\right)} - \pi = 0$$

$$f\left(\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{2}\,\eta\mu\frac{\pi}{2} + \cancel{2\sigma\upsilon\nu\frac{\pi}{2}} - \pi = 0.$$

$$f'(x) = 2\cancel{\eta\mu x} + 2x\,\sigma\upsilon\nu x - 2\cancel{\eta\mu x}$$



$$f'(x) = 2x\,\underset{+}{\sigma\upsilon\nu x}$$

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$
f'	$-$		$+$
f	$\searrow$		$\nearrow$

Αφού η $f(x)$ έχει δύο διαστήματα μονοτονίας και είναι συνεχής έχει το πολύ 2 ρίζες.

Αφού βρήκα ήδη δύο έχει δύο.

56. $f(x) = \dfrac{\ln x}{\sqrt{x}}$, $x > 0$.

(α) Η f συνεχής $[4,16]$ ως π.σ.σ

Η f παρ/μη $(4,16)$ ως π.π.σ.

$$f(4) = \frac{\ln 4}{\sqrt{4}} = \frac{\ln 2^2}{2} = \frac{2\ln 2}{2} = \ln 2$$

$$f(16) = \frac{\ln 16}{\sqrt{16}} = \frac{\ln 4^2}{4} = \frac{2\ln 4}{4} = \frac{\ln 2^2}{2}$$

$$= \frac{2\ln 2}{2} = \ln 2$$

$f(4) = f(16)$

Rolle $\exists\, \xi \in (4,16)$ τ.ω $f'(\xi) = 0$,

(β) $$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \,\frac{1}{2\sqrt{x}}}{x} = \frac{2x - x\ln x}{2x^2\sqrt{x}}$$

$$f'(x) = \frac{2 - \ln x}{2x\sqrt{x}}$$

$\longrightarrow f'(x) = 0 \Rightarrow 2 - \ln x = 0$

$2 = \ln x$

$x = e^2$

x		e^2
f'	$+$	$-$
f	$\nearrow$	$\searrow$

$f(x) \le e^2$

$f(x) \le \frac{2}{e}$

(1) $x^2 = 2^{2\sqrt{x}}, \quad x > 0$

$$\ln x^2 = \ln 2^{2\sqrt{x}}$$

$$2\ln x = 2\sqrt{x}\,\ln 2$$

$$\frac{\ln x}{\sqrt{x}} = \ln 2$$

$$f(x) = \ln 2$$

$\underline{\underline{x=4}}$ $\quad$ $\underline{\underline{x=16}}$

γιατί $f(4) = \ln 2$

$f(16) = \ln 2$.

Αφού έχω δύο διαστήματα
μονοτονίας έχω το πολύ 2 ρίζες

57. $f(x) = \dfrac{e^x}{x}$, $x > 0$

(α) $f(\ln 2) = \dfrac{e^{\ln 2}}{\ln 2} = \dfrac{2}{\ln 2}$

$f(\ln 4) = \dfrac{e^{\ln 4}}{\ln 4} = \dfrac{4}{\ln 2^2} = \dfrac{4}{2\ln 2} = \dfrac{2}{\ln 2}$

$f(\ln 2) = f(\ln 4)$ Rolle ✓

(β) $f'(x) = \dfrac{e^x \cdot x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2}$

x		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$f(x) \geq e$

(γ) $2^{e^x} = e^{2x}$ $(0, +\infty)$

$\ln 2^{e^x} = \ln e^{2x}$

$e^x \cdot \ln 2 = 2x \ln e$

$\dfrac{e^x}{2x} = \dfrac{1}{\ln 2} \iff \dfrac{e^x}{x} = \dfrac{2}{\ln 2}$ $f(x) = \dfrac{2}{\ln 2}$ προφανείς $x = \ln 2$ $x = \ln 4$

60. $f(x) = \begin{cases} e^{x^2} - 1, & x \leq 0 \\ x^2 e^{-x}, & x > 0 \end{cases}$

(α) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} e^{x^2} - 1 = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x^2 e^{-x} = 0$

Άρα $\boxed{\lim\limits_{x \to 0} f(x) = 0 \quad F(0) = 0}$ Συνεχής στο 0.

$\underline{x < 0}$

$f_1(x) = e^{x^2} - 1$

$f_1'(x) = 2x\, e^{x^2}$

$\rightarrow 2x e^{x^2} = 0$

$2x = 0$

$\boxed{x = 0}$

$\underline{x > 0}$

$f_2(x) = x^2 e^{-x}$

$f_2'(x) = 2x e^{-x} - x^2 e^{-x}$

$f_2'(x) = e^{-x}(2x - x^2)$

$\rightarrow 2x - x^2 = 0$

$x(2 - x) = 0$

$x = 0$ $x = 2$

x		0		2	
f_1'	$-$	0	$+$		$+$
f_2'	$-$	0	$+$	0	$-$
f'	$-$		$+$		$-$
f	↘		↗		↘

(B) $f(e^{-2x}-1) < f(-2x) \quad \forall x>0.$

• $x>0 \Rightarrow -2x<0 \Rightarrow e^{-2x} < e^{0}$

$e^{-2x} < 1$

$e^{-2x} - 1 < 0$

• $x>0 \Rightarrow -2x<0$

Στο $(-\infty, 0)$ η f ↘

$$e^{-2x} - 1 > -2x$$

- $e^{x} \geq x+1$

$$e^{-2x} \geq -2x+1$$

$$e^{-2x} - 1 \geq -2x$$

(8) $f(2e^{2-x}) = f(6-2x)$ $[0,2]$

- $0 \leq x \leq 2 \Rightarrow 0 \geq -x \geq -2$

$$2 \geq 2-x \geq 0$$

$$e^{2} \geq e^{2-x} \geq 1$$

$$\boxed{2e^{2} \geq 2e^{2-x} \geq 2}$$

- $0 \leq x \leq 2 \Rightarrow 0 \geq -2x \geq -4$

$$\boxed{6 \geq 6-2x \geq 2}$$

$\forall x > 2$ u $f \downarrow$

$$2e^{2-x} = 6-2x$$

$$e^{2-x} = 3-x$$

- $e^x \geq x+1$

$$e^{2-x} \geq 2-x+1$$

$$\boxed{e^{2-x} \geq 3-x}$$

To "=" $2-x=0$

$x=2$

⑧ $f\left(\frac{4}{x}\right) > f\left(\frac{8}{x^2}\right)$ $\forall x > 2$

- $x > 2 \Rightarrow \frac{1}{x} < \frac{1}{2} \Rightarrow \frac{4}{x} < 2 \Rightarrow 0 < \frac{4}{x} < 2$
- $x > 2 \Rightarrow x^2 > 4 \Rightarrow \frac{1}{x^2} < \frac{1}{4} \Rightarrow 0 < \frac{8}{x^2} < 2$

$\forall x \in (0,2)$ $f \nearrow$

$$\frac{4}{x} > \frac{8}{x^2}$$

$$4x^2 > 8x$$

$$x^2 > 2x$$

$$\underline{x > 2}$$

62. $f'(x) \neq 0$ $\qquad$ f συν + παρ

$f'(0) < 0$ $\qquad$ f' συν

ⓐ Νδο $f \downarrow$

Αφού $f'(x) \neq 0$ $\forall x \in \mathbb{R}$ και συνεχής

Τότε $f'(x) < 0$ ή $f'(x) > 0$

$\forall x \in \mathbb{R}$.

$f'(0) < 0$

$f'(x) < 0$. $\quad f \downarrow$

ⓑ · Νδο $f(2e^x) < f(1-x^2)$ $\forall x \in \mathbb{R}$.

$2e^x > 1 - x^2$.

• $e^x \geq x+1 \Rightarrow 2e^x \geq 2x+2 \Rightarrow 2e^x + x^2 \geq x^2 + 2x + 2$

$2e^x + x^2 - 1 \geq x^2 + 2x + 1$

$2e^x + x^2 - 1 \geq (x+1)^2 \geq 0$ $\iff$ $2e^x + x^2 - 1 > 0$

$2e^x > 1 - x^2$.

(γ) i) $f(3x+1)+1=f(x+1)+e^x$

Προφανής ρίζα το $x=0$

• Αν $x>0$

→ $3x>x \Rightarrow 3x+1>x+1 \Rightarrow f(3x+1)<f(x+1)$

→ Αν $x>0 \Rightarrow e^x>e^0 \Rightarrow e^x>1$

$$\left.\begin{aligned} f(3x+1)&<f(x+1) \\ 1&<e^x \end{aligned}\right\} \oplus \quad \boxed{f(3x+1)+1<f(x+1)+e^x}$$

• Αν $x<0$

• $3x<x \Rightarrow 3x+1<x+1 \Rightarrow f(3x+1)>f(x+1)$

• Αν $x<0 \Rightarrow e^x<e^0 \Rightarrow e^x<1$

$$\left.\begin{aligned} f(3x+1)&>f(x+1) \\ 1&>e^x \end{aligned}\right\} \oplus \quad \boxed{f(3x+1)+1>f(x+1)+e^x}$$

1. $f(x^7) - \ln x = f(x^6)$

$f(x^7) - \ln \frac{x^7}{x^6} = f(x^6)$

$f(x^7) - \ln x^7 = f(x^6) - \ln x^6$

$\varphi(x) = f(x) - \ln x$

$\varphi'(x) = f'(x) - \frac{1}{x} < 0$

$\varphi \searrow$

$\varphi(x^7) = \varphi(x^6)$

$x^7 = x^6$

$x^6(x-1) = 0$

$\cancel{x=0}$ $x=1$

iii) $f(\eta\mu^2 x) + e^{x^2} = f(x^2) + 1$

$x = 0$
προφανής

- $x^2 > 0 \Rightarrow e^{x^2} > e^0 \Rightarrow e^{x^2} > 1$
- $|\eta\mu x| < |x| \Rightarrow \eta\mu^2 x < x^2$

$f(\eta\mu^2 x) > f(x^2)$

$(+)$

$f(\eta\mu^2 x) + e^{x^2} > f(x^2) + 1$

Ασκησεις

για 10/12/25 Τριτη

24

49

58

59

66

67

69

70

71