23.

Ενότητα 2G

$f(x) = x^6 + 6x + 7$

(α) $f'(x) = 6x^5 + 6 = 6(x^5+1)$

$\rightarrow f'(x) = 0 \Rightarrow x^5 + 1 = 0$

$x^5 = -1$

$x = -1$

x		-1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(-1)$

$f(x) \geq 2$

(β) $\lim\limits_{x \to -1} \dfrac{1}{f(x)-2} = +\infty$

$\rightarrow f(x) \geq 2 \Rightarrow f(x) - 2 \geq 0$

(γ) $f(x) + (x+1)^2 = 2$

$\underbrace{f(x) - 2}_{\oplus} + \underbrace{(x+1)^2}_{\oplus} = 0$

• $f(x) - 2 \geq 0$ Το "=" για $x = -1$

• $(x+1)^2 \geq 0$ Το "=" για $x = -1$

Γενικά $f(x) - 2 + (x+1)^2 \geq 0$

Το "=" για $x = -1$

δ) $f(f(x)) = 83$

$f(f(x)) = f(2)$

• $2 > -1$
• $f(x) \geq 2$
$\Big\}\ \forall x > -1$ $f \nearrow \Rightarrow f$ 1-1

$f(x) = 2$

$\underline{\underline{x = -1}}$

Γνωρίζω ότι

$f(x) \geq 2$

Το "=" $x = -1$

(ε) $3f(\alpha) + 2f(\ln\beta) = 10$

Γνωρίζω ότι $f(x) \geq 2 \quad \forall x \in R$

$$\left.\begin{array}{l} \cdot\ f(\alpha) \geq 2 \Rightarrow 3f(\alpha) \geq 6 \\ \cdot\ f(\ln\beta) \geq 2 \Rightarrow 2f(\ln\beta) \geq 4 \end{array}\right\} \oplus$$

$$3f(\alpha) + 2f(\ln\beta) \geq 10$$

Το "=" $\alpha = -1$

$\beta = \frac{1}{e}$.

$f(\alpha) \geq 2$

$\Rightarrow \alpha = -1$ Το "="

$f(\ln\beta) \geq 2$

$\Rightarrow \ln\beta = -1$

$\beta = \frac{1}{e}$

33. $f(1)=2$

$f'(1)=0$

$f''(x)>0$

(a)

x		1	
f''	$+$		$+$
f'	$\nearrow\ -$	0	$+\ \nearrow$
f	$\searrow$		$\nearrow$

$f(x)\geq f(1)$

$f(x)\geq 2$

(β) $\dfrac{f(\alpha)-2}{x-2}+\dfrac{f(\beta)-2}{x}=0 \qquad (0,2)$

$g(x)=x\left(f(\alpha)-2\right)+(x-2)\left(f(\beta)-2\right)$

$$\left.\begin{array}{l} g(0)=-2\left(f(\beta)-2\right)<0 \\ g(2)=2\left(f(\alpha)-2\right)>0 \end{array}\right\}$$

$g(0)g(2)<0$

Bolzano

$\exists\,\xi\in(0,2)$

τ.ω

$g(\xi)=0.$

Αφου $f(x)\geq 2 \quad \forall x\in R$

$f(\alpha)\geq 2 \Rightarrow f(\alpha)-2>0$

$f(\beta)\geq 2 \Rightarrow f(\beta)-2>0$

H $g(x) = x(f(\alpha)-2) + (x-2)(f(\beta)-2)$

Είναι πολυωνυμο 1ου βαθμου

Εχει το πολυ 1 ριζα.

Εχουμε ηδη βρει 1 απο

τον Bolzano η οποια

είναι και μοναδικη.

| 32. $f(x) = \dfrac{\ln x}{x}$

ⓐ $f'(x) = \dfrac{\frac{1}{x}x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$

$\rightarrow f'(x) = 0 \Rightarrow 1 - \ln x = 0$ $\underline{x = e}$

x		e	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

$f(x) \leq f(e)$

$f(x) \leq \frac{1}{e}$

$e f(x) \leq 1$

ⓑ $e f\left(\frac{1+e^2}{e} - f(x)\right) - 1 = 0$

$f\left(\frac{1+e^2}{e} - f(x)\right) = \frac{1}{e}$

$\frac{1+e^2}{e} - f(x) = e$

$\frac{1+e^2}{e} - e = f(x) \Rightarrow \frac{1+e^2}{e} - \frac{e^2}{e} = f(x)$

$f(x) = \frac{1}{e} \Leftrightarrow$ $\underline{x = e}$

① $f(x) + f\left(\frac{x^2}{e}\right) = \frac{2}{e}$

$\left.\begin{array}{l} \cdot\ f(x) \le \frac{1}{e} \\ \\ \cdot\ f\left(\frac{x^2}{e}\right) \le \frac{1}{e} \end{array}\right\} \oplus\ f(x) + f\left(\frac{x^2}{e}\right) \le \frac{2}{e}$

Το "=" $\underline{\underline{x = e}}$

Αφού $f(x) \le \frac{1}{e}$ το "=" $\underline{x = e}$

Αφού $f\left(\frac{x^2}{e}\right) \le \frac{1}{e}$ το "=" $\frac{x^2}{e} = e$

$x^2 = e^2$

$\underline{x = e}$ $\quad$ $\underline{x = -e}$

31. $f(x) = 2e^{x-2} - x^2 + 2x - 2$

(a) $f'(x) = 2e^{x-2} - 2x + 2$

$f'(x) = 2\left(e^{x-2} - x + 1\right) \geq 0$

$\oplus$

$e^x \geq x+1$

$f \nearrow$

$e^{x-2} \geq x-2+1$

$e^{x-2} \geq x-1$

$e^{x-2} - x + 1 \geq 0$

(B) $\begin{cases} g(x) = h(x) \\ g'(x) = h'(x) \end{cases} \Rightarrow \begin{cases} 2e^{x-2} = x^2 - 2x + 2 \\ 2e^{x-2} = 2x - 2 \end{cases}$

$\Rightarrow \begin{cases} f(x) = 0 \\ f'(x) = 0 \end{cases}$ $\Rightarrow f(x) = f(2) \Rightarrow x = 2$

$\Rightarrow$

$f'(2) = 0$ ✓

24. $f(x) = e^{x-1} - \ln x + 1 \quad , x>0$

ⓐ $f'(x) = e^{x-1} - \frac{1}{x} \qquad f'(1)=0$

$f''(x) = e^{x-1} + \frac{1}{x^2} > 0$

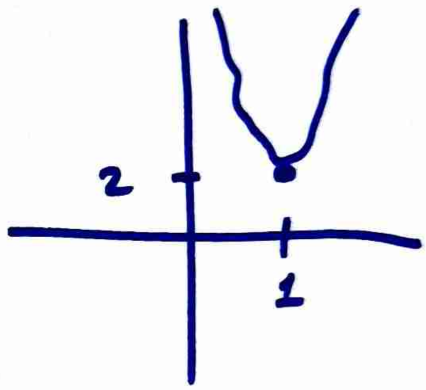
x	0	1	
f''		$+$	$+$
f'		$\nearrow$ $-$	0 $+$ $\nearrow$
f		$\searrow$	$\nearrow$

$x<1 \Rightarrow f'(x) < f'(1) \Rightarrow f'(x) < 0$

$x>1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x) > 0$

$f(x) \geq f(1)$

$\underline{\underline{f(x) \geq 2}}$



ⓑ $f\left(f(x^2+1) - 1\right) = 2$

1

$f(x^2+1) - 1 = 1$

$f(x^2+1) = 2$

1

$x^2+1=1 \qquad x^2=0 \qquad \underline{\underline{x=0}}$

(γ) $f(x) + f(x^{2020}) = 4$

$\left.\begin{array}{l} \cdot\ f(x) \geq 2 \\ \\ \cdot\ f(x^{2020}) \geq 2 \end{array}\right\} \oplus$

Το "=" $x=1$

Το "=" $x^{2020} = 1$

$\Leftrightarrow x=1$ ή $x=-1$

$$\boxed{f(x) + f(x^{2020}) \geq 4}$$

Λύση : $\underline{\underline{x=1}}$

(δ) $\lim\limits_{x \to 1} \dfrac{\ln x}{(x-1)(f(x)-2)} = \lim\limits_{x\to 1} \dfrac{\ln x}{x-1} \dfrac{1}{f(x)-2}$

$\rightarrow \lim\limits_{x \to 1} \dfrac{\ln x}{x-1} = \lim\limits_{x \to 1} \dfrac{1}{x} = 1$ $\qquad = 1 \cdot (+\infty)$

$= +\infty$

ε) $f(f(x)) > e + \ln\frac{e}{2}$

$f(f(x)) > e + \ln e - \ln 2$

$f(f(x)) > e + 1 - \ln 2$

$f(f(x)) > f(2)$

- $f(x) \geq 2$
- $2 > 1$

$\Big\}\ \forall x > 1$ η f ↗

$f(x) > 2$

Λυσαι: $x \in R - \{1\}$.

Γνωρίζω ότι $f(x) \geq 2 \ \forall x \in R$

το "=" για $x = 1$

Ενότητα 27

8. $f(x) = x^3+2x+1$

(α) $f'(x) = 3x^2+2>0$

$f \nearrow$

x	$-\infty$ $\quad$ $+\infty$
$f(x)$	$-\infty$ $\nearrow$ $+\infty$

$\Sigma T_f = \mathbb{R}$

$\lim\limits_{x\to-\infty} f(x) = -\infty$

$\lim\limits_{x\to+\infty} f(x) = +\infty$

- f συνεχής
- $f \nearrow$
- $\Sigma T_f = \mathbb{R}$

$\Big\}$ Το $0 \in \Sigma T_f$ άρα $\exists!\ \xi \in \mathbb{R}$ τ.ω. $f(\xi)=0$

Έστω $\xi \geq 0 \Rightarrow f(\xi) \geq f(0)$

$0 \geq 1$

Άτοπο!

Άρα $\xi<0$

13

$f(x) = f(-1)$

f 1-1

$x \cancel{=} -1$ στο $(0, +\infty)$

Αδύνατη.

12. (a) $f(x) = e^x$

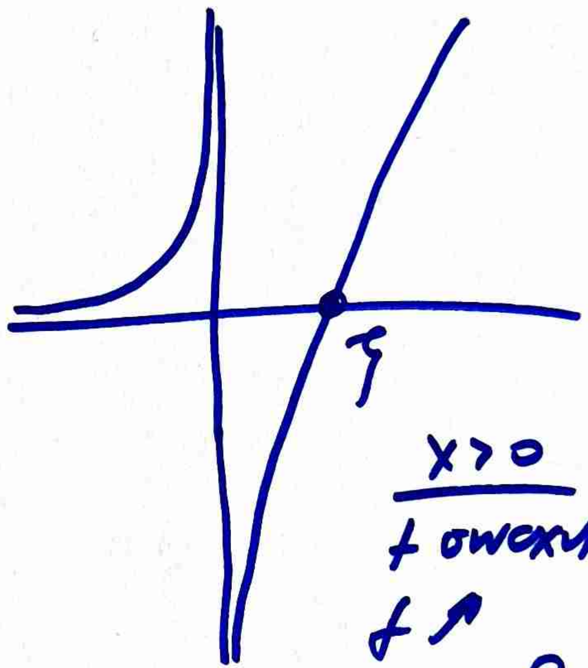
$g(x) = \frac{1}{x}$

$f(x) = g(x)$

$e^x = \frac{1}{x}$

$\underbrace{e^x - \frac{1}{x}}_{h(x)} = 0$

x	$-\infty$		0		$+\infty$
$h(x)$	0	$\nearrow$	$+\infty$ ‖ $-\infty$	$\nearrow$	$+\infty$



ξ

$h'(x) = e^x + \frac{1}{x^2} > 0$

$h \nearrow$

$\lim\limits_{x \to -\infty} h(x) = 0$

$\lim\limits_{x \to 0^-} h(x) = +\infty$

$\lim\limits_{x \to +\infty} h(x) = +\infty$

$\lim\limits_{x \to 0^+} h(x) = -\infty$

$\underline{x < 0}$

$\Sigma T_f = (0, +\infty)$

Δεν έχω
ρίζα
εδώ

$\underline{x > 0}$

f συνεχής

$f \nearrow$

$\Sigma T_f = \mathbb{R}$

Το $0 \in \Sigma T_f$

άρα $\exists!\, \xi$

τ.ω. $h(\xi) = 0$

13. ⓐ $2x + \ln x = 1 \qquad (0,1)$

$g(x) = 2x + \ln x - 1$

$g'(x) = 2 + \frac{1}{x} > 0 \quad g \nearrow$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^+} g(x) = -\infty \\ \lim\limits_{x \to 1^-} g(x) = 1 \end{array}\right\} \Sigma T_g = (-\infty, 1)$$

- g συνεχ
- $g \nearrow$
- $\Sigma T_f = (-\infty, 1)$

το $0 \in \Sigma T_g$ άρα

$\exists!\ \xi \in (0,1)$

τ.ω $g(\xi) = 0$

11. (a) $e^{x} = 3x$

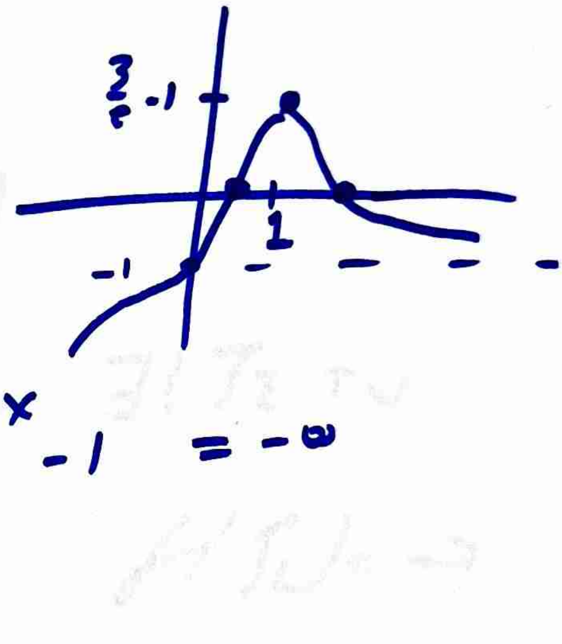
$$1 = \frac{3x}{e^{x}}$$

$$1 = 3xe^{-x}$$

$$\underbrace{3xe^{-x} - 1}_{f(x)} = 0$$

$$f'(x) = 3e^{-x} - 3xe^{-x} = 3e^{-3x}(1-x)$$

x		1	
f'	$+$		$-$
f	$\nearrow$		$\searrow$



$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3xe^{-x} - 1 = -\infty$$

$$f(1) = \frac{3}{e} - 1 = \frac{3-e}{e} > 0$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} 3xe^{-x} - 1 = \lim_{x\to+\infty} \frac{3x}{e^{x}} - 1$$

$$= -1$$

$x < 1$

- $f \uparrow$
- f conv (κυρτή)
- $\Sigma T_f = \left(-\infty, \frac{3}{e} - 1\right]$

Το $0 \in \Sigma T_f$ άρα $\exists ! \ \xi_1$ τ.ω $f(\xi_1) = 0$

$x > 1$

- $f \downarrow$
- f conv (κυρτή)
- $\Sigma T_f = \left(-\infty, \frac{3}{e} - 1\right]$

Το $0 \in \Sigma T_f$ άρα $\exists ! \ \xi_2$ τω

$f(\xi_2) = 0$

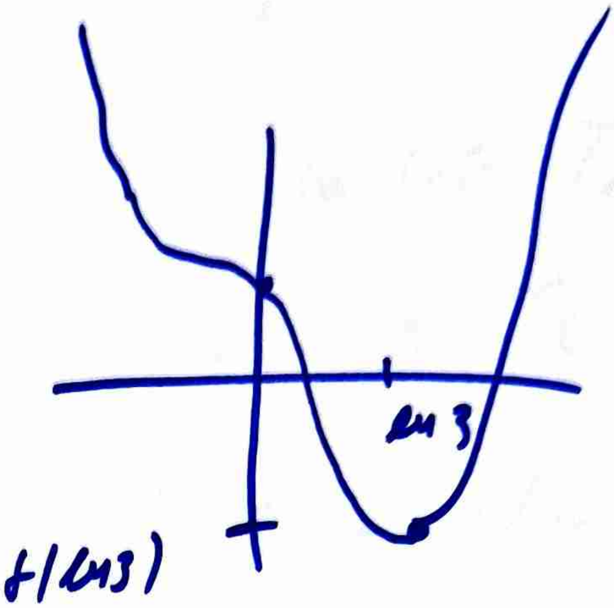
B' τρόπος

$$e^x = 3x$$

$$\underbrace{e^x - 3x}_{f(x)} = 0$$

$$f'(x) = e^x - 3$$

x		$\ln 3$	
f'	$-$		$+$
f	$\searrow$		$\nearrow$



$$\lim_{x \to -\infty} f(x) = +\infty$$

$$f(\ln 3) = 3 - 3\ln 3 = 3(1 - \ln 3) = 3(\ln e - \ln 3)$$
$$= 3\left(\ln \frac{e}{3}\right) < 0$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^x - 3x$$

$$= \lim_{x \to +\infty} e^x\left(1 - \frac{3x}{e^x}\right) = +\infty$$

$$\to \lim_{x \to +\infty} \frac{3x}{e^x} = \lim_{x \to +\infty} \frac{3}{e^x} = 0$$

$x < \ln 3$

f συνεχής

$f \downarrow$

$\Sigma T_f = [3\ln\frac{e}{3}, +\infty)$

Το $0 \in \Sigma T_f$

άρα $\exists!\ \xi_1$ τ.ω

$f(\xi_1) = 0$

$x > \ln 3$

f συνεχής

$f \uparrow$

$\Sigma T_f = [3\ln\frac{e}{3}, +\infty)$

Το $0 \in \Sigma T_f$

άρα $\exists!\ \xi_2$

τ.ω $f(\xi_2) = 0$

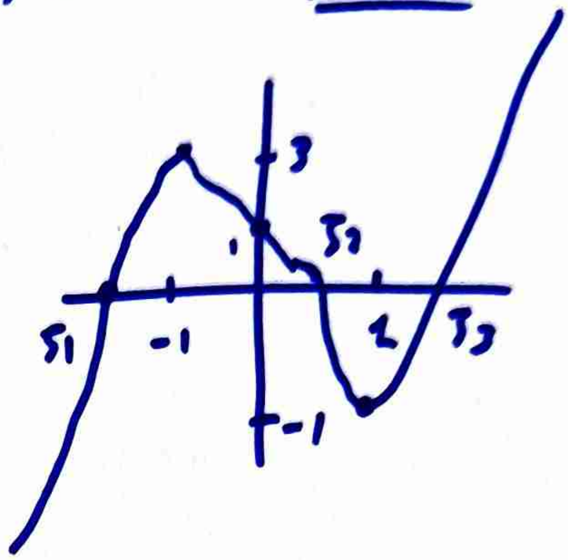
10. ⓐ $x^3-3x+1=0$

$f(x)=x^3-3x+1$

$f'(x)=3x^2-3=3(x^2-1)$

$\to f'(x)=0 \Rightarrow x^2-1=0$ $\underline{\underline{x=\pm 1}}$

x		-1		1	
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$



$\lim_{x\to-\infty} f(x)=-\infty$

$f(-1)=3$

$f(1)=-1$

$\lim_{x\to+\infty} f(x)=+\infty$

$\underline{x<-1}$

- f συνεχής
- $f \nearrow$
- $\Sigma T f=(-\infty,3)$

Το $0\in \Sigma Tf$
άρα $\exists!\ \xi_1<0$
τ.ω $f(\xi_1)=0$

$\underline{-1\le x\le 1}$

- f συνεχής
- $f \searrow$
- $\Sigma Tf=[-1,3]$

Το $0\in\Sigma Tf$
άρα $\exists!\ \xi_2$
τ.ω $f(\xi_2)=0$

Έστω $\xi_2\le 0$
$f(\xi_2)\ge f(0)$
$0\ge 1$
Άτοπο $\xi_2>0$

$\underline{x>1}$

- f συνεχής
- $f\nearrow$
- $\Sigma Tf=[-1,+\infty)$

$\left.\right\}$ Το $0\in\Sigma Tf$
άρα $\exists!\ \xi_3>0$
τ.ω $f(\xi_3)=0$

9. $f(x) = x^4 - 4x + 1$

(α) $f'(x) = 4x^3 - 4 = 4(x^3 - 1)$

$\rightarrow f'(x) = 0 \Rightarrow x^3 - 1 = 0 \Rightarrow x^3 = 1$

$\underline{\underline{x = 1}}$

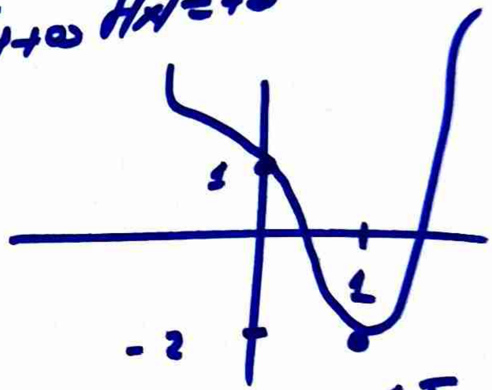
x		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$\lim_{x \to -\infty} f(x) = +\infty$

$\lim_{x \to +\infty} f(x) = +\infty$

$f(x) \geq f(1)$

$\underline{\underline{f(x) \geq -2}}$



$\Sigma T_f = [-2, +\infty)$

(β) $\dfrac{4x}{x^4+1} = 1$

$4x = x^4 + 1 \Rightarrow f(x) = 0$

$\underline{x < 1}$

- f συνεχής
- $f \downarrow$
- $\Sigma T_f = [-2, +\infty)$

Το $0 \in \Sigma T_f$ άρα $\exists!\, \xi_1$ τ.ω $f(\xi_1) = 0$

$\underline{x \geq 1}$

- f συνεχής
- $f \uparrow$
- $\Sigma T_f = [-2, +\infty)$

Το $0 \in \Sigma T_f$ άρα $\exists!\, \xi_2$ τ.ω $f(\xi_2) = 0$

Επομενο Μαθημα

27	32
21	2 α
35	3 γ
40	
41	