10.

Ενότητα 36

(B) $E: C_f, C_g, x=1, x=2$

$$E = \int_1^2 |f(x) - g(x)|\, dx =$$

$$= \int_1^2 |\ln x - x|\, dx = \int_1^2 x - \ln x\, dx$$

- $\ln x \leq x - 1$

$\ln x - x \leq -1$

$\ln x - x < 0$

$$= \int_1^2 x\, dx - \int_1^2 \ln x\, dx$$

$$= \frac{1}{2}\left(x^2\right)_1^2 - \int_1^2 1 \cdot \ln x\, dx$$

$$= \frac{3}{2} - \int_1^2 (x)' \ln x\, dx$$

$$= \frac{3}{2} - \left((x \ln x)_1^2 - \int_1^2 x \cdot \frac{1}{x}\, dx \right)$$

$$= \frac{3}{2} - \left(2\ln 2 - (x)_1^2 \right)$$

$$= \frac{3}{2} - 2\ln 2 + 1$$

11. $f(x) = x^3$ $\qquad$ $g(x) = 4x$

(01) $\mathcal{E} : C_f, C_g$

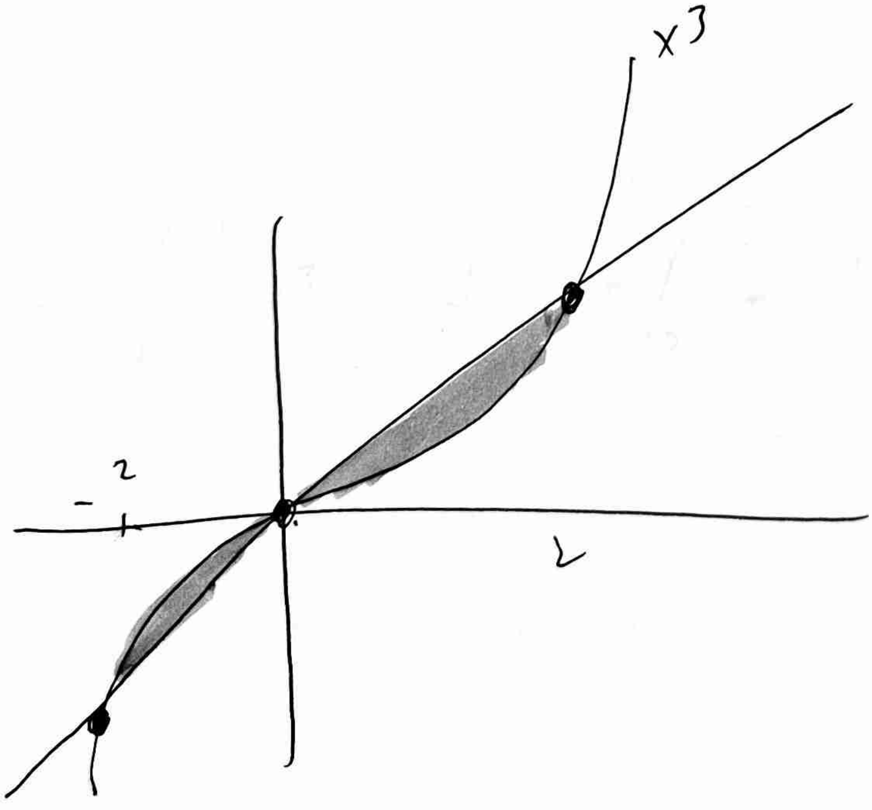
$f(x) = g(x) \iff x^3 = 4x \Rightarrow x^3 - 4x = 0$

$x(x^2-4) = 0$

x		-2		0		2	
x	$-$		$-$	0	$+$		$+$
x^2-4	$+$	0	$-$		$-$	0	$+$
x^3-4x	$-$		$+$		$-$		$+$

$x=0$ $\quad$ $x=2$ $\quad$ $x=-2$

$$\mathcal{E} = \int_{-2}^{2} |f(x) - g(x)|\, dx = \int_{-2}^{0} f(x) - g(x)\, dx + \int_{0}^{2} -f(x) + g(x)\, dx.$$



(B) i) $C_f, C_g, x=2, x=3$.

$$E = \int_2^3 |f(x)-g(x)|\,dx = \int_2^3 |x^3-4x|\,dx$$

$$= \int_2^3 x^3-4x\,dx = \cdots$$

ii). $E: C_f, C_g, x=-2, x=3$.

$$E = \int_{-2}^3 |f(x)-g(x)|\,dx = \int_{-2}^3 |x^3-4x|\,dx$$

$$= \int_{-2}^0 x^3-4x\,dx + \int_0^2 -x^3+4x\,dx + \int_2^3 x^3-4x\,dx$$

$$= \_\_\_$$

13. E : C_f, C_g, $x=2$

δ) $f(x)=\frac{1}{x}$, $x>0$ $g(x)=\sqrt{x}$

$f(x)=g(x)$

$\frac{1}{x}=\sqrt{x}$

$\frac{1}{x^2}=x$

$1=x^3$

$x=1$

$$E=\int_1^2 |f(x)-g(x)|\,dx=$$

$$=\int_1^2 \Big|\underbrace{\overset{\ominus}{\frac{1}{x}-\sqrt{x}}}_{h(x)}\Big|\,dx=\int_1^2 \sqrt{x}-\frac{1}{x}\,dx$$

κτλ.

$$h'(x)=-\frac{1}{x^2}-\frac{1}{2\sqrt{x}}<0$$

$h \downarrow$

$1<x<2$

$h\downarrow$

$h(1)>h(x)>h(2)$

$0>h(x)$

5c $\quad E = \int_1^2 |f(x) - g(x)|\,dx = \int_1^2 |\overset{\ominus}{h(x)}|\,dx \quad (*)$

$f(x) = g(x)$

$\frac{1}{x} = 1 + \ln x$

$\underbrace{\frac{1}{x} - 1 - \ln x}_{h(x)} = 0$

$h(x) = 0$

$h(x) = h(1)$

h 1-1

$x = 1$

$h'(x) = -\frac{1}{x^2} - \frac{1}{x} < 0$

$h \downarrow$

$1 < x < 2$

$h \downarrow$

$h(1) > h(x) > h(2)$

$0 > h(x)$

$(*) \quad \int_1^2 \ln x + 1 - \frac{1}{x}\,dx$

κτλ.

17. $f(x) = x - 2 + \dfrac{e^x}{e^{2x} - 1}$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x - 2 + \frac{e^x}{e^{2x} - 1} = +\infty$$

δεν σχυ ορίζονται

$$\to \lim_{x \to +\infty} \frac{e^x}{e^{2x} - 1} = \lim_{x \to +\infty} \frac{e^x}{2e^{2x}} = \lim_{x \to +\infty} \frac{1}{2e^x} = 0$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{x - 2 + \frac{e^x}{e^{2x} - 1}}{x}$$

$$= \lim_{x \to +\infty} \frac{x}{x} - \frac{2}{x} + \frac{\frac{e^x}{e^{2x} - 1}}{x}$$

$$= \lim_{x \to +\infty} 1 - \cancel{\frac{2}{x}}^{\,0} + \cancel{\frac{1}{x}}^{\,0} \cancel{\frac{e^x}{e^{2x} - 1}}^{\,0} = 1$$

$$\lim_{x \to +\infty} (f(x) - x) = \lim_{x \to +\infty} \cancel{x} - 2 + \frac{e^x}{e^{2x} - 1} - \cancel{x} = -2$$

$y = x - 2$

$E : C_f,\ y = x-2,\ x=1,\ x=e$

$$E = \int_1^e \left| f(x) - (x-2) \right| dx$$

$$= \int_1^e \left| \cancel{x-2} + \frac{e^x}{e^{2x}-1} \cancel{-x} \cancel{+2} \right| dx$$

$$= \int_1^e \left| \overset{+}{\frac{e^x}{e^{2x}-1}} \right| dx = \int_1^e \frac{e^x}{e^{2x}-1} dx$$

$\bullet\ 2x > 0 \Rightarrow e^{2x} > e^0 \Rightarrow e^{2x} > 1$

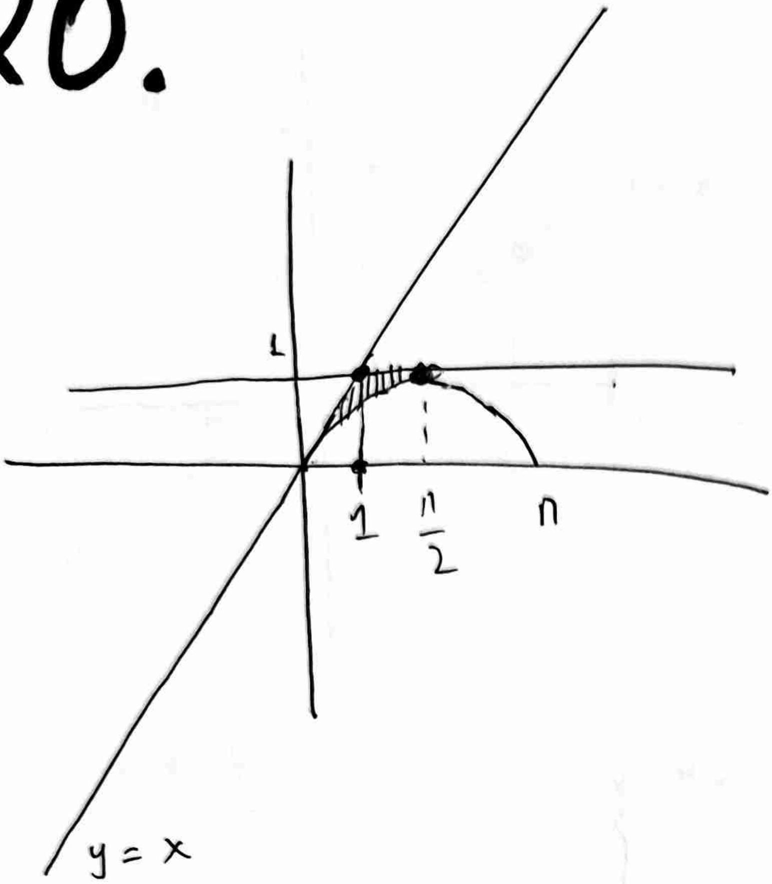
$e^{2x} - 1 > 0$

$$I = \int_1^e \frac{e^x}{e^{2x}-1} dx \xrightarrow[e^x dx = dt]{e^x = t} \int_e^{e^e} \frac{1}{t^2-1} dt$$

A, B.

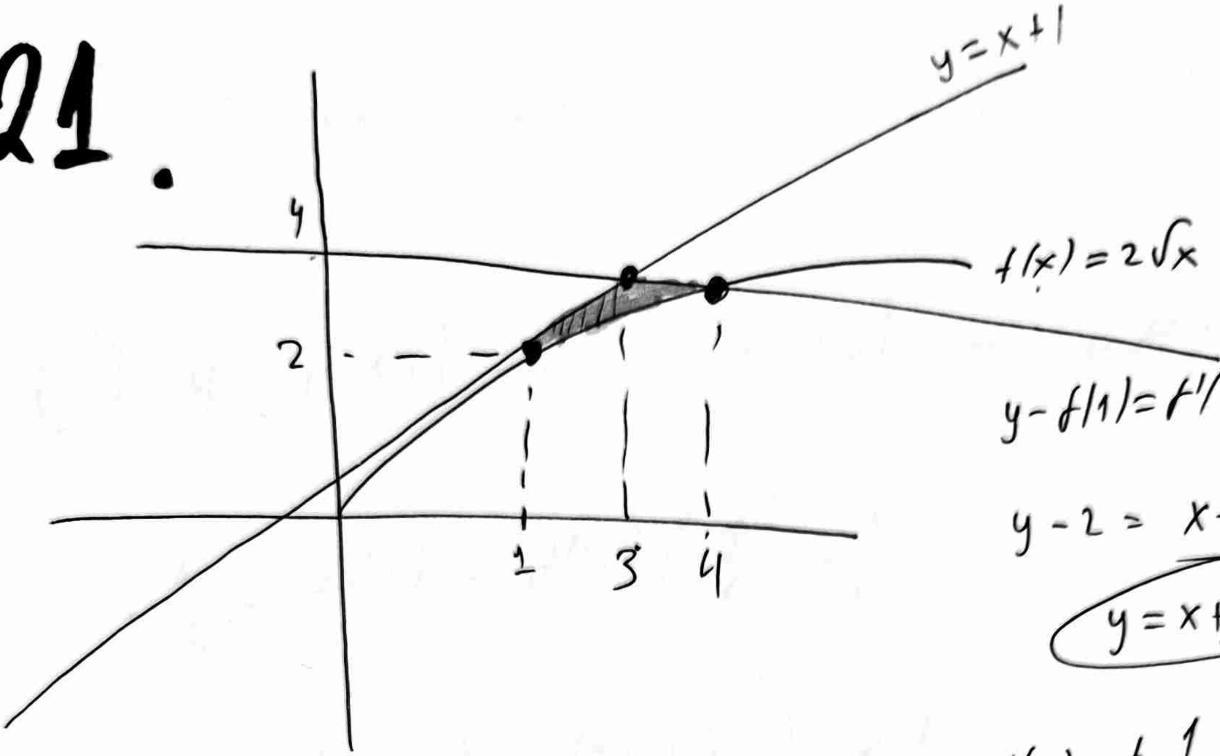
20.

$$\begin{cases} y = 1 \\ y = x \end{cases} \Rightarrow x = 1$$



$$E = \int_0^1 x - n\gamma x \, dx + \int_1^{\frac{n}{2}} 1 - n\gamma x \, dx.$$

21.



$y - f(1) = f'(1)(x-1)$

$y - 2 = x - 1$

$y = x + 1$

$f'(x) = \cancel{2}\frac{1}{\cancel{2}\sqrt{x}}$

$$\begin{cases} y = 4 \\ y = x + 1 \end{cases}$$

$4 = x + 1$

$x = 3$

$$\begin{cases} y = 4 \\ f(x) = 2\sqrt{x} \end{cases}$$

$4 = 2\sqrt{x}$

$16 = 4x$

$x = 4$

$$E = \int_1^3 x + 1 - 2\sqrt{x}\ dx + \int_3^4 4 - 2\sqrt{x}\ dx.$$

22. $f(x) = x^2 - x \qquad f'(x) = 2x - 1$

(a) $y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow M\left(\frac{1}{2}, -\frac{1}{2}\right)$

$$-\frac{1}{2} - f(x) = f'(x)\left(\frac{1}{2} - x\right)$$

$$-\frac{1}{2} - (x^2 - x) = (2x-1)\left(\frac{1}{2} - x\right)$$

$$\cancel{-\frac{1}{2}} - x^2 + \cancel{x} = \cancel{x} - 2x^2 \cancel{-\frac{1}{2}} + x$$

$$-x^2 + 2x^2 - x = 0$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

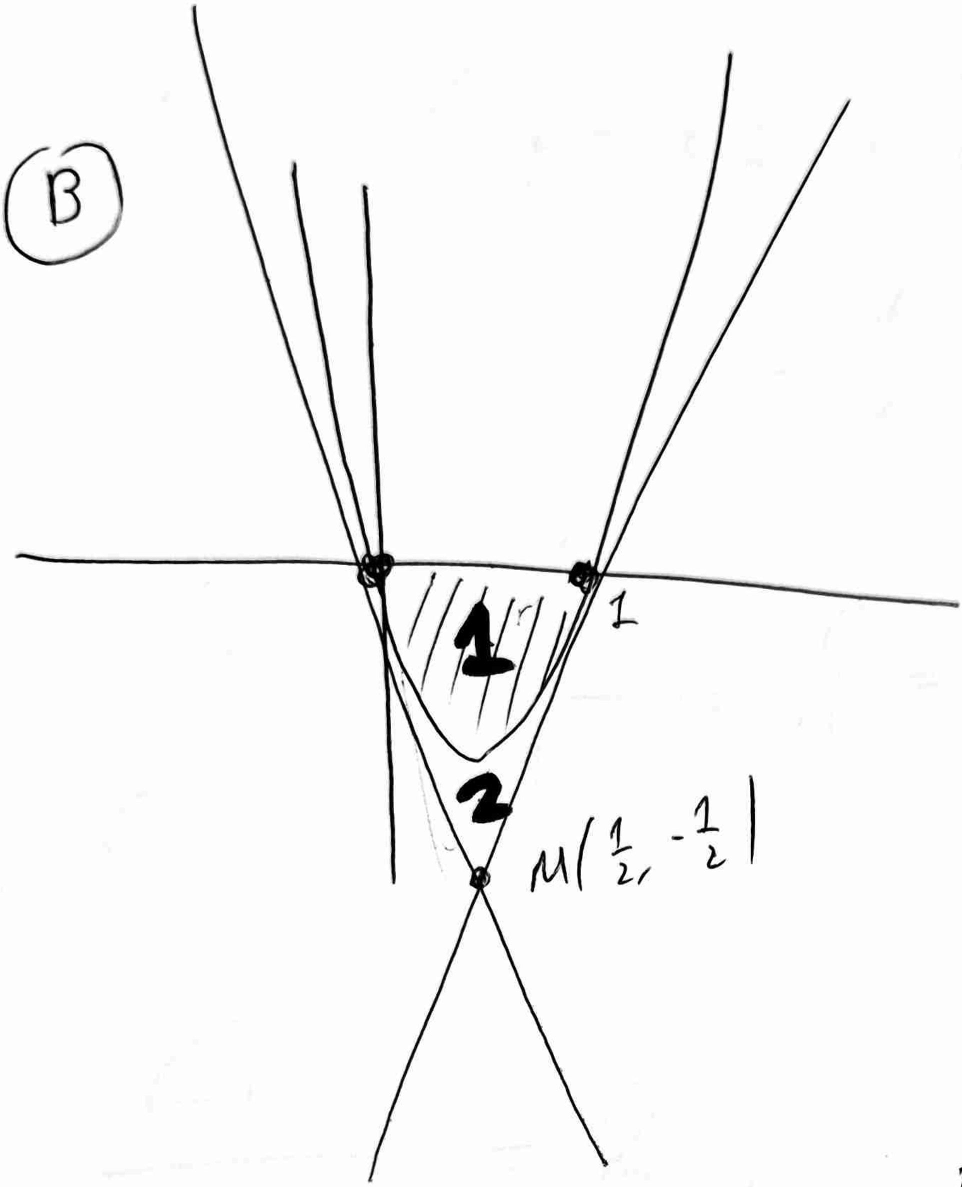
$x = 0 \qquad x = 1$.

$y - f(0) = f'(0)(x - 0) \qquad\qquad y - f(1) = f'(1)(x-1)$

$y - 0 = -x \qquad\qquad y - 0 = x - 1$

$y = -x \qquad\qquad y = x - 1$

B



$$E_{\tau\rho\iota\gamma} = \frac{B \cdot U}{2} = \frac{1 \cdot \frac{1}{2}}{2} = \frac{1}{4}$$

$$E = \int_0^1 |\overset{-}{f(x)}|\, dx = \int_0^1 -x^2 + x \, dx$$

$$= -\frac{1}{3}\left(x^3\right)_0^1 + \frac{1}{2}\left(x^2\right)_0^1 = -\frac{1}{3} + \frac{1}{2}$$

$$= \frac{1}{6}$$

$$\epsilon_2 = \epsilon_{Tp} - \epsilon_1 = \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12}$$

$$\frac{\epsilon_2}{\epsilon_1} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{6}{12} = \frac{1}{2}$$

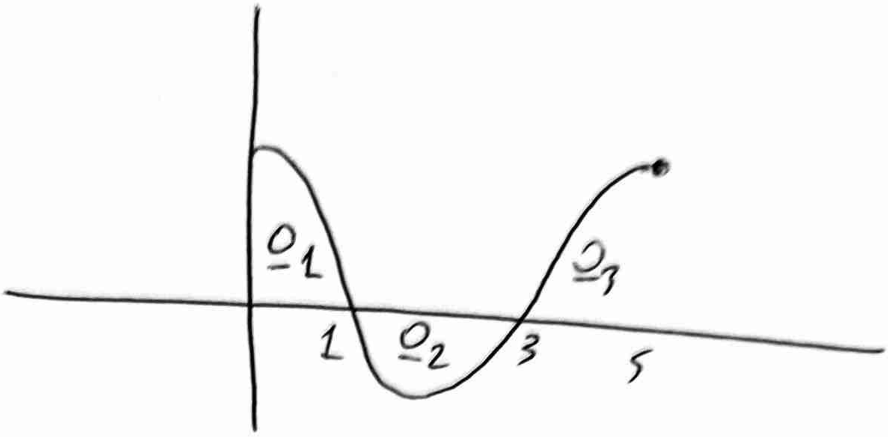
$$\frac{\epsilon_1}{\epsilon_2} = 2 ,$$

23.

$E(\Omega_1) = 1$

$E(\Omega_2) = 2$

$E(\Omega_3) = 3$



(a) $\int_0^1 f(x)\,dx$

$$E(\Omega_1) = \boxed{\int_0^1 f(x)\,dx = 1}$$

$$E(\Omega_2) = \int_1^3 |f(x)|\,dx = -\int_1^3 f(x)\,dx$$

$$2 = -\int_1^3 f(x)\,dx$$

$$\int_1^3 f(x)\,dx = -2$$

δ $E(\Omega_3) = \int_3^5 f(x)\,dx = 3$

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= 1 - 2 + 3 = 2.$$

Ενότητα 38

1. $f(x) = \dfrac{e^x}{x+1}, \quad x > -1$

F παράγουσα της f

$F(0) = 0.$

F

f

f'

f''

(α) $F'(x) = f(x) = \dfrac{e^x}{x+1} > 0$

$F \nearrow$

$$F''(x) = f'(x) = \frac{e^x(x+1) - e^x}{(x+1)^2} = \frac{e^x x}{(x+1)^2}$$

x	-1		0	
F''		$-$	0	$+$
F		$\frown$		$\smile$

$y - F(0) = F'(0)(x - 0)$

$y - 0 = f(0)x$

$\boxed{y = x}$

(β) $F(x) = x$

$\forall x > 0$ η F κυρτή $F(x) > x$

$\forall x < 0$ η F κοίλη $F(x) < x$

Το "$=$" για $x = 0$

(γ) Νδο $\displaystyle\int_0^1 \frac{F(x)}{x^2+1}\,dx > \frac{\ln 2}{2}$.

$\forall x > 0$ η $F(x) > x$

$$\frac{F(x)}{x^2+1} > \frac{x}{x^2+1}$$

$$\int_0^1 \frac{F(x)}{x^2+1}\,dx > \int_0^1 \frac{x}{x^2+1}\,dx$$

$$\rightarrow \int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\Big(\ln|x^2+1|\Big)_0^1$$

$$= \frac{1}{2}\ln 2 \quad \checkmark$$

2. $f(x) = \dfrac{e^x}{x-1}$

F παράγουσα της f στο $(1,+\infty)$

$F(2)=0$

F

f

f'

f''

(α) $F'(x) = f(x) = \dfrac{e^x}{x-1} > 0$

$F \nearrow$

$F(x^2+2) > 0$

$F(x^2+2) > F(2)$

$F \nearrow$

$x^2+2 > 2$

$x^2 > 0$

$x \neq 0$

(β) $F'' = f'(x) = \dfrac{e^x(x-1)-e^x}{(x-1)^2} = \dfrac{e^x(x-2)}{(x-1)^2}$

x	1	2	
F''		$-$ 0 $+$	
F		$\cap$	$\cup$

$y - F(2) = F'(2)(x-2)$

$y - 0 = f(2)(x-2)$

$\boxed{y = e^2(x-2)}$

(γ) E : C_F , $x'x$, $x=3$ νδο $2E > e^2$

$F(x) = 0$

$F(x) = F(2)$

F 1-1

$x=2$

$$E = \int_2^3 |F(x)|\, dx = \int_2^3 F(x)\, dx$$

Αν $x > 2$

$F \nearrow$

$F(x) > F(2)$

$F(x) > 0$

Αρκει νδο $2\int_2^3 F(x)\, dx > e^2$

$$\int_2^3 F(x)\, dx > \frac{e^2}{2}$$

$\forall x > 2$ $\quad F(x)$ κυρτή

$$F(x) > e^2(x-2)$$

$$\int_2^3 F(x)\,dx > \int_2^3 e^2(x-2)\,dx$$

$$\rightarrow \int_2^3 e^2(x-2)\,dx = e^2\left(\int_2^3 x\,dx - \int_2^3 2\,dx\right)$$

$$= e^2\left(\frac{1}{2}\left(x^2\right)_2^3 - 2\,(x)_2^3\right)$$

$$= e^2\left(\frac{5}{2} - 2\right) = e^2\,\frac{1}{2} = \frac{e^2}{2}$$

3.

$f(x) = (x-1)\, e^{x^2}$

$F(1) = 0$

F

f

f'

f''

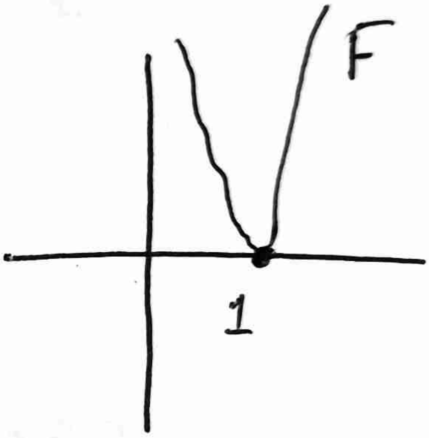
(a) $F'(x) = f(x) = (x-1)\, e^{x^2}$

x		1	
F'	$-$		$+$
F	$\searrow$		$\nearrow$

$F(x) \geq F(1)$

$F(x) \geq 0$

(b) $F\left(\underbrace{F'(x) - 2019}_{1}\right) = 0$



$F'(x) - 2019 = 1$

$F'(x) = 2020$

$f(x) = 2020$.

$f'(x) = e^{x^2} + (x-1)\, 2x\, e^{x^2} = e^{x^2}\left(1 + 2x(x-1)\right)$

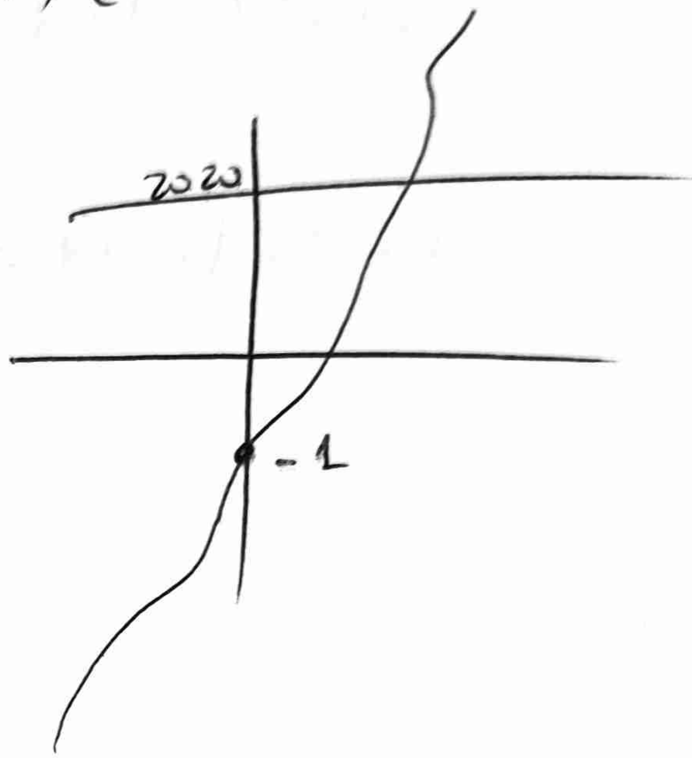
$f'(x) = \left(\underset{+}{2x^2 - 2x + 1}\right) e^{x^2}$

$f \nearrow$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x-1)\, e^{x^2} = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x-1)\, e^{x^2} = +\infty$$

$$\Sigma T_f = \mathbb{R}.$$



- $f \nearrow$
- f συνεχής
- $\Sigma T_f = \mathbb{R}$

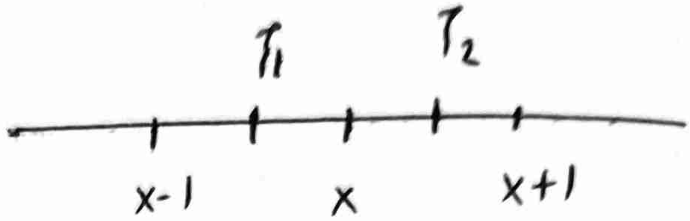
Το $2020 \in \Sigma T_f$

άρα $\exists!\ \xi$ τ.ω. $f(\xi) = 2020$.

β) Νδο F κυρτή

$F''(x) = f'(x) > 0$ $\quad F$ κυρτή.

νδο $F(x-1) + F(x+1) > 2F(x)$ $\quad \forall x \in \mathbb{R}$.



$$F'(\xi_1) = \frac{F(x) - F(x-1)}{x-(x-1)} = F(x) - F(x-1)$$

$$F'(\xi_2) = \frac{F(x+1) - F(x)}{x+1-x} = F(x+1) - F(x)$$

$$\xi_1 < \xi_2$$

F κυρτη $\Rightarrow F' \nearrow$

$$F'(\xi_1) < F'(\xi_2)$$

$$F(x) - F(x-1) < F(x+1) - F(x)$$

$$2F(x) < F(x+1) + F(x-1)$$

| 9. $f \downarrow$ συνεχ

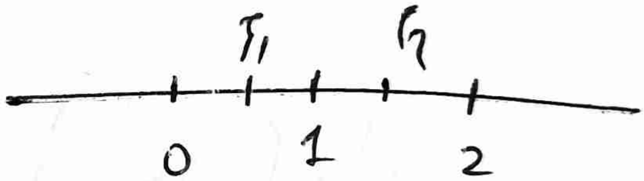
(α) Νδο $\int_0^1 f(x)\,dx > \int_1^2 f(x)\,dx$

Έστω $F(x)$ παράγουσα.

$$\big(F(x)\big)_0^1 > \big(F(x)\big)_1^2$$

$$F(1) - F(0) > F(2) - F(1)$$

$$2F(1) > F(2) - F(0),$$



$$F'(\xi_1) = \frac{F(1)-F(0)}{1-0} = F(1) - F(0)$$

$$F'(\xi_2) = \frac{F(2)-F(1)}{2-1} = F(2) - F(1)$$

$$\xi_1 < \xi_2 \Rightarrow F'(\xi_1) > F'(\xi_2)$$

$F' \downarrow$ $\quad F(1) - F(0) > F(2) - F(1)$

$$2F(1) > F(2) + F(0)$$

(B) Νδο $$\int_0^1 f(x)\,dx > 2\int_{\frac{1}{2}}^{1} f(2x)\,dx$$

$$\int_{\frac{1}{2}}^{1} f(2x)\,dx \;\underset{2dx=dt}{\overset{2x=t}{=\!=\!=}}\; \int_1^2 f(t)\,\frac{1}{2}\,dt$$

$$dx = \frac{1}{2}\,dt$$

$$\int_0^1 f(x)\,dx > \cancel{2}\int_1^2 f(t)\,\frac{1}{\cancel{2}}\,dt$$

$$\int_0^1 f(x)\,dx > \int_1^2 f(x)\,dx$$

14. $f:(0,+\infty) \longrightarrow \mathbb{R}$ συνεχή

F παράγουσα f $\quad F(0)=0$

Αν $F(x) \leq e^x \ln(x+1) \quad \forall x>-1$.

Νδο $f(0)=1$.

$$\underbrace{F(x) - e^x \ln(x+1)}_{\varphi(x)} \leq 0$$

$\varphi(x) \leq 0$

$\varphi(x) \leq \varphi(0)$

Ακρότατο στο 0

Fermat $\varphi'(0)=0$

$$\varphi'(x) = f(x) - \left(e^x \ln(x+1) + e^x \frac{1}{x+1} \right)$$

$$\varphi'(0) = f(0) - 0 - 1 = 0$$

$f(0)=1$.

6. $\int_{f(0)}^{f(1)-2} \ln(1+e^x)\,dx = 0$

Νδο $\exists \xi \in (0,1)$ τ.ω $f'(\xi)=2$.

$e^x > 0$

$1+e^x > 1$

$\ln(1+e^x) > \ln 1$

$$\boxed{\ln(1+e^x) > 0}$$

Άρα $f(1)-2 = f(0)$

$f(1)-f(0)=2$

α' τρόπος

$f'(x)-2=0$

$(\underbrace{f(x)-2x}_{h(x)})'=0$

$h(1)=f(1)-2=f(0)$

$h(0)=f(0)$

$h(1)=h(0)$ Rolle ok.

$\exists \xi \in (0,1)$ τ.ω $h'(\xi)=0$ $f'(\xi)=2$

Β' τρόπος

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0} = \frac{2}{1} = 2$$ ✓

17. $F(1)=0$

$\varepsilon: y = 2x-2$.

εφαπτομενη τη C_f

στο $x_0=1$

$f'(1)=2$

$f(1)=0$

$$\lim_{x\to 1}\frac{F(x)}{x^2-2x+1} =$$

$$= \lim_{x\to 1}\frac{f(x)}{2x-2} = \lim_{x\to 1}\frac{\frac{f(x)-f(1)}{x-1}}{\frac{2\cancel{(x-1)}}{\cancel{x-1}}}$$

$$= \frac{f'(1)}{2} = \frac{2}{2} = 1.$$

18. $f(0)=0$ $\qquad$ $\int \cos x\,dx$

$f'(0)=1$ $\qquad$ $F(0)=0.$

$$\lim_{x\to 0} \frac{xF(x)}{x-\sin x} = \lim_{x\to 0} \frac{F(x)+xf(x)}{1-\cos x}$$

$$= \lim_{x\to 0} \frac{F(x)+xf(x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{F(x)}{x^2}+\frac{f(x)}{x}}{\frac{1-\cos x}{x^2}}$$

$$= \lim_{x\to 0} \frac{\frac{F(x)}{x^2}+\frac{f(x)}{x}}{\frac{1-\cos x}{x^2}}$$

$$= \frac{\frac{1}{2}+1}{\frac{1}{2}} = \frac{1+2}{1} = 3.$$

$$\rightarrow \lim_{x\to 0} \frac{F(x)}{x^2} = \lim_{x\to 0} \frac{f(x)}{2x} = \lim_{x\to 0} \frac{f(x)}{x} \cdot \frac{1}{2}$$

$$= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \; \frac{1}{2} = f'(0)\,\frac{1}{2} = \frac{1}{2}$$

$$\rightarrow \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0) = 1$$

$$\rightarrow \lim_{x\to 0} \frac{1-\sigma\upsilon\nu x}{x^2} = \lim_{x\to 0} \frac{\eta\mu x}{2x} = \frac{1}{2}$$