Ενότητα 23

6. (B) $f(x) = -\frac{4x^3}{3} + 2x^2 - x - 4$

$f'(x) = -4x^2 + 4x - 1$

$f'(x) = -(2x-1)^2 \leq 0$

↓

9. (B) $f(x) = \sqrt{4 - x^2}$

x	-2		0		2
f'	$+$	$\varnothing$	$-$		
f		↗		↘	

$f'(x) = \frac{-2x}{2\sqrt{4-x^2}}$

$\longrightarrow f'(x) = 0 \Rightarrow \frac{-x}{\sqrt{4-x^2}} = 0$ $\quad (x=0)$

10. (D) $f(x) = (x-2)e^x$

$f'(x) = e^x + (x-2)e^x = e^x(1 + x - 2)$

$f'(x) = e^x(x-1)$

$\longrightarrow f'(x) = 0 \rightsquigarrow x - 1 = 0$

$(x = 1)$

x		1	
f'	$-$	$\varnothing$	$+$
f	↘		↗

13. (B) $f(x) = 2xe^{-x} + (x-1)^2$

$$f'(x) = 2e^{-x} - 2xe^{-x} + 2(x-1)$$

$$f'(x) = 2e^{-x}(1-x) - 2(1-x)$$

$$\boxed{f'(x) = 2(1-x)(e^{-x} - 1)}$$

$1-x=0$ ve $e^{-x}-1=0$

$\boxed{x=1}$ $e^{-x}=1$

$\boxed{x=0}$

x		0		1	
$1-x$	+		+	0	−
$e^{-x}-1$	+	0	−		−
f'	+		−		+
f	↗		↘		↗

13. ⓐ $f(x) = (x-2)e^x - \frac{1}{2}x^2 + x - 1$

$f'(x) = e^x + (x-2)e^x - \frac{1}{2}2x + 1$

$f'(x) = e^x(1+x-2) - x + 1$

$\boxed{f'(x) = e^x(x-1) - x + 1}$

$f'(x) = (x-1)(e^x - 1)$

$\rightarrow f'(x) = 0$

$x - 1 = 0$; $e^x - 1 = 0$

$e^x = 1$

$\boxed{x = 1}$ $\boxed{x = 0}$

x		0		1	
$x-1$	−		−	0	+
e^x-1	−	0	+		+
f'	+		−		+
f	↗		↘		↗

12. ① $f(x) = x\ln^2 x - 3x + 2$

$$f'(x) = \ln^2 x + x \cdot 2\ln x \cdot \frac{1}{x} - 3$$

$$f'(x) = \ln^2 x + 2\ln x - 3$$

$\longrightarrow f'(x) = 0 \longrightarrow \ln^2 x + 2\ln x - 3 = 0$

$\ln x = t$

$t^2 + 2t - 3 = 0$

$t = -3 \qquad t = 1$

$\ln x = -3 \qquad \ln x = 1$

$x = e^{-3} \qquad x = e$

x		e^{-3}		e	
f'	$+$		$-$		$+$
f	↗		↘		↗

11. (a) $f(x) = e^{2x} - 3e^x + x - 1$

$f'(x) = 2e^{2x} - 3e^x + 1$

$\longrightarrow f'(x) = 0 \quad \Rightarrow 2e^{2x} - 3e^x + 1 = 0$

$\Delta = 1$

$e^x = \frac{3 \pm 1}{4} \begin{cases} 1 \\ \frac{1}{2} \end{cases}$

x	$-\ln 2$		0	
f'	$+$	0	$-$	0 $+$
f	$\nearrow$		$\searrow$	$\nearrow$

$e^x = 1 \qquad e^x = \frac{1}{2}$

$x = 0 \qquad x = -\ln 2$

(b) $f(x) = e^x - e^{-x} - 2x$

$f'(x) = e^x + e^{-x} - 2 = e^x + \frac{1}{e^x} - 2$

$f'(x) = \frac{e^{2x} + 1 - 2e^x}{e^x} = \frac{(e^x - 1)^2}{e^x} \geq 0$

$f \nearrow$

11. ⓐ $f(x) = e^{-2x} + x - 1$

$$f'(x) = -2e^{-2x} + 1$$

$$\rightarrow f'(x) = 0 \Rightarrow -2e^{-2x} + 1 = 0$$

$$1 = 2e^{-2x}$$

$$e^{-2x} = \frac{1}{2}$$

$$-2x = \ln \frac{1}{2}$$

$$x = -\frac{1}{2} \ln \frac{1}{2} \qquad x = -\ln \sqrt{\frac{1}{2}}$$

$$x = -\ln \frac{\sqrt{2}}{2}$$

x		$-\ln\frac{\sqrt{2}}{2}$	
f'	$+$	0	$-$
f	↗	max	↘

10. ⓔ $f(x) = \dfrac{x^2}{e^x}$

$$f'(x) = \frac{2x\,e^x - x^2 e^x}{e^{2x}} = \frac{e^x(2x - x^2)}{e^{2x}}$$

$$f'(x) = \frac{2x - x^2}{e^x}$$

$\longrightarrow f'(x) = 0 \Rightarrow 2x - x^2 = 0$

$x(2 - x) = 0$

$x = 0$ $\quad$ $x = 2$

x		0		2	
f'	$-$	0	$+$	0	$-$
f	↘		↗		↘

20. ⓐ $f(x) = e^{x-1} - \ln x$, $x > 0$

$$f'(x) = e^{x-1} - \frac{1}{x} \qquad f'(1) = 0$$

$$f''(x) = e^{x-1} + \frac{1}{x^2} > 0$$

x	0	1	
f''	$+$	$+$	
f'	$\not\exists$	$-$	0 $+$
f	$\searrow$		$\nearrow$

ⓑ $f(x) = e^{x-1} + x\ln x - 2x$, $x > 0$

$$f'(x) = e^{x-1} + \ln x + 1 - 2$$

$$f'(x) = e^{x-1} + \ln x - 1 \qquad f'(1) = 0$$

$$f''(x) = e^{x-1} + \frac{1}{x} > 0$$

x	0	1	
f''	$+$ $\not\exists$	$+$	
f'	$\not\exists$	$-$	0 $+$
f	$\searrow$		$\nearrow$

21. ⓐ $f(x) = x - \dfrac{\ln x}{x}$, $x > 0$

$$f'(x) = 1 - \frac{\frac{1}{x}\,x - \ln x}{x^2} = 1 - \frac{1 - \ln x}{x^2}$$

$$\boxed{f'(x) = \frac{x^2 - 1 + \ln x}{x^2}}$$

$g(x) = x^2 - 1 + \ln x$ $\quad g(1) = 0.$

$g'(x) = 2x + \dfrac{1}{x} > 0$

x	0	1	$+\infty$
g'	$+$	$+$	
g	↗ $-$	0 $+$ ↗	
f'	$-$	$+$	
f	↘	↗	

22. ① $f(x) = 2e^x - x^2 - 2x$

$f'(x) = 2e^x - 2x - 2 = 2(e^x - x - 1)$

$\oplus$

• $e^x \geq x + 1$

$e^x - x - 1 \geq 0$

$f \nearrow$

② $f(x) = x \ln x - \frac{x^2}{2}$

$f'(x) = \ln x + x \cdot \frac{1}{x} - \frac{1}{2} 2x$

$f'(x) = \ln x + 1 - x \leq 0$

$f \searrow$

$\ln x \leq x - 1$

$\ln x - x + 1 \leq 0$

23. ⓐ $f(x) = 2e^x - x^2$

$$f'(x) = 2e^x - 2x = 2(e^x - x) > 0$$ (E)

$f \uparrow$

- $e^x \geq x+1 \Rightarrow e^x - x \geq 1$

$e^x - x > 0$

ⓑ $f(x) = \frac{1}{2}x^2 + x - x\ln x$

$f'(x) = x + 1 - \ln x - 1$

$f'(x) = x - \ln x > 0$

$f \uparrow$

$\ln x \leq x - 1$

$1 \leq x - \ln x$

$0 < x - \ln x$

28. ⓐ $f(x) = \begin{cases} (x-2)e^{x-1}+2, & x<1 \\ -x^2+2x, & x \geq 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x-2)e^{x-1}+2 = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -x^2+2x = 1$

$\lim_{x \to 1} f(x) = f(1) = 1$ Συνεχής

$x<1$

$f_1(x) = (x-2)e^{x-1}+2$

$f_1'(x) = e^{x-1} + (x-2)e^{x-1}$

$f_1'(x) = e^{x-1}(x-1)$

$\rightarrow f_1'(x) = 0$

$x = 1$

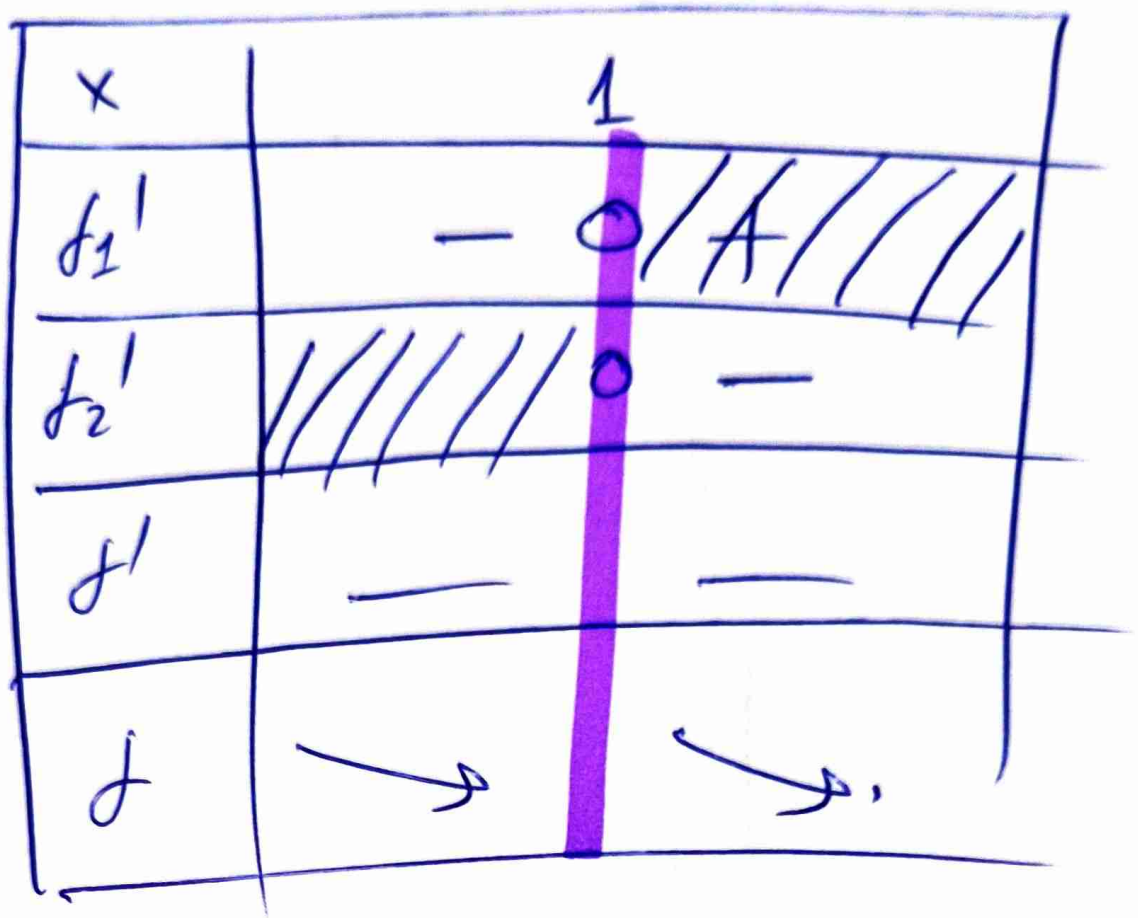
$x \geq 1$

$f_2(x) = -x^2+2x$

$f_2'(x) = -2x+2$

$\rightarrow -2x+2 = 0$

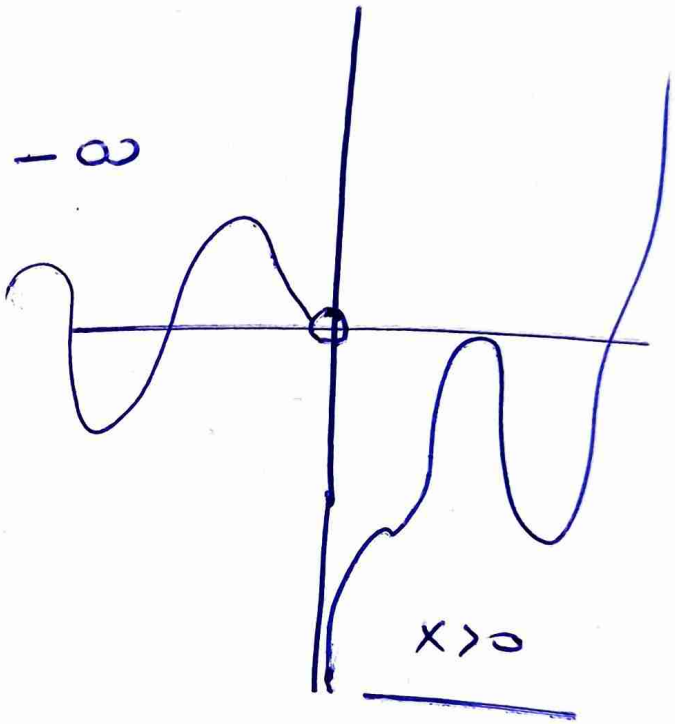
$x = 1$


x
1
f_1'
f_2'
f'
f

28. ① $f(x) = \begin{cases} \ln(e^x - x), & x < 0 \\ \ln x - x, & x > 0. \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \ln(e^x - x) = 0$

$\lim_{x \to 0^+} \ln x - x = -\infty$



$x < 0$

$f_1(x) = \ln(e^x - x)$

$f_1'(x) = \dfrac{e^x - 1}{e^x - x}$

$\rightarrow e^x - 1 = 0$

$x = 0$

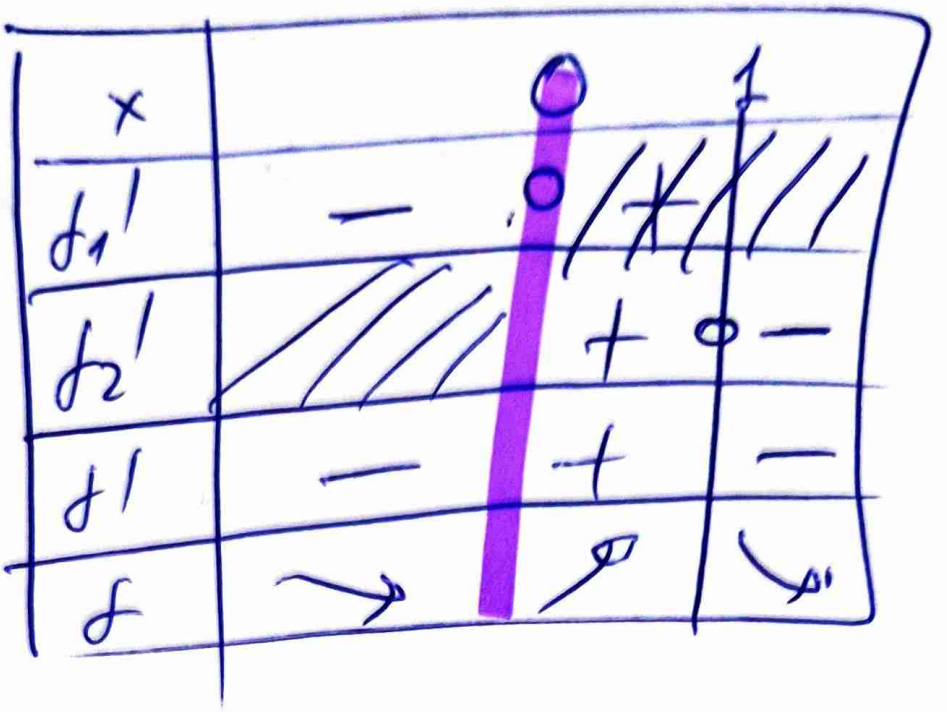
$x > 0$

$f_2(x) = \ln x - x$

$f_2'(x) = \dfrac{1}{x} - 1$

$f_2'(x) = \dfrac{1 - x}{x}$

$\rightarrow \dfrac{1-x}{x} = 0 \quad x = 1$


x
0
1
f_1'
−
0
+
+
0
−
f_2'
f'
−
+
−
f

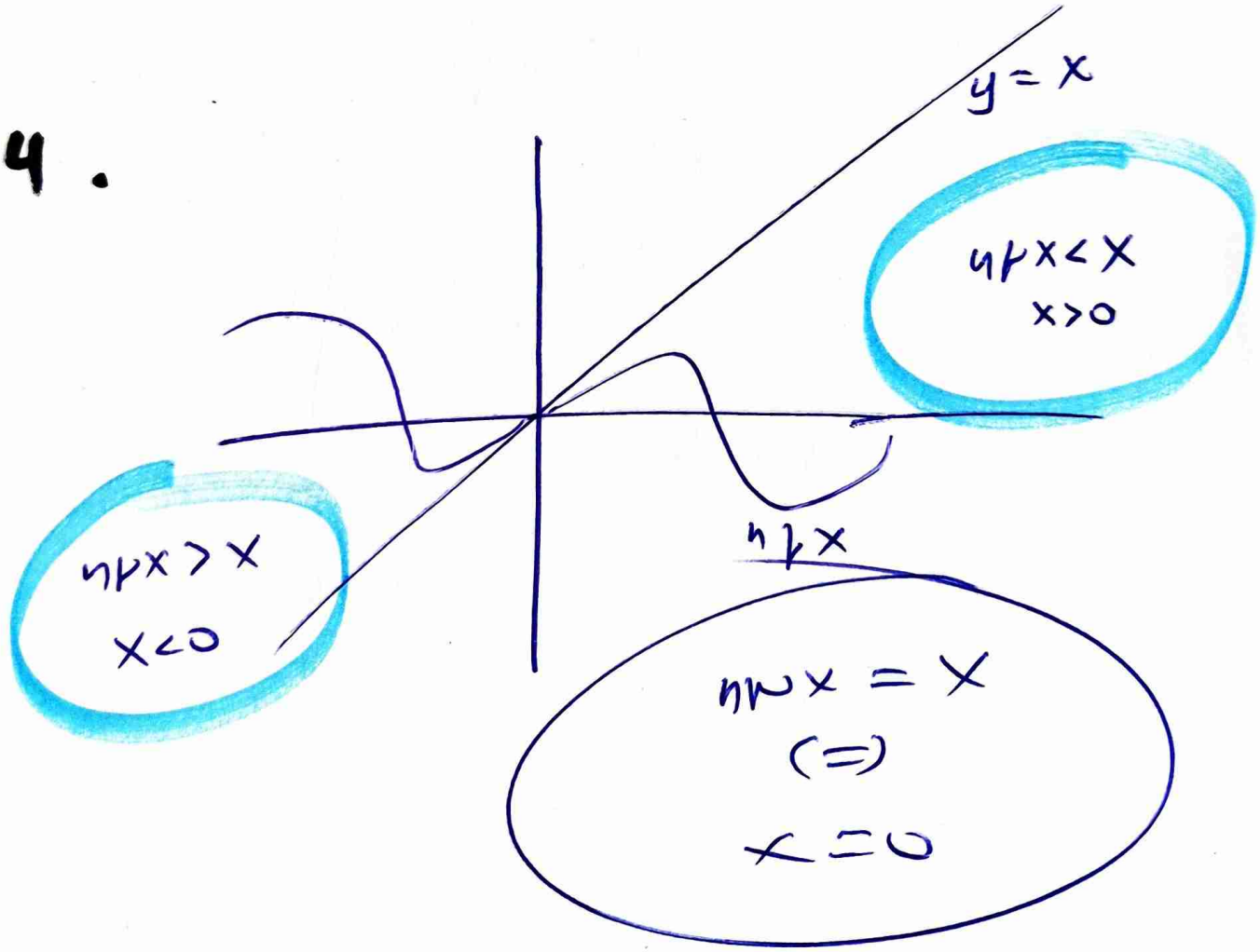
Βασικές Ανισότητες

1. $e^x \geq x+1$ Το "=" $x=0$

2. $\ln x \leq x-1$ Το "=" $x=1$

3. $|\eta\mu x| \leq |x|$ Το "=" $x=0$

4.



5. $f(x) = \frac{x^3}{x^2+1}$ Ενότητα 24

(01) $f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2}$

$f'(x) = \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} > 0$

$f \nearrow$

(B) i) Αντίστοιχα $f(5x^3) > f(8x^2+8)$.

$f \nearrow$

$5x^3 > 8x^2 + 8.$

$5x^3 > 8(x^2+1)$

$\frac{x^3}{x^2+1} > \frac{8}{5}$

$f(x) > f(2)$

$x > 2$

11). $(x^2+1)\,f(e^x-1) - x^3 = 0$.

$$(x^2+1)\,f(e^x-1) = x^3$$

$$f(e^x-1) = \frac{x^3}{x^2+1}$$

$$f(e^x-1) = f(x)$$

$$f: 1-1$$

$$e^x - 1 = \text{✗}$$

$e^x \geq x+1$ Το "=" για $x=0$

---

⑧ $$\lim_{x\to 0} \frac{\ln x}{f(2x+1) - f(x+1)} =$$

$$= \lim_{x\to 0^+} \ln x \cdot \frac{1}{f(2x+1) - f(x+1)} = -\infty \cdot (+\infty) = -\infty$$

(+)

• $x < 2x \Rightarrow x+1 < 2x+1 \Rightarrow f(x+1) < f(2x+1)$

6. $f(x) = x - \ln(1+e^x)$

(a) $f'(x) = 1 - \dfrac{e^x}{1+e^x} = \dfrac{1+e^x-e^x}{1+e^x}$

$f'(x) = \dfrac{1}{1+e^x} > 0$ $f \nearrow$

$f''(x) = \dfrac{-e^x}{(e^x+1)^2} < 0$ $f' \searrow$

(B) i) $2f'(f(x)+\ln 2) < 1$

$f'(f(x)+\ln 2) < \frac{1}{2}$

$f'(f(x)+\ln 2) < f'(0)$

$f' \searrow$

$f(x)+\ln 2 > 0$

$f(x) > -\ln 2$

$f(x) > f(0)$ $\forall x > 0$

11). $f^{-1}\left(\ln\frac{1+e^x}{1+e^{x^2}}\right) = f^{-1}(x-x^2)$

f^{-1} bijektiv $\Rightarrow$ f bijektiv

$$\ln\frac{1+e^x}{1+e^{x^2}} = x-x^2$$

$$\ln(1+e^x) - \ln(1+e^{x^2}) = x-x^2$$

$$x^2 - \ln(1+e^{x^2}) = x - \ln(1+e^x)$$

$$f(x^2) = f(x)$$

f bijektiv $\Rightarrow$

$$x^2 = x$$

$x=0$ $x=1$

iii) $f(x) + 2f'(1-e^x) = \ln \frac{e}{2}$

$f(x) + 2f'(1-e^x) = \ln e - \ln 2$

$f(x) + 2f'(1-e^x) = 1 - \ln 2$

$f(x) + 2f'(1-e^x) - 1 + \ln 2 = 0$

$g(x)$. $x = 0$

$g(0) = f(0) + 2f'(0) - 1 + \ln 2$

$g(0) = -\ln 2 + 2\frac{1}{2} - 1 + \ln 2 = 0$

- $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $x_1 < x_2 \rightarrow 1 - e^{x_1} > 1 - e^{x_2} \Rightarrow f'(1-e^{x_1}) < f'(1-e^{x_2})$

(+)

g ↗

① $\lim\limits_{x \to 0} \dfrac{\ln x}{f(x) + \ln 2} =$

$= \lim\limits_{x \to 0^+} \ln x \cdot \dfrac{1}{f(x) + \ln 2}$

$= -\infty \cdot \dfrac{1}{(+\infty)} = -\infty$

Για $x > 0$

$f(x) > f(0)$

$f(x) > -\ln 2$

$f(x) + \ln 2 > 0$

8. $f(x) = e^{1-x} + \ln x - 1, \quad x > 0.$

(a) Р1 Ц - пробую

$$f'(x) = -e^{1-x} + \frac{1}{x}$$

$f(x) = 0$

$f(x) = f(1)$

$f \nearrow -1$

$x = 1$

$$f'(x) = -\frac{1}{e^{x-1}} + \frac{1}{x}$$

$$f'(x) = \frac{-x}{xe^{x-1}} + \frac{e^{x-1}}{xe^{x-1}}$$

$$f'(x) = \frac{e^{x-1} - x}{xe^{x-1}} \quad (+)$$

$f \nearrow$.

$e^x \geq x + 1$

$e^{x-1} \geq x - 1 + 1$

$e^{x-1} \geq x$

$e^{x-1} - x \geq 0$

11). $e^{1-x} - e^{1-x^2} > \ln x$

$$e^{1-x} - e^{1-x^2} > \ln \frac{x^2}{x}$$

$$e^{1-x} - e^{1-x^2} > \ln x^2 - \ln x$$

$$e^{1-x} + \ln x - 1 > e^{1-x^2} + \ln x^2 - 1$$

$$f(x) > f(x^2)$$

$f \nearrow$

$$x > x^2$$

$$x^2 - x < 0$$

x		0		1
x^2-x	$+$	$\|$	$-$	$\|$ $+$

$$x \in (0,1).$$

(B) i) $e^{1-\eta\rho x} - e^{1-\sigma\omega x} = -\ln \varepsilon\varphi x$

$$e^{1-\eta\rho x} - e^{1-\sigma\omega x} = -\ln \frac{\eta\rho x}{\sigma\omega x} \quad \left(0, \frac{\pi}{2}\right).$$

$$e^{1-\eta\rho x} - e^{1-\sigma\omega x} = -(\ln \eta\rho x - \ln \sigma\omega x)$$

$$e^{1-\eta\rho x} - e^{1-\sigma\omega x} = \ln \sigma\omega x - \ln \eta\rho x$$

$$e^{1-\eta\rho x} + \ln \eta\rho x - 1 = e^{1-\sigma\omega x} + \ln \sigma\omega x - 1$$

$$f(\eta\rho x) = f(\sigma\omega x)$$

f 3/-1

$$\eta\rho x = \sigma\omega x$$

$$\frac{\eta\rho x}{\sigma\omega x} = 1$$

$$\varepsilon\varphi x = 1$$

$$x = \frac{\pi}{4}$$

⑧ $\lim_{x \to 1^+} \frac{\ln x}{(x-1)f(x)} = \lim_{x \to 1^+} \frac{\ln x}{x-1} \cdot \frac{1}{f(x)}$

$= 1 \cdot (+\infty)$

$\longrightarrow \lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1 \qquad = +\infty$

x	1	
$f(x)$	$-$	$+$

Εποχή 24

1

2

3

4