17

20. $f(x) = x^2 - x + \frac{5}{4}$

$g(x) = e^{-2x}$

$f'(x) = 2x - 1$

$f'(-\frac{1}{2}) = -1 - 1 = -2$

$f(-\frac{1}{2}) = \frac{1}{4} + \frac{1}{2} + \frac{5}{4} = 2$

$y - f(-\frac{1}{2}) = f'(-\frac{1}{2})(x + \frac{1}{2})$

$y - 2 = -2(x + \frac{1}{2})$

$y - 2 = -2x - 1$

$\boxed{y = -2x + 1}$

Αρκει νδο η $y = -2x + 1$ εφαπτεται

της C_g

Αρκει νδο το συστημα $\begin{cases} g'(x) = -2 \\ g(x) = -2x + 1 \end{cases}$

εχει λυση.

$$\begin{cases} -2e^{-2x} = -2 \quad \Rightarrow \quad e^{-2x} = 1 \quad x=0 \\ e^{-2x} = -2x+1 \end{cases}$$

$$e^{0} = 1$$

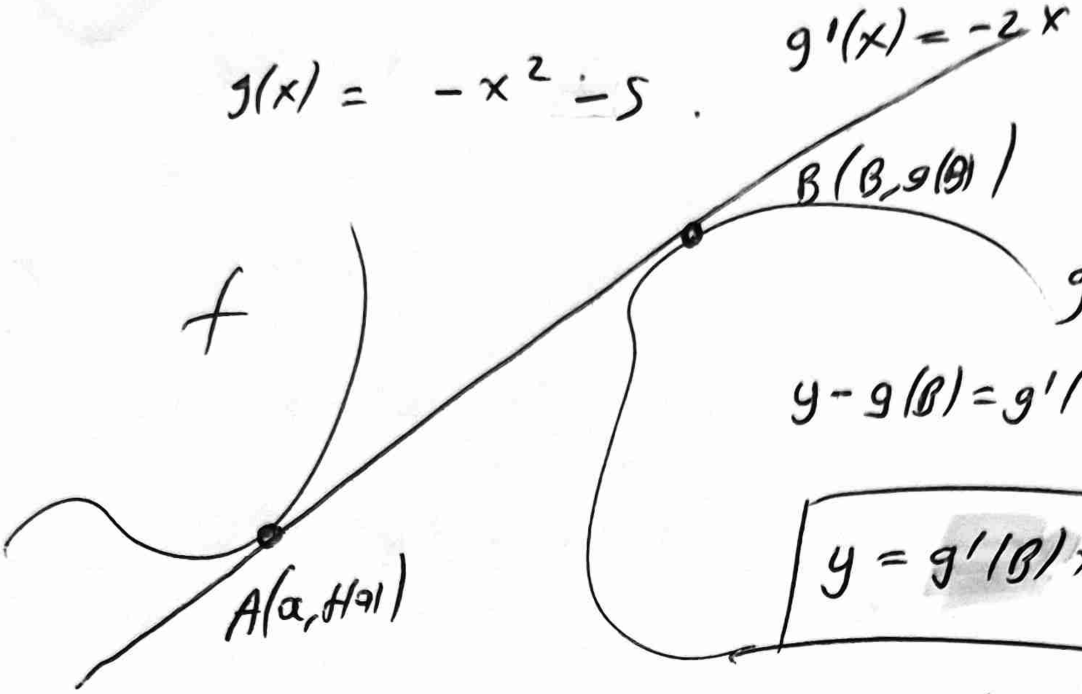
$$1 = 1 \quad \checkmark$$

$f(x) = x^2 - 2x$

28. $f(x) = x(x-2)$ $\qquad f'(x) = 2x - 2$

$g'(x) = -2x$

$g(x) = -x^2 - 5$



$y - g(\beta) = g'(\beta)(x - \beta)$

$$\boxed{y = g'(\beta)x + g(\beta) - \beta g'(\beta)}$$

$y - f(a) = f'(a)(x - a)$

ιδια

$$\boxed{y = f'(a)x + f(a) - a f'(a)}$$

$$\begin{cases} f'(a) = g'(\beta) \\ \\ f(a) - a f'(a) = g(\beta) - \beta g'(\beta) \end{cases}$$

$$\begin{cases} 2a - 2 = -2\beta \\ \\ a^2 - 2a - a \cdot (2a - 2) = -\beta^2 - 5 - \beta(-2\beta) \end{cases}$$

$$\begin{cases} a - 1 = -\beta \\ a^2 - \cancel{2a} - 2a^2 + \cancel{2a} = -\beta^2 - 5 + 2\beta^2 \end{cases}$$

$$\begin{cases} \boxed{\beta = 1-a} \\ \\ -a^2 = \beta^2 - 5 \end{cases}$$

$$-a^2 = (1-a)^2 - 5$$

$$-a^2 = 1 - 2a + a^2 - 5$$

$$0 = 2a^2 - 2a - 4$$

$$0 = a^2 - a - 2$$

$a = 2$ $\qquad$ $a = -1$

$\beta = -1$ $\qquad$ $\beta = 2$

$\varepsilon_1 :\ y - f(2) = f'(2)(x-2) \Rightarrow y - 0 = 2(x-2)$

$$\boxed{y = 2x - 4}$$

$\varepsilon_2 :\ y - f(-1) = f'(-1)(x-(-1))$

$$y - 3 = -4(x+1)$$

$$y = -4x - 1$$

27. $f(x) = x^2 - 3x + 3$ $\qquad f'(x) = 2x - 3$

$g(x) = \frac{1}{x}$ $\qquad g'(x) = -\frac{1}{x^2}$

(α) Έστω $f(x) < g(x)$

$$x^2 - 3x + 3 < \frac{1}{x}$$

$$x^3 - 3x^2 + 3x - 1 < 0$$

$$(x-1)^3 < 0$$

$$x - 1 < 0$$

$$x < 1$$

Όταν $x < 1 \rightarrow f(x) < g(x)$

Όταν $x > 1 \Rightarrow f(x) > g(x)$

Όταν $x = 1$ τέμνονται,

(β) $\begin{cases} f'(1) = g'(1) \\ f(1) = g(1) \end{cases} \rightarrow \begin{cases} -1 = -1 \\ 1 = 1 \end{cases}$ ✓

26. $f(x) = \dfrac{2x}{x^2+1}$ $\qquad f'(x) = \dfrac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$

$g(x) = ax^2 + Bx + 1$ $\qquad g'(x) = 2ax + B$

$$\begin{cases} f'(-1) = g'(-1) \\ f(-1) = g(-1) \end{cases} \Leftrightarrow \begin{cases} 0 = B - 2a \\ -1 = a - B + 1 \end{cases} +$$

$-1 = -a + 1$

$\underline{\underline{a = 2}}$

$\underline{\underline{B = 4}}$

22. $f(x) = \beta x^2 - ax + 2$ $\qquad f'(x) = 2\beta x - a$

$\eta: y = ax + 1$ εφαπτεται στο -1

$$\begin{cases} f'(-1) = a \\ f(-1) = a(-1) + 1 \end{cases} \Leftrightarrow \begin{cases} -2\beta - a = a \\ \beta + a + 2 = -a + 1 \end{cases}$$

$$\Rightarrow \begin{cases} -2\beta = 2a \\ \beta + 2a = -1 \end{cases} \Rightarrow \boxed{a = -\beta}$$

$\beta + 2(-\beta) = -1$

$-\beta = -1$

$\beta = 1$ $\qquad a = -1$

21. $f(x) = x^2+x+2$

$y=\lambda x+1$ εφάπτεται της C_f

$$\begin{cases} f'(x)=\lambda & \Rightarrow \boxed{2x+1=\lambda} \\ f(x)=\lambda x+1 & \Rightarrow x^2+x+2=\lambda x+1 \end{cases}$$

$x^2+x+2=(2x+1)x+1$

$x^2+\cancel{x}+2=2x^2+\cancel{x}+1$

$1=x^2$

$x=1$ ή $x=-1$

$\underline{\underline{\lambda=3}}$ $\qquad$ $\underline{\underline{\lambda=-1}}$

Ενότητα 23

9. (α) $f(x) = \sqrt{x^2+4}$ $\quad x \in R.$

$$f'(x) = \frac{2x}{2\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}$$

$\rightarrow f'(x) = 0 \Rightarrow \frac{x}{\sqrt{x^2+4}} = 0 \Rightarrow \underline{\underline{x=0}}$

x		0	
f'	$-$	0	$+$
f	↘		↗

$f(x) \geq f(0)$

$\underline{\underline{f(x) \geq 2}}$

(β) $f(x) = \sqrt{4-x^2}$

πρέπει $4-x^2 \geq 0$

x		-2		2	
$4-x^2$	$-$		$+$		$-$

$x \in [-2, 2]$

$$f'(x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2}}$$

x	-2		0		2
f'		$+$		$-$	
f		↗		↘	

$f(x) \leq f(0)$

$\underline{\underline{f(x) \leq 2}}$

δ $f(x) = x - 4\sqrt{x}$, $x \geq 0$

$$f'(x) = 1 - 4\frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{2\sqrt{x} - 4}{2\sqrt{x}} = \frac{\sqrt{x} - 2}{\sqrt{x}}$$

$\longrightarrow \frac{\sqrt{x}-2}{\sqrt{x}} = 0 \Rightarrow \sqrt{x} - 2 = 0 \Rightarrow \sqrt{x} = 2$

$x = 4$

x	0		4	
f'		$-$	0	$+$
f		$\searrow$		$\nearrow$

$f(x) \geq f(4)$

$f(x) \geq -4$

(ε) $f(x) = \sqrt{x} + \sqrt{2-x}$

πρέπει $\underline{\underline{x \geq 0}}$ και $2-x \geq 0$

$\underline{\underline{x \leq 2}}$

$D_f = [0,2]$.

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{-1}{2\sqrt{2-x}} = \frac{\sqrt{2-x} - \sqrt{x}}{2\sqrt{x}\sqrt{2-x}}$$

$\to f'(x) = 0 \Rightarrow \sqrt{2-x} - \sqrt{x} = 0$

$\sqrt{2-x} = \sqrt{x}$

$2-x = x$

$2 = 2x$

$\underline{\underline{x = 1}}$

x		1	
f'	+	0	−
f	↗		↘

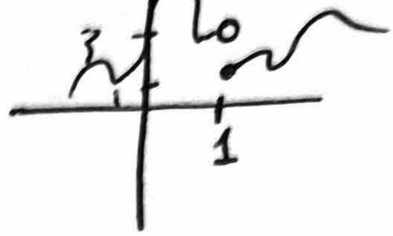
$f(x) \leq f(1)$

$f(x) \leq 2$.

Ενότητα 23

28. (δ) $f(x) = \begin{cases} x^2 e^{1-x} + 2, & x<1 \\ x - 2\sqrt{x-1}, & x \geq 1 \end{cases}$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2e^{1-x}+2 = 3$



$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} x-2\sqrt{x-1} = 1$

Δεν είναι συνεχής στο 1

x < 1

$f_1(x) = x^2 e^{1-x}+2$

$f_1'(x) = 2xe^{1-x} - x^2e^{1-x}$

$f_1'(x) = e^{1-x}(2x-x^2)$

$f_1'(x) = e^{1-x}(2-x)x$

2   0

x > 1

$f_2(x) = x-2\sqrt{x-1}$

$f_2'(x) = 1-2\dfrac{1}{2\sqrt{x-1}}$

$f_2'(x) = \dfrac{\sqrt{x-1}-1}{\sqrt{x-1}}$

$\to \sqrt{x-1}-1=0$

$\sqrt{x-1}=1$

$x-1=1$

$x=2$

x		0		1		2	
f_1'	−	0	+		~~+~~	~~0~~	~~−~~
f_2'	~~//~~		~~//~~		−	0	+
f'	−		+		−		+
f	↘		↗		↘		↗

Ενότητα 24

10. $f(x) = e^x + x + \sigma\upsilon\nu x$

(α) $f'(x) = e^x + \underbrace{1 - \eta\mu x}_{\oplus} > 0 \quad f \nearrow$

(β) i) Νδο $f(\varepsilon\varphi x) - e^x > x + \sigma\upsilon\nu x \quad \forall x \in (0, \frac{\pi}{2})$

$$f(\varepsilon\varphi x) > e^x + x + \sigma\upsilon\nu x$$

$$f(\varepsilon\varphi x) > f(x)$$

$$f \nearrow$$

$$\varepsilon\varphi x > x$$

$\underbrace{\varepsilon\varphi x - x}_{g(x)} > 0$ αρκει νδο $g(x) > 0$ $\forall x \in (0, \frac{\pi}{2})$

$$g'(x) = \frac{1}{\sigma\upsilon\nu^2 x} - 1 = \frac{1 - \sigma\upsilon\nu^2 x}{\sigma\upsilon\nu^2 x} = \frac{\eta\mu^2 x}{\sigma\upsilon\nu^2 x} > 0 \quad g \nearrow$$

Τελος

Αν $x > 0 \Rightarrow g(x) > g(0) \Rightarrow g(x) > 0$

ii). Νδο $f(e^{x-1}-1) > f(\ln x^x)$ $\forall x>1$

Αφου $f \nearrow$ $\quad e^{x-1}-1 > \ln x^x$

$$e^{x-1}-1 > x\ln x$$

$$\underbrace{e^{x-1}-1-x\ln x}_{h(x)} > 0$$

$\Rightarrow$ Αρκει νδο $h(x)>0$ $\forall x>1$

$h'(x)=e^{x-1}-\ln x-1$ $\qquad h'(1)=0$

$h''(x)=e^{x-1}-\frac{1}{x}$ $\qquad h''(1)=0$

$h'''(x)=e^{x-1}+\frac{1}{x^2}>0$

x		1	
h'''	$+$		$+$
h''	$\nearrow$ $-$	0	$+$ $\nearrow$
h'	$\searrow$ $+$	0	$+$ $\nearrow$
h	$\nearrow$		$\nearrow$

Τελος

Αν $x>1$

$h(x)>h(1)$

$h(x)>0$

$x<1 \Rightarrow h''(x)<h''(1) \Rightarrow h''(x)<0$

$h'(x)\geq h'(1) \Rightarrow h'(x)\geq 0$

B' τρόπος

$h'(x) = e^{x-1} - \ln x - 1$

- $e^x \geq x+1$

$e^{x-1} \geq x-1+1$

$\boxed{e^{x-1} \geq x}$

- $\ln x \leq x-1$

$\boxed{-\ln x \geq -x+1}$

$\oplus$

$e^{x-1} - \ln x \geq x - x + 1$

$e^{x-1} - \ln x - 1 \geq 0$

$h'(x) \geq 0$ $\qquad h \nearrow$

iii). Nδο $f(e^x) > f(1+\eta\mu x)$ $\forall x > 0$

Αφού $f \nearrow$ $\quad e^x > 1 + \eta\mu x$ $\quad$ Αρκεί νδο

$\underbrace{e^x - 1 - \eta\mu x}_{\varphi(x)} > 0 \Rightarrow \varphi(x) > 0$ $\forall x \geq 0$.

$\varphi'(x) = e^x - \sigma\upsilon\nu x = \underbrace{e^x - 1}_{\oplus} + \underbrace{1 - \sigma\upsilon\nu x}_{\oplus} > 0$ $\quad \varphi \nearrow$

- $x > 0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1$

Τέλος

Αν $x > 0$

$\varphi(x) > \varphi(0)$

$\varphi(x) > 0$

3. $f(x) = x - \ln(x^2+1)$, $x \in \mathbb{R}$.

(α) Μονοτονία

$$f'(x) = 1 - \frac{2x}{x^2+1} = \frac{x^2+1-2x}{x^2+1} = \frac{(x-1)^2}{x^2+1} \geq 0$$

$f \uparrow$

(β) i) $x = \ln(x^2+1)$

$$x - \ln(x^2+1) = 0$$

$$f(x) = 0$$

$$f(x) = f(0)$$

f 1-1

$$x = 0$$

ii). $f(x^2+1) > f(e^x)$

$f \uparrow$

$$x^2+1 > e^x$$

$$\ln(x^2+1) > x$$

$$0 > x - \ln(x^2+1)$$

$$f(0) > f(x)$$

$f \uparrow$

$$0 > x$$

iii). $e^{f(x)} - 1 = f^2(x)$

$$e^{f(x)} = f^2(x)+1$$

$$f(x) = \ln f^2(x)+1$$

$$f(x) - \ln(f^2(x)+1) = 0$$

$$f(f(x)) = f(0)$$

$$f(x) = 0$$

$\to f(x) = f(0)$

f 1-1

$x = 0$

① Ndo $\ln(n^2+1) + f(\ln n) < n$.

$$\boxed{f(x) = x - \ln(x^2+1)}$$

$$f(\ln n) < n - \ln(n^2+1)$$

$$f(\ln n) < f(n)$$

$$f \uparrow$$

$$\ln n < n$$

$\cdot$ $\ln x \leq x - 1$

$$\ln n \leq n - 1$$

$$\ln n - n \leq -1$$

$$\ln n - n < 0$$

$$\ln n < n \quad \checkmark$$

Basikal Avisoa...

$$e^x \geq x + 1$$

$$\ln x \leq x - 1$$

δ) $$\lim_{x\to 0} \frac{\ln x}{f(3x)-f(2x)} = \lim_{x\to 0^+} \ln x \cdot \frac{1}{\underbrace{f(3x)-f(2x)}_{+}}$$

$$= -\infty \cdot (+\infty) = -\infty$$

• $2x < 3x \Rightarrow f(2x) < f(3x)$

$$f(3x) - f(2x) > 0$$

ε) Αν $f(a) < 0$ νδο $e^a < 1$

$f(a) < 0$

$f(a) < f(0)$

$f \uparrow$

$a < 0$

$e^a < e^0$

$e^a < 1$

12. α) $e^x = x^2+1$

$$\frac{e^x}{x^2+1} = 1$$

$$\underbrace{\frac{e^x}{x^2+1} - 1}_{f(x)} = 0 \quad =1$$

$f(x) = 0$

$f(x) = f(0)$

f 1-1

$\underline{x=0}$

$$f'(x) = \frac{e^x(x^2+1) - e^x 2x}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2} \geq 0$$

$f \nearrow$

β) Νδο $2e^{\eta\mu\theta} \leq e(2-\sigma\upsilon\nu^2\theta)$

$$2e^{\eta\mu\theta} \leq 2e - e(1-\eta\mu^2\theta)$$

$$2e^{\eta\mu\theta} \leq 2e - e + e\,\eta\mu^2\theta$$

$$2e^{\eta\mu\theta} \leq e + e\,\eta\mu^2\theta$$

$$2e^{\eta\mu\theta} \leq e(1+\eta\mu^2\theta)$$

$$\frac{e^{n\mu l}}{1+n\gamma^2\theta} \le \frac{e}{2}$$

$$f(n\gamma\theta) \le f(1)$$

$$f \nearrow$$

$$n\gamma\theta \le 1$$

7. $f(x) = \dfrac{e^x}{x}, \ x \geq 1$

$$f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2} \geq 0 \qquad f \nearrow$$

$$f(x) + 1 > e + \ln f(x)$$

$$e^{f(x)+1} > e^{e + \ln f(x)}$$

$$e^{f(x)} \cdot e^1 > e^e \cdot e^{\ln f(x)}$$

$$e^{f(x)} e > e^e f(x)$$

$$\frac{e^{f(x)}}{f(x)} > \frac{e^e}{e}$$

$$f(f(x)) > f(e)$$

$f \nearrow$

$f(x) > e$

$f(x) > f(1)$

$f \nearrow \qquad x > 1.$

6. $f(x) = x - \ln(1+e^x)$, $x \in \mathbb{R}$

ⓐ $f'(x) = 1 - \dfrac{e^x}{1+e^x} = \dfrac{1+e^x - e^x}{1+e^x} = \dfrac{1}{1+e^x}$

$f \nearrow$

$f''(x) = \dfrac{-e^x}{(e^x+1)^2} < 0$

$f' \downarrow$

ⓑ i) $2f'(f(x)+\ln 2) < 1$

$f'(f(x)+\ln 2) < \frac{1}{2}$

$f'(f(x)+\ln 2) < f'(0)$

$f' \downarrow$

$f(x) + \ln 2 < 0$

$f(x) < -\ln 2$

$f(x) < f(0)$

$f \nearrow$

$x < 0$.

11).

$$f'\left(\ln\frac{1+e^{x}}{1+e^{x^2}}\right)=f'(x-x^2)$$

$$f' \; 1-1$$

$$\ln(1+e^{x})-\ln(1+e^{x^2})=x-x^2$$

$$x^2-\ln(1+e^{x^2})=x-\ln(1+e^{x})$$

$$f(x^2)=f(x)$$

$$f \; 1-1$$

$$x^2=x$$

$$x^2-x=0$$

$$x(x-1)=0$$

$$x=0 \quad ή \quad x=1$$

III). $f(x) + 2f'(1-e^x) = \ln \frac{e}{2}$

$f(x) + 2f'(1-e^x) = 1 - \ln 2$

$\boxed{\varphi(x) = f(x) + 2f'(1-e^x) - 1 + \ln 2}$

- $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $x_1 < x_2 \Rightarrow 1-e^{x_1} > 1-e^{x_2}$ (f)

$f'(1-e^{x_1}) < f'(1-e^{x_2})$

$\varphi \nearrow$

$\varphi(x) = 0$

$\varphi(x) = \varphi(0)$

φ 1-1

$x = 0$

$\varphi(0) = f(0) + 2f'(0) - 1 + \ln 2$

$\varphi(0) = -\ln 2 + \cancel{2}\frac{1}{\cancel{2}} - 1 + \ln 2$

$\varphi(0) = 0$

(δ) $\lim_{x \to 0} \frac{\ln x}{f(x) + \ln 2} = \lim_{x \to 0^+} \ln x \; \frac{1}{f(x) + \ln 2} = -\infty \; (+\infty)$

(+)

$= -\infty$

Αν $x > 0 \Rightarrow f(x) > f(0) \Rightarrow f(x) > -\ln 2$

$f(x) + \ln 2 > 0$

Επομενο Μαθημα

23

10 αβδε (παλια ασκηση)

11 αγ

12 αγεη

13 αγζ

14 αγ

15 α

16 α.

20 αδ

21 αγ

22 αγ

23 αγζ

24 αγ

25 αγ

28 αβγ.