Ενότητα 6

2. (α) $f(x) = 2\ln(x+1) - 3$ $\qquad D_f = (-1, +\infty)$

• $x_1 < x_2 \Rightarrow x_1 + 1 < x_2 + 1 \Rightarrow \ln(x_1+1) < \ln(x_2+1)$

$$2\ln(x_1+1) < 2\ln(x_2+1)$$

$$2\ln(x_1+1) - 3 < 2\ln(x_2+1) - 3$$

$$f(x_1) < f(x_2)$$

$f \nearrow \Rightarrow f$ 1-1

(B) $f(x) = 1 - \sqrt{3-2x}$

• $x_1 < x_2 \Rightarrow -2x_1 > -2x_2 \Rightarrow 3 - 2x_1 > 3 - 2x_2$

$$\sqrt{3-2x_1} > \sqrt{3-2x_2}$$

$$-\sqrt{3-2x_1} < -\sqrt{3-2x_2}$$

$$1 - \sqrt{3-2x_1} < 1 - \sqrt{3-2x_2}$$

$$f(x_1) < f(x_2)$$

$f \nearrow \Rightarrow f$ 1-1.

2. γ) $f(x) = \dfrac{x-3}{x+1}$

$f(x_1) = f(x_2)$

$$\frac{x_1-3}{x_1+1} = \frac{x_2-3}{x_2+1} \Leftrightarrow (x_1-3)(x_2+1) = (x_2-3)(x_1+1)$$

$$\cancel{x_1x_2} + x_1 - 3x_2 - \cancel{3} = \cancel{x_2x_1} + x_2 - 3x_1 - \cancel{3}$$

$$4x_1 = 4x_2 \qquad \Rightarrow f \; 1-1.$$

$$x_1 = x_2$$

ε) $f(x) = \ln \dfrac{x}{1-x}$

$$f(x_1) = f(x_2) \Rightarrow \ln \frac{x_1}{1-x_1} = \ln \frac{x_2}{1-x_2}$$

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1(1-x_2) = x_2(1-x_1)$$

$$x_1 - \cancel{x_1x_2} = x_2 - \cancel{x_1x_2}$$

$$x_1 = x_2 \qquad f \; 1-1.$$

4. $f: \mathbb{R} \to \mathbb{R}$ $\qquad f(2) = 3$ $\qquad f$: 1-1

α) $f(x) = 3$

$f(x) = f(2)$

f : 1-1

$\underline{\underline{x = 2}}$

β) $f(e^x + 1) = 3$

$f(e^x + 1) = f(2)$

f : 1-1

$e^x + 1 = 2$

$e^x = 1$

$\underline{\underline{x = 0}}$

γ) $f(x^2 - 3x) = f(x - 3)$

f : 1-1

$x^2 - 3x = x - 3$

$x^2 - 4x + 3 = 0$

$\underline{\underline{x = 1}}$ $\qquad \underline{\underline{x = 3}}$

δ) $f\left(f(x) - 1\right) = 3$

$f\left(f(x) - 1\right) = f(2)$

f : 1-1

$f(x) - 1 = 2$

$f(x) = 3$

$f(x) = f(2)$

f : 1-1 $\qquad x = 2$

5. $f(x) = x^5 + 2x^3 - 3$

(α) $x_1 < x_2 \Rightarrow x_1^5 < x_2^5$

$x_1 < x_2 \Rightarrow 2x_1^3 - 3 < 2x_2^3 - 3$ (+)

$f \nearrow \Rightarrow f$ 1-1

(β) $x^5 + 2x^3 = 3$

$x^5 + 2x^3 - 3 = 0$

$f(x) = 0$

$f(x) = f(1)$

f 1-1

$x = 1$

(γ) $\lambda^5 - \mu^5 = 2\mu^3 - 2\lambda^3$

$\lambda^5 + 2\lambda^3 = \mu^5 + 2\mu^3$

$\lambda^5 + 2\lambda^3 - 3 = \mu^5 + 2\mu^3 - 3$

$f(\lambda) = f(\mu)$

f 1-1

$\lambda = \mu$.

(8) $(3x-2)^5 + 2(3x-2)^3 = 3$

$(3x-2)^5 + 2(3x-2)^3 - 3 = 0$

$f(3x-2) = f(1)$

f 31-1

$3x-2=1$

$3x=3$

$\underline{x=1}$

6. $$f(x) = \frac{e^x}{e^{x+1} - 1}$$

(a) $$f(x_1) = f(x_2)$$

$$\frac{e^{x_1}}{e^{x_1+1} - 1} = \frac{e^{x_2}}{e^{x_2+1} - 1}$$

$$e^{x_1}\left(e^{x_2+1} - 1\right) = e^{x_2}\left(e^{x_1+1} - 1\right)$$

$$\cancel{e^{x_1+x_2+1}} - e^{x_1} = \cancel{e^{x_2+x_1+1}} - e^{x_2}$$

$$e^{x_1} = e^{x_1}$$

$f : 1\text{-}1$

$$x_1 = x_2$$

(β) $$\left(e - e^{-x}\right) f(x^2) = 1$$

$$\left(e - \frac{1}{e^x}\right) f(x^2) = 1$$

$$\left(\frac{e^1 \cdot e^x}{e^x} - \frac{1}{e^x}\right) f(x^2) = 1$$

$$\left(\frac{e^{x+1} - 1}{e^x}\right) f(x^2) = 1$$

$$\frac{e^{x+1} - 1}{e^x} \, f(x^2) = 1$$

$$f(x^2) = \frac{e^x}{e^{x+1} - 1}$$

$$f(x^2) = f(x)$$

f 1-1

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$x = 0$ $\quad$ $x = 1$

8. (α) $e^{x-1} + \ln x = 1$

$$\underbrace{e^{x-1} + \ln x - 1}_{f(x)} = 0 \Rightarrow f(x) = 0$$

$f(x) = f(1)$

f 1-1

$\underline{\underline{x = 1}}$

- $x_1 < x_2 \Rightarrow x_1 - 1 < x_2 - 1$

  $e^{x_1-1} < e^{x_2-1}$

(+)

- $x_1 < x_2 \Rightarrow \ln x_1 - 1 < \ln x_2 - 1$

$f \nearrow \Rightarrow f$ 1-1.

9. (α) $(3x-2)^5 + (3x-2)^3 - 2 = 0$

$$\boxed{\begin{gathered} f(x) = x^5 + x^3 - 2 \\ f \nearrow \end{gathered}}$$

$f(3x-2) = 0$

$f(3x-2) = f(1)$

f 1-1

$3x - 2 = 1$

$3x = 3$

$x = 1$

10. (a) $\ln \dfrac{2x^2+1}{x^2+2} = e^{x^2+2} - e^{2x^2+1}$

$$\ln(2x^2+1) - \ln(x^2+2) = e^{x^2+2} - e^{2x^2+1}$$

$$\ln(2x^2+1) + e^{2x^2+1} = \ln(x^2+2) + e^{x^2+2}$$

$$f(2x^2+1) = f(x^2+2)$$

f : 1-1

$$2x^2+1 = x^2+2$$

$$x^2 = 1$$

$x = 1$ $\quad$ $x = -1$

$f(x) = \ln x + e^x$

$f \nearrow$

(13) $\ln εφx = συνx - ημx \qquad x \in \left(0, \frac{\pi}{2}\right)$

$$\ln \frac{ημx}{συνx} = συνx - ημx$$

$$\ln ημx - \ln συνx = συνx - ημx$$

$$\ln ημx + ημx = \ln συνx + συνx$$

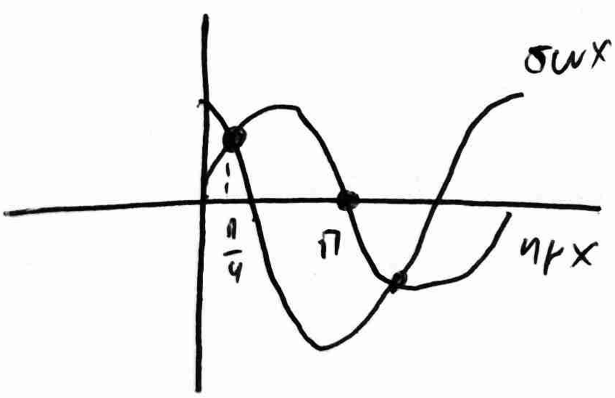
$$f(ημx) = f(συνx)$$

f 1-1

$$ημx = συνx$$

$$\underline{\underline{x = \frac{\pi}{4}}} \qquad x \in \left(0, \frac{\pi}{2}\right)$$

$$\boxed{\begin{array}{c} f(x) = \ln x + x \\ f \nearrow \end{array}}$$



(γ) $e^x - e^{x^5} = 4\ln x \quad , \quad x > 0$

$e^x - e^{x^5} = \ln x^4$

$e^x - e^{x^5} = \ln \frac{x^5}{x}$

$e^x - e^{x^5} = \ln x^5 - \ln x$

$e^x + \ln x = \ln x^5 + e^{x^5}$

$\boxed{f(x) = e^x + \ln x \qquad f \nearrow}$

$f(x) = f(x^5)$

f 1-1

$x = x^5$

$1 = x^4$

$x = 1 \qquad \cancel{x = -1}$

απορ/τε

12. $f: \mathbb{R} \to \mathbb{R} \ \downarrow$

ⓐ $g(x) = f(x) - x$

$$\left.\begin{array}{l} \bullet\ x_1 < x_2 \overset{f\downarrow}{\Rightarrow} f(x_1) > f(x_2) \\ \bullet\ x_1 < x_2 \Rightarrow -x_1 > -x_2 \end{array}\right\} \overset{(+)}{\Rightarrow} g(x_1) > g(x_2)$$

$g \downarrow$

ⓑ. $f(\lambda^2 - 3\lambda) - f(2\lambda - 6) = \lambda^2 - 5\lambda + 6$

$$f(\lambda^2 - 3\lambda) - (\lambda^2 - 3\lambda) = f(2\lambda - 6) - (2\lambda - 6)$$

$$g(\lambda^2 - 3\lambda) = g(2\lambda - 6)$$

g : 1-1

$$\lambda^2 - 3\lambda = 2\lambda - 6$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$\lambda = 2$ $\lambda = 3$

7. (7) $f(x)=\sqrt{1-\ln x}$

πρέπει $x>0$ και $1-\ln x \geq 0$

$1 \geq \ln x$

$e \geq x$

$D_f = (0, e]$

<u>Μονοτονία $f(x)$</u>

· $x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2 \Rightarrow -\ln x_1 > -\ln x_2$

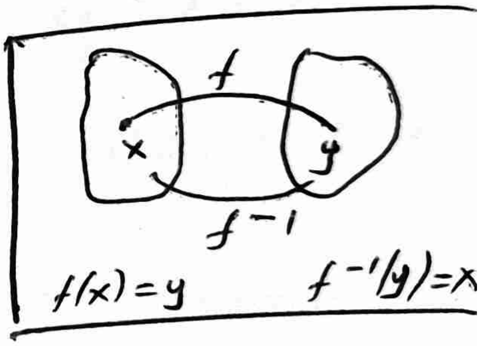
$1-\ln x_1 > 1-\ln x_2$

$\sqrt{1-\ln x_1} > \sqrt{1-\ln x_2}$

$f(x_1) > f(x_2)$.

$f\downarrow \Rightarrow f$ 1-1

άρα f αντιστρ.



Εύρεση αντίστροφης

Θέτω $f(x)=y$

$y=\sqrt{1-\ln x}$ $\quad y \geq 0$

$y^2 = 1-\ln x$

$\ln x = 1-y^2$

$e^{\ln x} = e^{1-y^2}$

$x = e^{1-y^2}$

Θέτω $x=f^{-1}(y)$

$f^{-1}(y) = e^{1-y^2}$

$\boxed{f^{-1}(x) = e^{1-x^2}}$

Τελος

$0 < x \le e$

$\underbrace{0 < e}_{\checkmark}^{1-y^2} \le e$

$e^{1-y^2} \le e^1$

$\cancel{1} - y^2 \le \cancel{1}$

$-y^2 \le 0$

$y^2 \geq 0$

που ισχυει.

7. δ) $f(x) = \dfrac{2e^{x}-1}{e^{x}+1}$, $x \in \mathbb{R}$.

$f(x_1) = f(x_2)$

$$\frac{2e^{x_1}-1}{e^{x_1}+1} = \frac{2e^{x_2}-1}{e^{x_2}+1}$$

$$(2e^{x_1}-1)(e^{x_2}+1) = (2e^{x_2}-1)(e^{x_1}+1)$$

$$\cancel{2e^{x_1}e^{x_2}} + 2e^{x_1} - e^{x_2} \cancel{-1} = \cancel{2e^{x_2}e^{x_1}} + 2e^{x_2} - e^{x_1} \cancel{-1}$$

$$3e^{x_1} = 3e^{x_2}$$

$$e^{x_1} = e^{x_2}$$

$$x_1 = x_2$$

f : 1-1

f αντιστρ.

Θετω $f(x)=y$

$$y=\frac{2e^x-1}{e^x+1}$$

$$y(e^x+1)=2e^x-1$$

$$ye^x+y=2e^x-1$$

$$ye^x-2e^x=-1-y$$

$$e^x(y-2)=-1-y$$

$$e^x=\frac{-1-y}{y-2}$$

$$x=\ln\frac{-1-y}{y-2}$$

πρεπει $\frac{-1-y}{y-2}>0$

y		-1		2	
$-1-y$	$+$	0	$-$		$-$
$y-2$	$-$		$-$	0	$+$
$\frac{-1-y}{y-2}$	$-$		$+$		$-$

$y\in(-1,2)$

Θετω $x=f^{-1}(y)$

$$f^{-1}(y)=\ln\frac{-1-y}{y-2}$$

$$f^{-1}(x)=\ln\frac{-1-x}{x-2},\ x\in(-1,2)$$

17. ε) $f(x) = \ln \frac{1+x}{1-x}$

πρέπει $\frac{1+x}{1-x} > 0$

x		-1		1	
$1+x$	$-$	0	$+$		$+$
$1-x$	$+$		$+$	0	$-$
$\frac{1+x}{1-x}$	$-$		$+$		$-$

$D_f = (-1, 1)$

Έστω $f(x_1) = f(x_2)$

$$\ln \frac{1+x_1}{1-x_1} = \ln \frac{1+x_2}{1-x_2}$$

$$\frac{1+x_1}{1-x_1} = \frac{1+x_2}{1-x_2} \Rightarrow (1+x_1)(1-x_2) = (1+x_2)(1-x_1)$$

$$\cancel{1} - x_2 + x_1 - \cancel{x_1 x_2} = \cancel{1} - x_1 + x_2 - \cancel{x_1 x_2}$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

f 1-1

$$f(x) = y$$

$$y = \ln \frac{1+x}{1-x}$$

$$e^{y} = \frac{1+x}{1-x}$$

$$e^{y}(1-x) = 1+x$$

$$e^{y} - xe^{y} = 1+x$$

$$e^{y} - 1 = x + xe^{y}$$

$$e^{y} - 1 = x(1+e^{y})$$

$$\frac{e^{y}-1}{e^{y}+1} = x$$

$$f^{-1}(y) = \frac{e^{y}-1}{e^{y}+1}$$

$$\boxed{f^{-1}(x) = \frac{e^{x}-1}{e^{x}+1}}$$

Τελος

$$-1 < x < 1$$

$$-1 < \frac{e^y - 1}{e^y + 1} < 1$$

$$-e^y - 1 < e^y - 1 < e^y + 1$$

$-e^y - \cancel{1} < e^y - \cancel{1}$ και $\cancel{e^y} - 1 < \cancel{e^y} + 1$

$0 < 2e^y$ που ισχυει!

$-1 < 1$ που ισχυει!

$$D_{f^{-1}} = \mathbb{R}.$$

8. $f(x) = (x-2)^2 + 3$, $x \geq 2$.

- $x_1 < x_2 \Rightarrow x_1 - 2 < x_2 - 2$

$$(x_1-2)^2 < (x_2-2)^2$$

$$(x_1-2)^2 + 3 < (x_2-2)^2 + 3$$

$$f(x_1) < f(x_2)$$

$f \nearrow \Rightarrow f$ 1-1

f αντιστρέφεται.

$f(x) = y$

$$y = (x-2)^2 + 3$$

$y > 0$

$$y - 3 = (x-2)^2$$

$y - 3 \geq 0$

$y \geq 3$

$$\sqrt{y-3}^{\,2} = (x-2)^2$$

$$|\sqrt{y-3}| = |x-2|$$

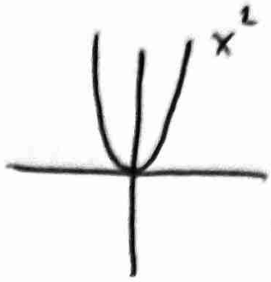
$$f^{-1}(x) = \sqrt{x-3} + 2$$

$$\sqrt{y-3} = x - 2$$

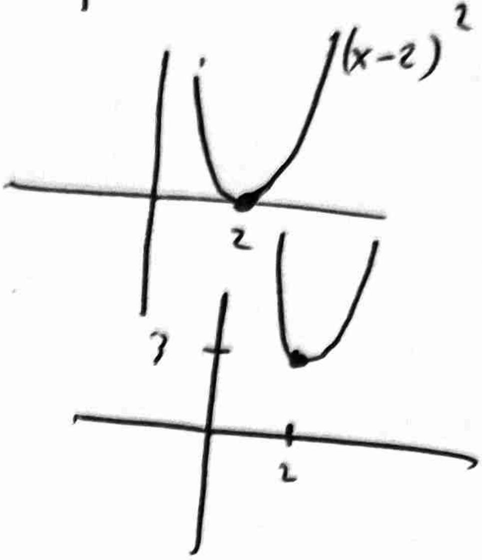
$$x = \sqrt{y-3} + 2$$

$$f^{-1}(y) = \sqrt{y-3} + 2$$

Τελος



$x \geq 2.$

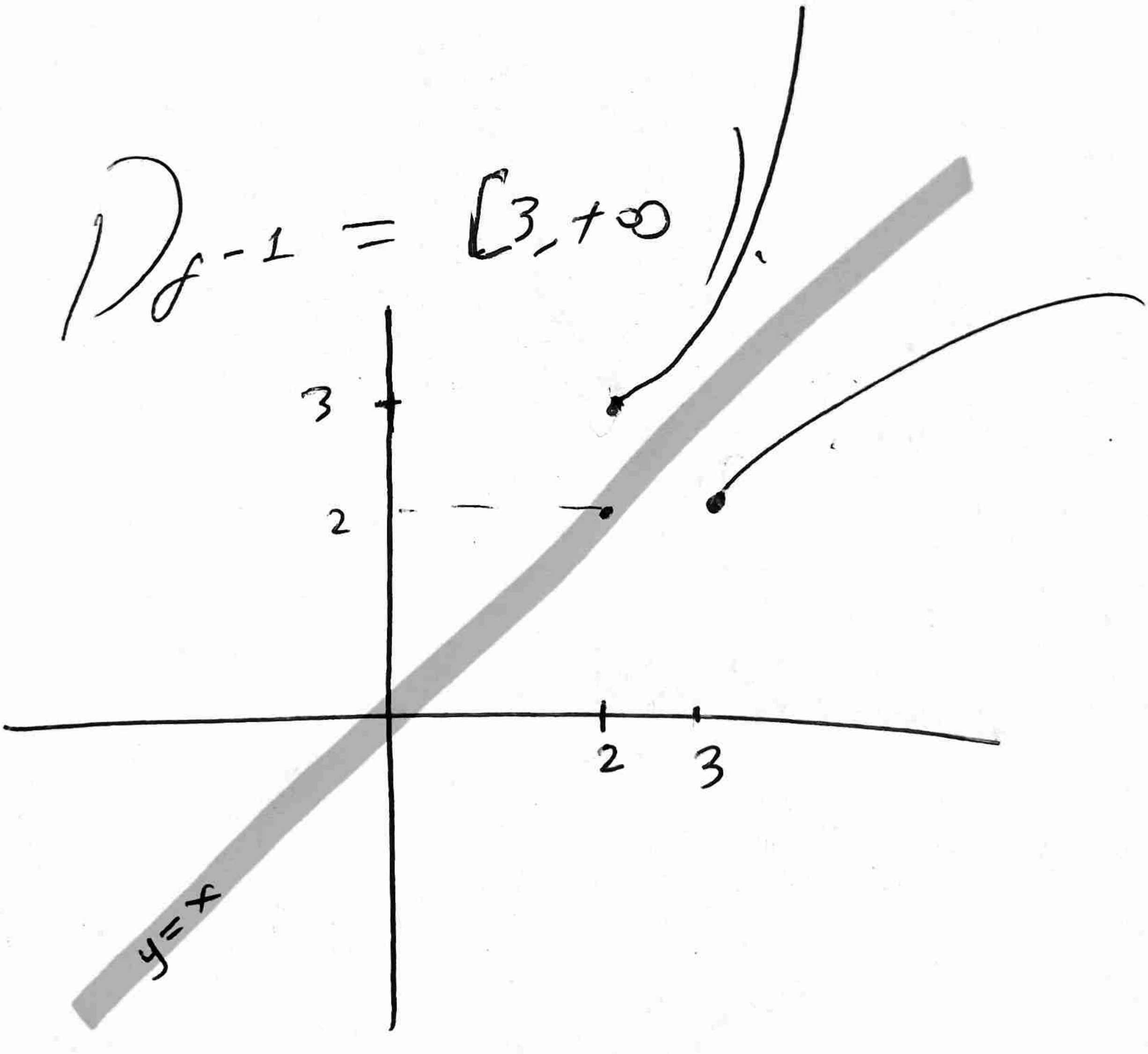


$\sqrt{y-3} + \cancel{2} \geq \cancel{2}$

$\sqrt{y-3} \geq 0$

που ισχυει.

$D_{f^{-1}} = [3, +\infty)$



5. $f(x) = \ln x + 1 \quad , \quad x > 0$

(α) $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad f \nearrow \Rightarrow f$ 1-1.

(β) Αφού 1-1 αντιστρ.

$f(x) = y$

$y = \ln x + 1$

$y - 1 = \ln x$

$e^{y-1} = x$

$f^{-1}(y) = e^{y-1}$

$$\boxed{f^{-1}(x) = e^{x-1}}$$

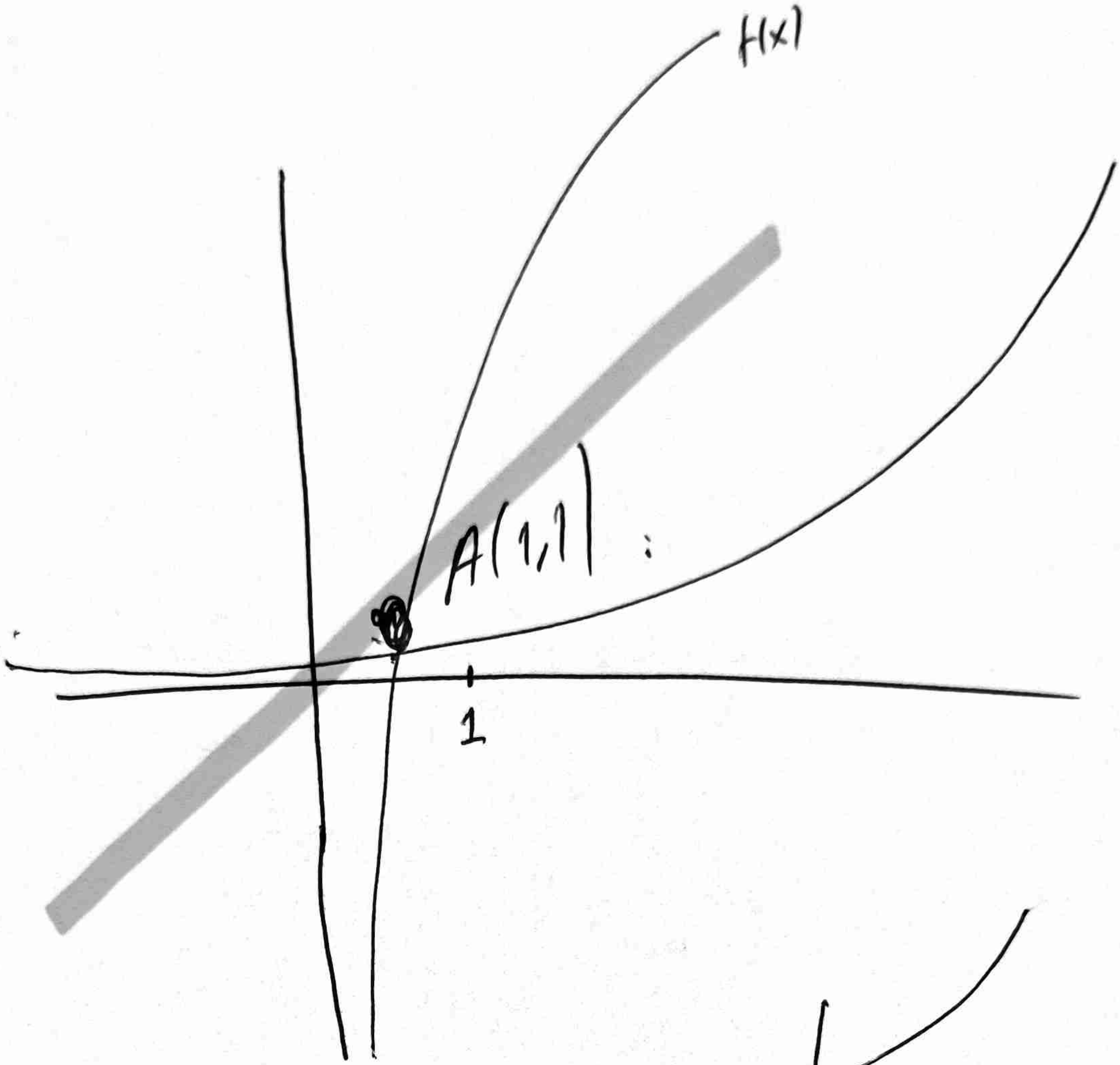
$D_{f^{-1}} = \mathbb{R}$.

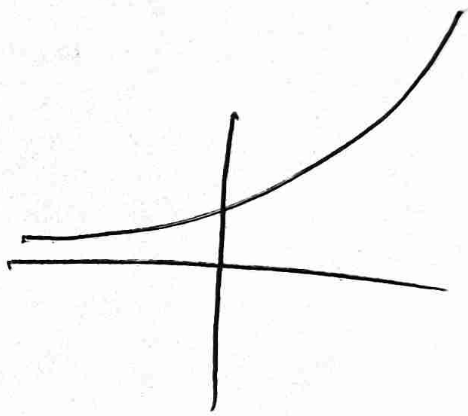
<u>Τέλος</u>

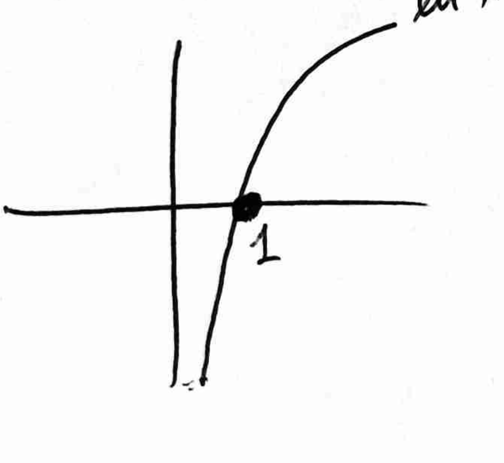
$x > 0$

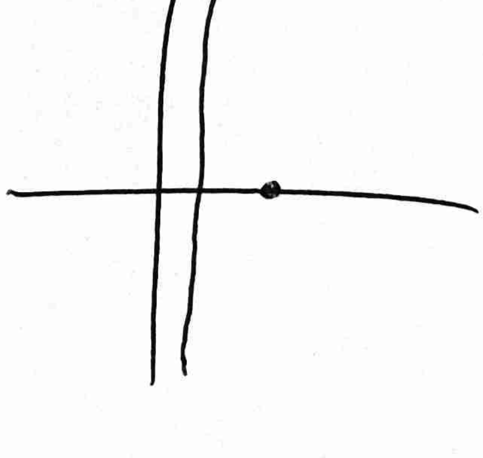
$e^{y-1} > 0$

που ισχύει.


f(x)
A(1,1)
1




ln x
1



17. $f(x) = e^x + x$ $\qquad f(\mathbb{R}) = \mathbb{R}$.

(α) $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

$(+)$

$f(x_1) < f(x_2)$ $\qquad f \nearrow \Rightarrow f$ 1-1

(β) $f^{-1}(x) = 0$

$f(f^{-1}(x)) = f(0)$

$\underline{\underline{x = 1}}$

$x < 1 \Rightarrow f^{-1}(x) < f^{-1}(1)$

$f^{-1}(x) < 0$

$x > 1 \Rightarrow f^{-1}(x) > f^{-1}(1)$

$f^{-1}(x) > 0$

x		1	
$f^{-1}(x)$	$\nearrow$ $-$	0	$+$ $\nearrow$

Έστω $f^{-1}(x_1) < f^{-1}(x_2)$

$f \nearrow$

$f(f^{-1}(x_1)) < f(f^{-1}(x_2))$

$x_1 < x_2$

$f^{-1} \nearrow$

20. $f(x) = e^{x-1} + 2x - 3$

(α) $x_1 < x_2 \Rightarrow e^{x_1-1} < e^{x_2-1}$

(+)

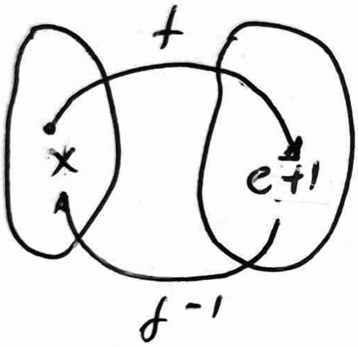
$x_1 < x_2 \Rightarrow 2x_1 - 3 < 2x_2 - 3$

$f \nearrow$

f 1-1

f αντιστρ.

(β) Βρες το $f^{-1}(e+1)$.



$f(x) = e+1$

$f(x) = f(2)$

f 1-1

$x = 2$

$f^{-1}(e+1) = 2$

(γ) i) $f\left(1 + f^{-1}(x+1)\right) = 0$

$f\left(1 + f^{-1}(x+1)\right) = f(1)$

f 1-1

$\cancel{1} + f^{-1}(x+1) = \cancel{1}$

$$f^{-1}(x+1) = 0$$

$$f(f^{-1}(x+1)) = f(0)$$

$$x+1 = e^{-1} - 3$$

$$\underline{\underline{x = \frac{1}{e} - 4}}$$

11). $f(x^2+x) + f^{-1}(e^2+3) > e+4$

$f(x) = e^2+3$

$f(x) = f(3)$

f 1-1

$\underline{\underline{x = 3}}$

$f^{-1}(e^2+3) = 3$

$f(x^2+x) + 3 > e+4$

$f(x^2+x) > e+1$

$f(x^2+x) > f(2)$

$f \nearrow$

$x^2+x > 2$

$x^2-x-2 > 0$

x		-1		2	
x^2-x-2	$+$		$-$		$+$

$x \in (-\infty,-1) \cup (2,+\infty)$

23. $f:\mathbb{R}\to\mathbb{R}$ $\quad f\nearrow$ $\quad f(1)=2$.

(α) Αφού $f\nearrow \Rightarrow f$: 1-1 $\Rightarrow f$ αντιστρέψιμη.

(β) $f(x) < 3 - f^{-1}(x+1)$.

$$\underbrace{f(x)-3+f^{-1}(x+1)}_{\varphi(x)} < 0$$

$\varphi(x)<0$

$\varphi(x)<\varphi(1)$

$\varphi\nearrow$

$x<1$

Μονοτονία $\varphi(x)$

- $x_1<x_2 \Rightarrow f(x_1)<f(x_2) \Rightarrow f(x_1)-3<f(x_2)-3$ $\quad +$
- $x_1<x_2 \Rightarrow x_1+1<x_2+1 \Rightarrow f^{-1}(x_1+1)<f^{-1}(x_2+1)$

γιατί $f^{-1}\nearrow$ επίσης.

$$\underbrace{f(x_1)-3+f^{-1}(x_1+1)}_{\varphi(x_1)} < \underbrace{f(x_2)-3+f^{-1}(x_2+1)}_{\varphi(x_2)}$$

$\varphi\nearrow$

$\varphi(1)=f(1)-3+f^{-1}(2)=2-3+1=0$

25. $f(x) = x^5 + x^3 + x$

$f(\mathbb{R}) = \mathbb{R}$.

(α) $x_1 < x_2 \Rightarrow x_1^5 < x_2^5$

$x_1 < x_2 \Rightarrow x_1^3 < x_2^3$ (+) $f \nearrow$

$x_1 < x_2$

f : 1-1

f αντιστρέφεται.

(β). $f(x^2-3x) - f(2x-6) = -x^2+5x-6$

$f(x^2-3x) + x^2-3x = f(2x-6) + 2x-6$

$$\boxed{\begin{array}{c} h(x) = f(x) + x \\ h \nearrow \Rightarrow h : 1\text{-}1 \end{array}}$$

$h(x^2-3x) = h(2x-6)$

h : 1-1

$x^2-3x = 2x-6$

$x^2-5x+6 = 0$

$x=2$ $x=3$

(γ). $\left[f^{-1}(\text{συν}x)\right]^5 + \left[f^{-1}(\text{συν}x)\right]^3 + f^{-1}(\text{συν}x) - x^2 = 1$

$$f\left(f^{-1}(\text{συν}x)\right) = x^2 + 1$$

$$\text{συν}x = x^2 + 1.$$

$$0 = \underbrace{x^2}_{+} + \underbrace{1 - \text{συν}x}_{+}$$

$$\begin{cases} x^2 = 0 \Rightarrow \underline{\underline{x = 0}} \\ 1 - \text{συν}x = 0 \quad \underline{\underline{x = 0}} \end{cases}$$

$$\underline{\underline{x = 0}}$$

Ελεύθερο Μάθημα

Δευτέρα 9:30

7

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