Ενότητα 7

5. $f(x) = \ln x + 1$

$D_f = (0, +\infty)$

(α) $x_1 < x_2 \Rightarrow \ln x_1 + 1 < \ln x_2 + 1$

$f(x_1) < f(x_2)$

$f \nearrow \Rightarrow f$ 1-1

(β) $y = \ln x + 1$

$y - 1 = \ln x$ $\qquad x > 0$

$e^{y-1} = e^{\ln x}$ $\qquad e^{y-1} > 0$ που ισχύει.

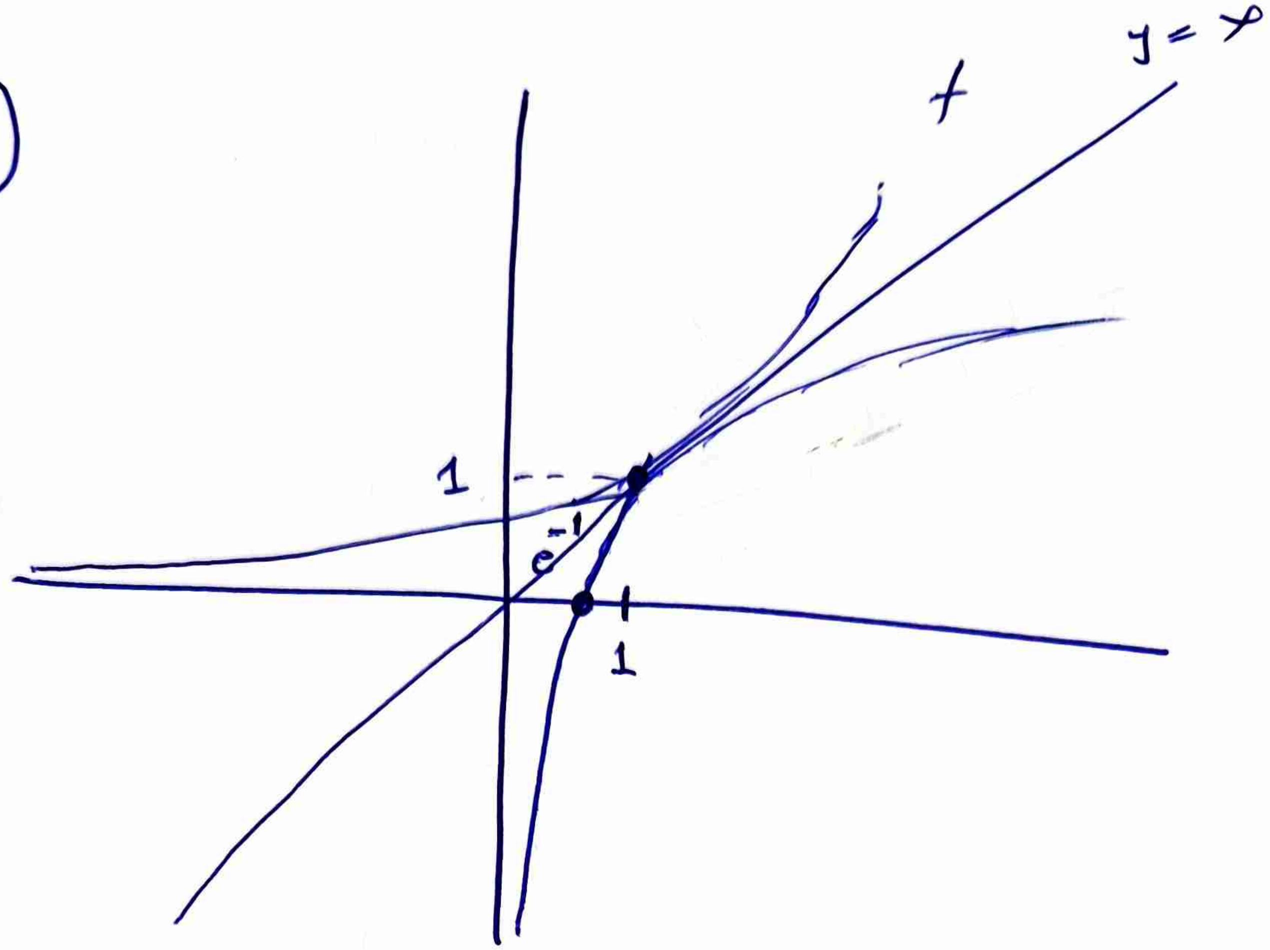
$e^{y-1} = x$

$f^{-1}(y) = e^{y-1}$

$\boxed{f^{-1}(x) = e^{x-1}}$

$D_{f^{-1}} = \mathbb{R}$.

(γ)



7. (α) $f(x) = 1 - \sqrt{1+e^x}\ , \ x \in R$

Ευκολα f ↓

$y = 1 - \sqrt{1+e^x}$

$y-1 = -\sqrt{1+e^x}$

$(y-1)^2 = 1+e^x$

$(y-1)^2 - 1 = e^x$

$x = \ln\left((y-1)^2 - 1\right)$

$f^{-1}(x) = \ln\left((x-1)^2 - 1\right)$

$D_{f^{-1}} = (-\infty, 0)$.

$y-1 < 0$

$\underline{y < 1}$

$(y-1)^2 - 1 > 0$

$(y-1)^2 > 1^2$

$|y-1| > |1|$

$|y-1| > 1$

$y-1 > 1$ ή $y-1 < -1$

$\underline{y > 2}$ $\quad$ $\underline{y < 0}$

$y \in (-\infty, 0) \cup (2,$

7. B $f(x) = 1 - \ln(1+e^x)$

Συνάρτ. $f \nearrow$

$y = 1 - \ln(1+e^x)$

$\ln(1+e^x) = 1-y$

$1+e^x = e^{1-y}$

$e^x = e^{1-y} - 1$

$x = \ln(e^{1-y} - 1)$

$f^{-1}(x) = \ln(e^{1-x} - 1)$

$D_{f^{-1}} = (-\infty, 1)$

$e^{1-y} - 1 > 0$

$e^{1-y} > 1$

$1-y > \ln 1$

$-y > -1$

$y < 1$

7. σε $f(x) = x - \ln(1+e^x)$

$$f(x) = \ln e^x - \ln(1+e^x)$$

$$f(x) = \ln\left(\frac{e^x}{1+e^x}\right)$$

Εύκολα $f(x_1)=f(x_2) \Rightarrow x_1 = x_2$

$$y = \ln \frac{e^x}{e^x+1}$$

$$e^y = \frac{e^x}{e^x+1}$$

$$e^y(e^x+1) = e^x$$

$$e^y e^x + e^y = e^x$$

$$e^y e^x - e^x = -e^y$$

$$e^x(e^y-1) = -e^y$$

$$e^x = \frac{-e^y}{e^y-1} \Rightarrow e^x = \frac{e^y}{1-e^y}$$

1

$$e^x = \frac{e^y}{1-e^y}$$

Präuч

$$\frac{e^y}{1-e^y} > 0$$

$$1-e^y > 0$$

$$1 > e^y$$

$$\underline{\underline{0 > y}}$$

$$x = \ln \frac{e^y}{1-e^y}$$

$$f^{-1}(x) = \ln \frac{e^x}{1-e^x}$$

$$D_{f^{-1}} = (-\infty, 0)$$

8. $f(x) = (x-2)^2 + 3 \quad , \; x \geq 2$

$f(x_1) = f(x_2)$

$(x_1-2)^2 + \cancel{3} = (x_2-2)^2 + \cancel{3}$

$|x_1 - 2| = |x_2 - 2|$

$x_1 - \cancel{2} = x_2 - \cancel{2}$ $\qquad f: 1-1$

$x_1 = x_2$

$y = (x-2)^2 + 3$ $\qquad y > 0$

$y - 3 = (x-2)^2$ $\qquad y - 3 \geq 0$

$y \geq 3$

$\sqrt{y-3}^{\,2} = (x-2)^2$

$|\sqrt{y-3}| = |x-2|$

$\sqrt{y-3} = x - 2$ $\qquad f^{-1}(x) = \sqrt{x-3} + 2$

$\sqrt{y-3} + 2 = x$ $\qquad D_{f^{-1}} = [3, +\infty)$.

Αφού $x \geq 2$

$$\sqrt{y-3} + \cancel{2} \geq \cancel{2}$$

$$\sqrt{y-3} \geq 0$$

που ισχυει!

Στο προηγουμενο μαθημα ειπαμε

$f(x) = x \iff f^{-1}(x) = x$

ειναι ισοδυναμες.

Τι λυνω τη μια

Τι λυνω την αλλη

Ειναι το ιδιο

Παραγουν τις ιδιες λυσεις.

Οι εξισωσει

$f(x) = f^{-1}(x) \iff f(x) = x \iff f^{-1}(x) = x$

Ειναι ισοδυναμει ΜΟΝΟ ΑΝ

$f \nearrow$

Αποδειξη

$f(x) = f^{-1}(x)$

f συναρτηση

$f(f(x)) = f(f^{-1}(x))$

$f(f(x)) = x$

$f(f(x)) + f(x) = f(x) + x$

$\boxed{g(x) = f(x) + x}$

$g(f(x)) = g(x)$

g: 1-1

$f(x) + x$

Μονοτονια $g(x)$

- $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $x_1 < x_2$ (+)

$x_1 + f(x_1) < x_2 + f(x_2)$

$g(x_1) < g(x_2)$

$g \nearrow$

g: 1-1.

20. $f(x) = e^{x-1} + 2x - 3 \quad , x \in \mathbb{R}$

(α) $x_1 < x_2 \Rightarrow x_1 - 1 < x_2 - 1 \Rightarrow e^{x_1-1} < e^{x_2-1}$

$x_1 < x_2 \Rightarrow 2x_1 - 3 < 2x_2 - 3$ (+)

$$e^{x_1-1} + 2x_1 - 3 < e^{x_2-1} + 2x_2 - 3$$

$$f(x_1) < f(x_2)$$

$$f \nearrow$$

Γνωρίζω
$f(\alpha) = b$
$\Leftrightarrow$
$f^{-1}(b) = \alpha$

(β) Βρω $f^{-1}(e+1)$

$f(\text{ποσο;}) = e+1$

$f(2) = e+1 \Leftrightarrow \underline{\underline{f^{-1}(e+1) = 2}}$

γ) i) $f\left(1+f^{-1}(x+1)\right)=0$

$$f\left(1+f^{-1}(x+1)\right)=f(1)$$

$$f \text{ 1-1}$$

$$1+f^{-1}(x+1)=1$$

$$f^{-1}(x+1)=0$$

$$f\left(f^{-1}(x+1)\right)=f(0)$$

$$x+1=e^{-1}-3$$

$$\underline{\underline{x=e^{-1}-4}}$$

ii) $f(x^2+x)+f^{-1}(e^2+3)>e+4$

$f(3)=e^2+3 \iff \underline{\underline{f^{-1}(e^2+3)=3}}$

$f(x^2+x)+3>e+4$

$f(x^2+x)>e+1$

$f(x^2+x)>f(2)$

$x^2+x>2$

$x^2+x-2>0$

$x\in(-\infty,-2)\cup(1,+\infty)$

25. $f(x) = x^5 + x^3 + x$

01 $x_1 < x_2 \Rightarrow x_1^5 < x_2^5$

$x_1 < x_2 \Rightarrow x_1^3 < x_2^3$ (+) $f \nearrow$

$x_1 < x_2$

B $f(x^2-3x) - f(2x-6) = -x^2+5x-6$

$f(x^2-3x) + x^2-3x = f(2x-6)+2x-6$

$$\boxed{g(x) = f(x)+x}$$

$g(x^2-3x) = g(2x-6)$

$x^2-3x = 2x-6$

$x^2-5x+6=0$

$x=2$ $x=3$

$\underbrace{x_1 < x_2 \Rightarrow f(x_1) < f(x_2)}$

(+)

$g(x_1) < g(x_2)$

$g \nearrow$

(γ) $$\underbrace{\left[f^{-1}(\sigma\upsilon\nu x)\right]^5+\left[f^{-1}(\sigma\upsilon\nu x)\right]^3+f^{-1}(\sigma\upsilon\nu x)}-x^2=1$$

$$f\left(f^{-1}(\sigma\upsilon\nu x)\right)=x^2+1$$

$$\sigma\upsilon\nu x=x^2+1$$

Προφανής ρίζα $x=0$

$$0=x^2+1-\sigma\upsilon\nu x$$

- $x^2>0 \quad \forall x\neq 0$
- $1>\sigma\upsilon\nu x \Rightarrow 1-\sigma\upsilon\nu x>0 \quad \forall x\neq 0.$

Άρα $x^2+1-\sigma\upsilon\nu x>0$

$\forall x\neq 0.$

(8) $f^{-1}(x) - f^{-1}(x^2) > \ln x \quad , x>0$

$$f^{-1}(x) - f^{-1}(x^2) > \ln \frac{x^2}{x}$$

$$f^{-1}(x) - f^{-1}(x^2) > \ln x^2 - \ln x$$

$$f^{-1}(x) + \ln x > \ln x^2 + f^{-1}(x^2)$$

$$\boxed{g(x) = f^{-1}(x) + \ln x}$$

$$g(x) > g(x^2)$$

- $x_1 < x_2 \Rightarrow f^{-1}(x_1) < f^{-1}(x_2)$

(+)

- $x_1 < x_2 \Rightarrow \ln x_1 < \ln x_2$

$g(x_1) < g(x_2) \qquad g \nearrow$

$x > x^2$

$1 > x$

Έστω ότι η $f \nearrow$

Έστω $f^{-1}(x_1) < f^{-1}(x_2)$

$f \nearrow$

$f(f^{-1}(x_1)) < f(f^{-1}(x_2))$

$x_1 < x_2$

$f^{-1} \nearrow$

Ότι μονοτονία έχει η μία έχει και η άλλη.

6. ① $f \nearrow$ $g \searrow$ $f(x) > 0$ $g(x) > 0$ Ενότητα 4

Νδο $\frac{f}{g} \nearrow$

$$\boxed{h(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}}$$

• $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ ⊙

• $x_1 < x_2 \Rightarrow g(x_1) > g(x_2) \Rightarrow \frac{1}{g(x_1)} < \frac{1}{g(x_2)}$

$$\frac{f(x_1)}{g(x_1)} < \frac{f(x_2)}{g(x_2)}$$

$$h(x_1) < h(x_2)$$

$$h \nearrow$$

7. (B) Av $f \uparrow$ $g \downarrow$ νδο $f \circ g \downarrow$

$$h(x) = (f \circ g)(x) = f(g(x))$$

- $x_1 < x_2 \Rightarrow g(x_1) > g(x_2) \Rightarrow f(g(x_1)) > f(g(x_2))$

$h(x_1) > h(x_2)$

$h \downarrow$.

18. $f(x) = a^x - x \qquad 0 < a < 1$

(a) $x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}$

(+)

$x_1 < x_2 \Rightarrow -x_1 > -x_2$

$a^{x_1} - x_1 > a^{x_2} - x_2$

$f(x_1) > f(x_2)$

$f \downarrow$.

(B) $a^{x^2+4} - a^{5x-2} < x^2-5x+6$

$0 < a < 1$

$a^{x^2+4} - (x^2+4) < a^{5x-2} - (5x-2)$

$f(x^2+4) < f(5x-2)$

$f \downarrow$

$x^2+4 > 5x-2$

$x^2-5x+6 > 0$

x		2		3	
x^2-5x+6	$+$		$-$		$+$

$x \in (-\infty, 2) \cup (3, +\infty)$

20. (γ) $x < \dfrac{2}{x^4+1}$

Ενότητα 4

$x(x^4+1) < 2$

$x^5 + x - 2 < 0 \Rightarrow f(x) < 0$

$\boxed{f(x) = x^5 + x - 2}$

$f(x) < f(1)$

Εύκολα

$f \nearrow$

$f \nearrow$

$x < 1$

22.

$f(x) = e^{x-1}\ ,\ x>0$

$g(x) = \frac{1}{x}\ ,\ x>0$

Σχεσηι Θεση

$f(x) < g(x) \Rightarrow e^{x-1} < \frac{1}{x}$

$\underbrace{e^{x-1} - \frac{1}{x}}_{h(x)} < 0$

$h(x) < 0$

$h(x) < h(1)$

$h \uparrow$

$\underline{0 < x < 1}$

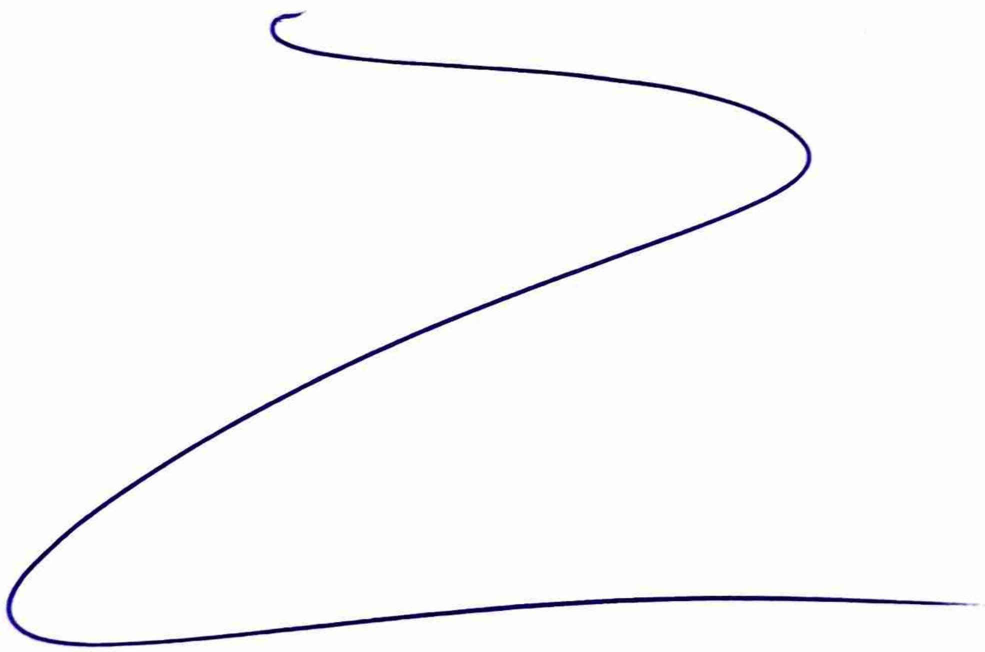
- $x_1 < x_2 \Rightarrow e^{x_1-1} < e^{x_2-1}$ ⊕
- $x_1 < x_2 \Rightarrow \frac{1}{x_1} > \frac{1}{x_2} \Rightarrow -\frac{1}{x_1} < -\frac{1}{x_2}$

$h \nearrow$

Όταν $x \in (0,1)$ $f(x) < g(x)$

Όταν $x > 1$ $f(x) > g(x)$

Όταν $x = 1$ $f(x) = g(x)$



25. α) f γνησίως μονότονη

$A(1,5)$ $\quad f(1)=5$

$B(5,-2)$ $\quad f(5)=-2$

$1<5$

$f \downarrow$

$\underset{\underset{5}{\parallel}}{f(1)} > \underset{\underset{-2}{\parallel}}{f(5)}$

Αφού f γν. μονότονη $\Rightarrow f\nearrow$ ή $f\downarrow$.

β) $x_1<x_2 \Rightarrow f(x_1)>f(x_2) \Rightarrow f(f(x_1))<f(f(x_2))$

$(f\circ f)(x_1)<(f\circ f)(x_2)$

$f\circ f \nearrow$.

(γ) $f(f(e^x)) < -2$

$f(f(e^x)) < f(5)$

$f \downarrow$

$f(e^x) > 5$

$f(e^x) > f(1)$

$f \downarrow$

$e^x < 1$

$\underline{\underline{x < 0}}$

(δ) $f(g(x)) > 5$

$g(x) = x + \ln x$

$g \nearrow$

$f(g(x)) > f(1)$

$f \downarrow$

$g(x) < 1$

$g(x) < g(1)$

$g \nearrow$

$0 < x < 1$ ✓

$\lim\limits_{x \to x_0^-} f(x)$ Αριστερό πλευρικό όριο

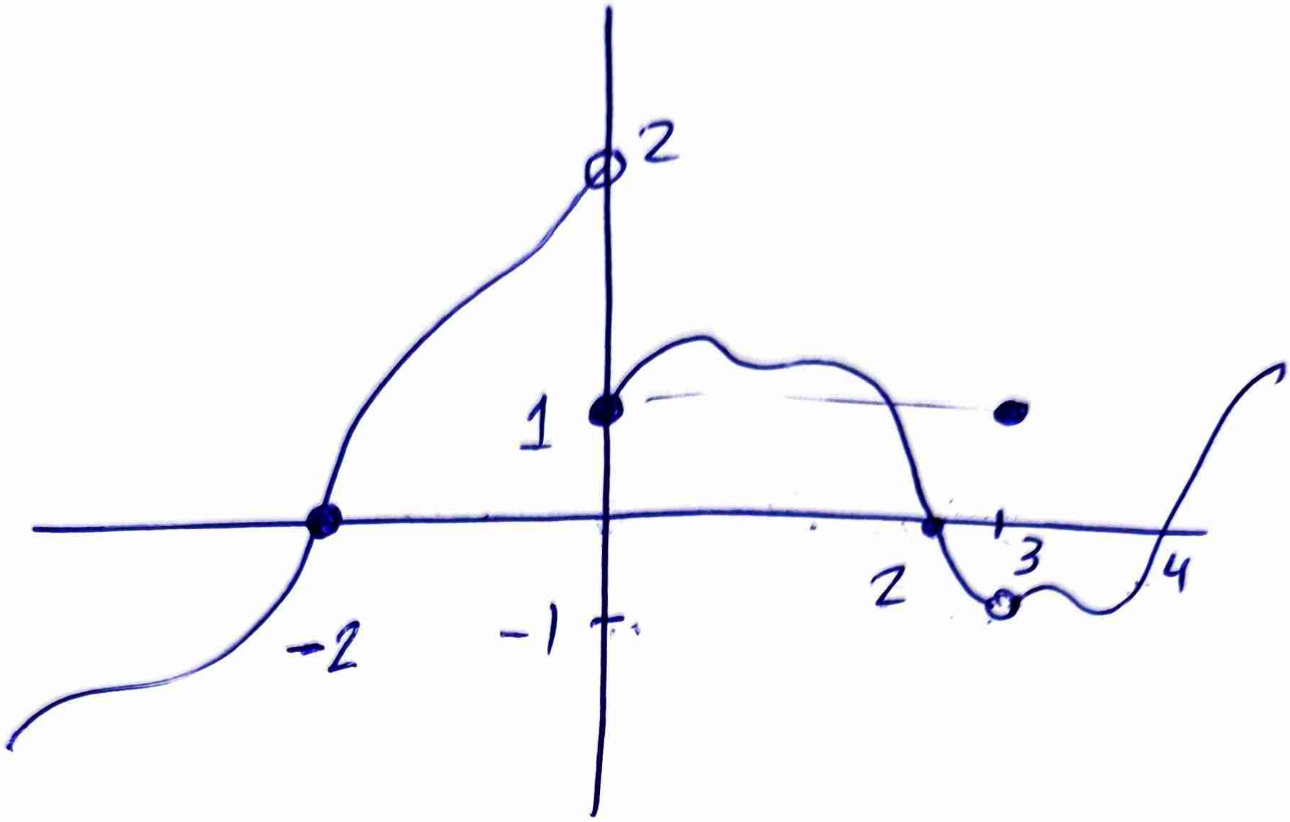
$\lim\limits_{x \to x_0^+} f(x)$ Δεξιό πλευρικό όριο,

Αν $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$

Τότε λέμε ότι το όριο

$\lim\limits_{x \to x_0} f(x)$ υπάρχει και

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x).$$



$$\lim_{x\to 0^-} f(x) = 2$$

$$\lim_{x\to 0^+} f(x) = 1$$

Το $\lim\limits_{x\to 0} f(x)$ δεν υπάρχει γιατί $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$.

$$\lim_{x\to 3^-} f(x) = -1$$

$$\lim_{x\to 3^+} f(x) = -1$$

$$\lim_{x\to 3} f(x) = -1$$

Επιλυση οριων

1. $$\lim_{x\to 1}\left(\frac{x^2-5x}{x+2}\right) = \frac{1^2-5\cdot 1}{1+2} = \frac{1-5}{3} = \frac{-4}{3}$$

2. $$\lim_{x\to 2}\frac{x^3-2x-4}{x^2-6x+8} \underset{}{\overset{\left(\frac{0}{0}\right)}{=\!=}}$$

1	0	−2	−4	2
↓	2	4	4	
1	2	2	0	

1	−6	8	2
↓	2	−8	
1	−4	0	

$$= \lim_{x\to 2}\frac{\cancel{(x-2)}(x^2+2x+2)}{\cancel{(x-2)}(x-4)} = \frac{4+4+2}{-2} = -5$$

3. $$\lim_{x \to 1} \left(\frac{x}{x-1} + \frac{x-2}{x^2-x} \right) =$$

$$= \lim_{x\to 1} \frac{x}{x-1} + \frac{x-2}{x(x-1)} =$$

$$= \lim_{x\to 1} \frac{x^2 + x - 2}{x(x-1)} = \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

$$= \lim_{x\to 1} \frac{x+2}{x} = 3.$$

4. $$\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x^2-1} \left(\frac{0}{0}\right) =$$

$$= \lim_{x\to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(x+1)(\sqrt{x+3}+2)} =$$

$$= \lim_{x\to 1} \frac{\cancel{\sqrt{x+3}}^{\cancel{2}} - 2^2}{(x-1)(x+1)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)(\sqrt{x+3}+2)} = \frac{1}{8}$$

5. $$\lim_{x\to 1} \frac{\sqrt{x^2+3}-3x+1}{x^2-1} \left(\frac{0}{0}\right) =$$

$$= \underset{x\to 1}{l} \frac{\sqrt{x^2+3}-(3x-1)}{x^2-1}$$

$$= \underset{x\to 1}{l} \frac{\left[\sqrt{x^2+3}-(3x-1)\right]\left[\sqrt{x^2+3}+3x-1\right]}{(x-1)(x+1)\left(\sqrt{x^2+3}+3x-1\right)}$$

$$= \underset{x\to 1}{l} \frac{\cancel{\sqrt{x^2+3}}^{\,\cancel{2}} - (3x-1)^2}{(x-1)(x+1)\left(\sqrt{x^2+3}+3x-1\right)}$$

$$= \underset{x\to 1}{l} \frac{x^2+3-(9x^2-6x+1)}{(x-1)(x+1)\left(\sqrt{x^2+3}+3x-1\right)}$$

$$= \lim_{x \to 1} \frac{-8x^2 + 6x + 2}{(x-1)(x+1)(\sqrt{x^2+3} + 3x - 1)}$$

$$= \lim_{x \to 1} \frac{-2(4x^2 - 3x - 1)}{(x-1)(x+1)(\sqrt{x^2+3} + 3x - 1)}$$

4	-3	-1	1
↓	4	1	
4	1	0	

$$= \lim_{x \to 1} \frac{-2 \cdot \cancel{(x-1)}(4x+1)}{\cancel{(x-1)}(x+1)(\sqrt{x^2+3} + 3x - 1)}$$

$$= \frac{-\cancel{2} \cdot 5}{\cancel{2} \cdot 4} = -\frac{5}{4}.$$