Θεμα 21

$$f(x) = \frac{e^x}{e^x - 1} \qquad D_f = \mathbb{R}^*$$

(a) $f(x_1) = f(x_2) \Leftrightarrow \dfrac{e^{x_1}}{e^{x_1}-1} = \dfrac{e^{x_2}}{e^{x_2}-1} \Leftrightarrow$

$$e^{x_1}(e^{x_2}-1) = e^{x_2}(e^{x_1}-1) \Leftrightarrow \cancel{e^{x_1}e^{x_2}} - e^{x_1} = \cancel{e^{x_1}e^{x_2}} - e^{x_2}$$

$e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2 \Rightarrow f$ 1-1 άρα αντιστρ.

$$y = \frac{e^x}{e^x-1} \Leftrightarrow y(e^x-1) = e^x \Leftrightarrow ye^x - y = e^x$$

$$ye^x - e^x = y \Leftrightarrow e^x(y-1) = y \Leftrightarrow e^x = \frac{y}{y-1}$$

$y \neq 1$

$x = \ln\left(\dfrac{y}{y-1}\right)$ πρέπει $\dfrac{y}{y-1} > 0$.

y		0		1	
y	$-$	0	$+$		$+$
$y-1$	$-$		$-$	0	$+$
$\frac{y}{y-1}$	$+$		$-$		$+$

$f^{-1}(x) = \ln\left(\dfrac{x}{x-1}\right)$

$D_{f^{-1}} \longrightarrow$

$y \in (-\infty, 0) \cup (1, +\infty)$.

(B) $f'(x) = \dfrac{e^x(e^x-1) - e^x e^x}{(e^x-1)^2}$

$f'(x) = \dfrac{\cancel{e^{2x}} - e^x - \cancel{e^{2x}}}{(e^x-1)^2} = -\dfrac{e^x}{(e^x-1)^2} < 0$

$f \downarrow$ στο $(-\infty, 0)$ και στο $(0, +\infty)$.

(Γ). $f''(x) = -\dfrac{e^x(e^x-1)^2 - e^x\,2(e^x-1)e^x}{(e^x-1)^4}$

$f''(x) = -\dfrac{e^x(e^x-1) - 2e^{2x}}{(e^x-1)^3}$

$f''(x) = -\dfrac{-e^x - e^{2x}}{(e^x-1)^3} = \dfrac{e^x(e^x+1)}{(e^x-1)^3}$

x		0	
f''	$-$		$+$
f	$\frown$		$\smile$

(γ) · $f^{-1}(x) = \ln\left(\dfrac{x}{x-1}\right)$ $\qquad D_{f^{-1}} = (-\infty, 0) \cup (1, +\infty)$

$\quad\; \parallel$

$\quad \varphi(x)$

$$\varphi'(x) = \frac{1}{\frac{x}{\cancel{x-1}}} \cdot \frac{x-1-x}{(x-1)^{\cancel{2}}} = \frac{1}{x} \cdot \frac{-1}{x-1}$$

$$\varphi'(x) = \frac{-1}{\underbrace{x(x-1)}_{+}} < 0 \qquad \varphi \searrow \text{ στο } (-\infty, 0) \text{ και } (1, +\infty)$$

$$\varphi''(x) = \frac{x-1+x}{x^2(x-1)^2} = \frac{2x-1}{x^2(x-1)^2}$$

x		0		$\frac{1}{2}$		1	
φ''	$-$		$\cancel{-}$	$\cancel{0}$	$\cancel{+}$		$+$
φ	$\frown$						$\smile$

Θεμα 22

$$f(x)=\begin{cases} e^{x-1}+\alpha, & x\le 1 \\ 1+\frac{\ln x}{x}, & x>1 \end{cases}$$

ⓐ Αφου η f ειναι συνεχης

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x) \Leftrightarrow$$

$$\lim_{x\to 1^-}\left(e^{x-1}+\alpha\right)=\lim_{x\to 1^+}\left(1+\frac{\ln x}{x}\right)$$

$$1+\alpha=1 \Leftrightarrow \underline{\underline{\alpha=0}}$$

$$\boxed{f(x)=\begin{cases} e^{x-1}, & x\le 1 \\ 1+\frac{\ln x}{x}, & x>1 \end{cases}}$$

ⓑ $$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{e^{x-1}-1}{x-1}=\lim_{x\to 1^-}\frac{e^{x-1}}{1}=1.$$

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{1+\frac{\ln x}{x}-1}{x-1}=\lim_{x\to 1^+}\frac{\ln x}{x^2-x}$$

$$=\lim_{x\to 1^+}\frac{\frac{1}{x}}{2x-1}=\frac{1}{1}=1 \qquad f'(1)=1=\varepsilon\varphi\omega$$

$\omega=45$

(β). $\underline{x \leq 1}$ $\qquad\qquad\qquad$ $\underline{x > 1}$

$f_1'(x) = e^{x-1} > 0$ $\qquad\qquad$ $f_2'(x) = \dfrac{1 - \ln x}{x^2}$

$\rightarrow 1 - \ln x = 0$

$1 = \ln x$

$\boxed{x = e}$

x		1	e
$f_1'(x)$	+	//////	//////
$f_2'(x)$	//////	+	0 −
$f'(x)$	+	+	−
$f(x)$	↗	↗	↘

$f(x) \leq f(e)$

$\boxed{f(x) \leq 1 + \frac{1}{e}.}$

Αφού $D_f = \mathbb{R}$ και $f(x)$ συνεχ. δν έχει κατακόρυφη ασύμπτωτη.

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} e^{x-1} = 0$ $\qquad$ $\varepsilon_1 : y = 0$ $\quad -\infty$

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} 1 + \dfrac{\ln x}{x} = 1$ $\qquad$ $\varepsilon_2 : y = 1$ $\quad +\infty$

$\rightarrow \lim\limits_{x \to +\infty} \dfrac{\ln x}{x} = \lim\limits_{x \to +\infty} \dfrac{1}{x} = 0$

(δ) $\displaystyle\int_1^2 f(x)\,dx = \int_1^2 1 + \frac{\ln x}{x}\,dx = \int_1^2 1\,dx + \int_1^2 \frac{\ln x}{x}\,dx = (x)_1^2 + \int_0^{\ln 2} t\,dt$

$\boxed{\begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \end{array}}$ $\displaystyle = 1 + \frac{1}{2}(t^2)_0^{\ln 2} = \frac{\ln^2 2}{2} + 1$

Θεμα 23

• $f(x) = e^{x-1} - \ln x$, $D_f = (0, +\infty)$

ⓐ $f'(x) = e^{x-1} - \frac{1}{x}$ $\quad f'(1) = 0$

$f''(x) = e^{x-1} + \frac{1}{x^2} > 0$

x	0		1	
f''		+		+
f'		↗ −	0	+ ↗
f		↘		↗

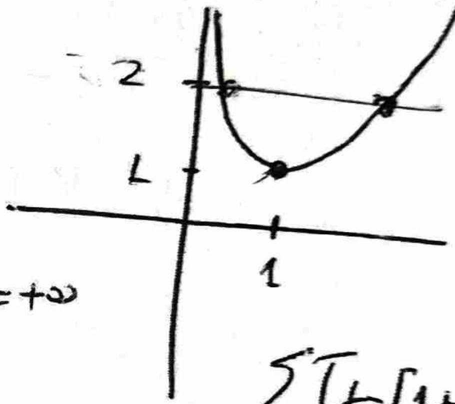
ⓑ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (e^{x-1} - \ln x) = e^{-1} - (-\infty) = +\infty$

$f(x) \geq f(1)$

$f(x) \geq 1$.

$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} (e^{x-1} - \ln x) =$

$= \lim_{x\to +\infty} e^{x-1} \left(1 - \cancel{\frac{\ln x}{e^{x-1}}}^{\,0}\right) = +\infty$



$\Sigma T_f = [1, +\infty)$.

$\rightarrow \lim_{x\to +\infty} \frac{\ln x}{e^{x-1}} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{e^{x-1}} = \lim_{x\to +\infty} \frac{1}{x e^{x-1}} = 0$

ⓖ

$x \leq 1$

• f συνεχ.

• f↓

• $\Sigma T_f = [1, +\infty)$

Το $2 \in \Sigma T_f$ άρα $\exists x_1$ τ.ω $f(x_1) = 2$ μοναδικό.

$x > 1$

• f συνεχ.

• f↗

• $\Sigma T_f = [1, +\infty)$

Το $2 \in \Sigma T_f$ άρα $\exists x_2$ μοναδικό τ.ω $f(x_2) = 2$

δ) $f(x) + |x-1| = 1$.

$$\underbrace{f(x)-1}_{\oplus} + \underbrace{|x-1|}_{\oplus} = 0$$

- $f(x) \geq 1 \Rightarrow f(x)-1 \geq 0$ Το "=" για $x=1$.
- $|x-1| \geq 0$ Το "=" για $x=1$

Μοναδική λύση $x=1$.

Θεμα 24

• $f(x) = \alpha e^{x-1} + \beta x^2$

(α) Η εφ/νη εφαπτομενη στο 1.

↳ $y = 0 \cdot x + 1$

$$\begin{cases} f'(1) = 0 \\ f(1) = 1 \end{cases} \Leftrightarrow \begin{cases} \alpha + 2\beta = 0 \\ \alpha + \beta = 1 \end{cases} \Leftrightarrow \begin{cases} -\alpha - 2\beta = 0 \\ \alpha + \beta = 1 \end{cases} (+)$$

$$-\beta = 1$$

$$\underline{\beta = -1}$$

$$\underline{\alpha = 2}$$

$f'(x) = \alpha e^{x-1} + 2\beta x$

$$\boxed{f(x) = 2e^{x-1} - x^2}$$

(β) $f'(x) = 2e^{x-1} - 2x$. $f'(1) = 0$

$f''(x) = 2e^{x-1} - 2$

$\longrightarrow 2e^{x-1} - 2 = 0$

$2e^{x-1} = 2$

$e^{x-1} = 1$

$x - 1 = 0$

$x = 1$.

x		1	
f''	$-$	0	$+$
f'	↘ $+$		$+$ ↗
f	↗		↗

Αρα η f ↗ αρα 1-1 αρα αντιστρεψιμη.

$D_{f^{-1}} = ΣΤ_f$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(2e^{x-1} - x^2\right) = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(2e^{x-1} - x^2\right) = \lim_{x\to+\infty} x^2\left(2\frac{e^{x-1}}{x^2} - 1\right) = +\infty$$

$$\rightarrow \lim_{x\to+\infty} \frac{e^{x-1}}{x^2} = \lim_{x\to+\infty} \frac{e^{x-1}}{2x} = \lim_{x\to+\infty} \frac{e^{x-1}}{2} = +\infty$$

$$\Sigma T_f = \mathbb{R} .$$

(8)

x		1	
f''	$-$	0	$+$
f	$\frown$		$\smile$

$A(1, f(1))$ $\Sigma.K$.

(5) . $f'(2e^{x-1}+1) > f'(x^2+2)$.

$\boxed{e^{x-1} > 0 \Rightarrow 2e^{x-1} > 0 \Leftrightarrow 2e^{x-1}+1 > 1}$

$\boxed{x^2 \geq 0 \Rightarrow x^2+2 > 2}$

$\forall x > 1$, $f' \nearrow$

$2e^{x-1}+1 > x^2+2$

$2e^{x-1} - x^2 > 1$.

$f(x) > 1 \Rightarrow f(x) > f(1) \Rightarrow \underline{\underline{x > 1}}$

Θεμα 25

$$f(x) = x + \frac{2\ln x}{x} \quad , \quad D_f = (0,+\infty)$$

ⓐ $$f'(x) = 1 + 2\,\frac{\frac{1}{x}x - \ln x}{x^2} = 1 + 2\,\frac{1-\ln x}{x^2} = \frac{x^2+2-2\ln x}{x^2}$$

Θετω $\boxed{\varphi(x) = x^2+2-2\ln x}$ $\quad \varphi(1)=0$

$$\varphi'(x) = 2x - \frac{2}{x} = \frac{2x^2-2}{x} = 2\,\frac{x^2-1}{x}$$

$\varphi'(x)=0 \iff x=1$ ή $x=-1$

x	0		1	
φ'		$-$		$+$
φ		$\searrow$ $+$	3	$+$ $\nearrow$
f'		$+$		$+$
f		$\nearrow$		$\nearrow$

$\varphi(x) \geq \varphi(1)$

$\underline{\varphi(x) \geq 3}$

ⓑ $$f'(x) = 1 + 2\,\frac{1-\ln x}{x^2}$$

$$f''(x) = 2\,\frac{-\frac{1}{x}x^2 - (1-\ln x)2x}{x^4} = 2\,\frac{-x - 2x + 2x\ln x}{x^4}$$

$$f''(x) = 2\,\frac{-3+2\ln x}{x^3}$$

$\rightarrow$ $2\ln x - 3 = 0$

$\ln x = \frac{3}{2}$

$x = e^{3/2} = \sqrt{e^3} = e\sqrt{e}$

x	0		$e\sqrt{e}$	
f''		$-$	0	$+$
f		$\frown$		$\smile$

(8) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + \frac{2\ln x}{x} = \lim_{x\to 0^+} x + 2\ln x \frac{1}{x} =$

$= 0 + 2(-\infty)(+\infty) = -\infty.$

$\boxed{\varepsilon_1 : x = 0}$

$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \left(x + \frac{2\ln x}{x} \right) = +\infty$

$\to \lim_{x\to +\infty} \frac{2\ln x}{x} = \lim_{x\to +\infty} \frac{2}{x} = 0$

$\lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to +\infty} \frac{x + \frac{2\ln x}{x}}{x} = \lim_{x\to +\infty} 1 + \frac{2\ln x}{x^2} = \underline{1}$

$\to \lim_{x\to +\infty} \frac{2\ln x}{x^2} = \lim_{x\to +\infty} \frac{\frac{2}{x}}{2x} = \lim_{x\to +\infty} \frac{2}{2x^2} = 0$

$\lim_{x\to +\infty} (f(x) - x) = \lim_{x\to +\infty} \left(x + \frac{2\ln x}{x} - x \right) = 0$

$\boxed{\varepsilon_2 : y = x}$

(δ) · $E = \int_1^e |f(x) - x| \, dx = \int_1^e \left| \frac{2\ln x}{x} \right| dx$ (Α)

• $f(x) = x \Rightarrow x + \frac{2\ln x}{x} = x \Leftrightarrow \frac{2\ln x}{x} = 0 \Leftrightarrow x = 1$

• $f(x) - x = \frac{2\ln x}{x}$

(Α) $\int_1^e 2 \frac{\ln x}{x} \, dx = 2 \int_1^e \frac{\ln x}{x} \, dx$

Θέτω $\ln x = t$

$\frac{1}{x} \cdot dx = dt$

$= 2 \int_0^1 t \, dt =$

$= (t^2)_0^1 = 1.$

Θεμα 26

• $f(x) = \begin{cases} \dfrac{ημx}{x}, & x \in (0,π) \\ α, & x = 0. \end{cases}$ Σωστό!.

(α) νδο $α = 1$.

$$\lim_{x \to 0^+} f(x) = f(0) \Leftrightarrow \lim_{x \to 0^+} \frac{ημx}{x} = α \Leftrightarrow α = 1$$

(β). $ε: y - f(0) = f'(0)(x - 0)$

• $f(0) = 1$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{ημx}{x} - 1}{x} = \lim_{x \to 0} \frac{ημx - x}{x^2}$$

$$= \lim_{x \to 0} \frac{συνx - 1}{2x} = \lim_{x \to 0} \frac{-ημx}{2} = 0$$

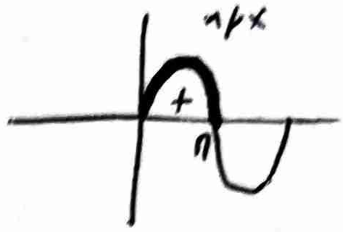
$ε: y - 1 = 0(x - 0)$ $\boxed{ε: y = 1}$.

(γ) $f'(x) = \dfrac{x\,συνx - ημx}{x^2}$

$$\boxed{φ(x) = x\,συνx - ημx}$$

$φ'(x) = συνx - x\,ημx - συνx$

$φ'(x) = -\underset{\oplus}{x}\,\underbrace{ημx}_{\oplus} < 0$ στο $(0,π)$



x	0 $\qquad\qquad$ $π$
$φ'$	$-$
$φ$	$\searrow$ $-$
f'	$-$
f	$\searrow$

$x > 0 \Rightarrow φ(x) < φ(0) \Rightarrow φ(x) < 0$

(δ) $\lim\limits_{x\to 0}\left[ημx\,\ln x\right] = \lim\limits_{x\to 0} \dfrac{ημx}{x} \cdot x\ln x = 1 \cdot 0 = 0$

$\rightarrow \lim\limits_{x\to 0} x\ln x = \lim\limits_{x\to 0} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = 0.$

Θεμα 27

• $f(x)= a^x - x$

• $f(x) \geq 1 \quad \forall x \in R$.

(α) νδο $a=e$.

$f(x) \geq 1$

$f(x) \geq f(0)$

$\left.\begin{array}{l}\text{Ακροτατο στο } 0\\ f \text{ παρ/μη στο } 0\\ \text{Το } 0 \text{ εσωτερικο του } R\end{array}\right\}$ Fermat $f'(0)=0$

$f'(x)= a^x \ln a - 1$

$f'(0)= \ln a - 1 = 0 \quad \Rightarrow \ln a = 1 \quad \Leftrightarrow a=e$

$$\boxed{f(x)= e^x - x}$$

(β). $e<\pi \Rightarrow f(e)<f(\pi) \Rightarrow e^e - e < e^\pi - \pi$

$\pi - e < e^\pi - e^e$

$1 < \dfrac{e^\pi - e^e}{\pi - e}$.

$f(x)=e^x-x$

$f'(x)=e^x-1$

x		0	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0) \Rightarrow f(x) \geq 1$

(Γ) $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty} \dfrac{e^x - x}{x} = \lim\limits_{x\to-\infty} \dfrac{e^x - 1}{1} = -1$

$\lim\limits_{x\to-\infty} (f(x)+x) = \lim\limits_{x\to-\infty} e^x = 0$ $\quad y = -x$ $\quad -\infty$

(Δ) $I = \int_0^\pi (e^x - x)\,\text{συν}x\,dx = \int_0^\pi e^x\,\text{συν}x\,dx - \int_0^\pi x\,\text{συν}x\,dx$

$I = J - R = -\dfrac{e^\pi + 1}{2} + 2.$

$J = \int_0^\pi e^x\,\text{συν}x\,dx = \int_0^\pi (e^x)'\,\text{συν}x\,dx = (e^x\,\text{συν}x)_0^\pi + \int_0^\pi e^x\,\text{ημ}x\,dx$

$J = -e^\pi - 1 + \int_0^\pi (e^x)'\,\text{ημ}x\,dx$

$J = -e^\pi - 1 + \overset{0}{\cancel{(e^x\,\text{ημ}x)_0^\pi}} - \int_0^\pi e^x\,\text{συν}x\,dx$

$J = -e^\pi - 1 - J \iff 2J = -e^\pi - 1 \iff J = -\dfrac{e^\pi + 1}{2}.$

$R = \int_0^\pi x\,\text{συν}x\,dx = \int_0^\pi x\,(\text{ημ}x)'\,dx = (x\,\text{ημ}x)_0^\pi - \int_0^\pi \text{ημ}x\,dx$

$R = +(\text{συν}x)_0^\pi = -2$

Θεμα 28

- $f: R \rightarrow R$ παρ/μη
- $f(0) = 1$
- $f(x) \geq 2e^x - x - 1 \quad \forall x \in R$.

ⓐ $\underbrace{f(x) - 2e^x + x + 1}_{\varphi(x)} \geq 0 \Leftrightarrow \varphi(x) \geq 0 \Leftrightarrow \varphi(x) \geq \varphi(0)$

Fermat

$\varphi'(0) = 0$

γιατι ελαχιστο στο 0

παρ/μη στο 0

Οεσωτερικο του R.

$\varphi'(x) = f'(x) - 2e^x + 1$

$\varphi'(0) = f'(0) - 1 = 0$

$f'(0) = 1$

$\varepsilon\varphi\omega = 1 \Leftrightarrow \underline{\underline{\omega = 45}}$

ⓑ $f(x) \geq 2e^x - x - 1$

$\lim_{x\to+\infty} f(x) \geq \lim_{x\to+\infty} 2e^x - x - 1 \Rightarrow \lim_{x\to+\infty} f(x) \geq +\infty$

$\lim_{x\to+\infty} f(x) = +\infty$.

$\rightarrow \lim_{x\to+\infty} 2e^x - x - 1 = \lim_{x\to+\infty} x\left(2\frac{e^x}{x} - 1 - \cancel{\frac{1}{x}}\right) = +\infty$

$\rightarrow \lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} e^x = +\infty$

(γ) νδο $2\int_0^1 f(x)\,dx \geq 4e-7$

Αφού $f(x) \geq 2e^x - x - 1$

$$\int_0^1 f(x)\,dx \geq \int_0^1 2e^x - x - 1$$

$$\Leftrightarrow \int_0^1 f(x)\,dx \geq 2\,(e^x)_0^1 - \frac{1}{2}(x^2)_0^1 - (x)_0^1$$

$$\int_0^1 f(x)\,dx \geq 2(e-1) - \frac{1}{2} - 1$$

$$\Leftrightarrow 2\int_0^1 f(x)\,dx \geq 4e - 4 - 1 - 2$$

$$2\int_0^1 f(x)\,dx \geq 4e - 7 .$$

(δ) $\lim\limits_{x\to 0}(f(x)-1)\ln x = \lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \cdot x\ln x$

$$= f'(0)\cdot 0 = 0$$

γιατί $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$$= \lim_{x\to 0^+} -\frac{x^{\cancel{2}}}{\cancel{x}} = 0 .$$

Θεμα 29

• $f(x) = 2e^x - x^2 - 2x$

ⓐ $f'(x) = 2e^x - 2x - 2 = 2(e^x - x - 1) \geq 0$ $f \nearrow$

• $e^x \geq x+1 \Rightarrow e^x - x - 1 \geq 0$

ⓑ $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} (2e^x - x^2 - 2x) = -\infty$

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} (2e^x - x^2 - 2x) =$

$= \lim\limits_{x \to +\infty} e^x \left(2 - \cancel{\frac{x^2}{e^x}}^{0} - \cancel{\frac{2x}{e^x}}^{0}\right) = +\infty$

$\to \lim\limits_{x \to +\infty} \frac{x^2}{e^x} = \lim\limits_{x \to +\infty} \frac{2x}{e^x} = \lim\limits_{x \to +\infty} \frac{2}{e^x} = 0$

$\Sigma T_f = \mathbb{R}.$

ⓖ i) $\lim\limits_{x \to 0} \frac{f(x) - 2}{\eta\mu x} = \lim\limits_{x \to 0} \frac{\frac{f(x) - f(0)}{x - 0}}{\frac{\eta\mu x}{x}} = \frac{f'(0)}{1} = 0.$

11/. $\lim_{x\to 0} (f(x)-2)\ln x =$

$= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \times \ln x = f'(0)\cdot 0 = 0$

$\rightarrow \lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0.$

⑧ NSO $f(e^x) > f(x)$ $\forall x \in \mathbb{R}$.

$\cdot\ e^x \geq x+1$

$e^x > x$

$f \nearrow$

$f(e^x) > f(x)$.

Θεμα 30

$$f(x)=\begin{cases} 2x\ln x - x^2+2 & , x>0 \\ 2 & , x=0 \end{cases}$$

(α) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left[2x\ln x - x^2+2\right] = 2$

$\to \lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} =$

$= \lim\limits_{x\to 0^+} -\dfrac{x^2}{x} = 0$

$f(0)=2$ αρα συνεχης στο 0.

Επισης συνεχης στο $(0,+\infty)$ ως πραξεις συνεχων συναρτησεων.

(B) $f'(x) = 2\ln x + 2 - 2x = 2\overset{\ominus}{(\ln x - x + 1)} \leq 0$

• $\ln x \leq x-1 \Rightarrow \ln x - x + 1 \leq 0$

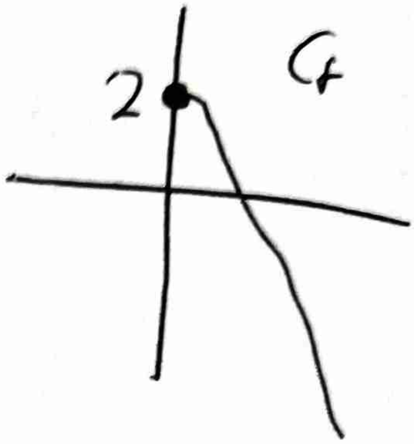
$f \downarrow$

(γ) $f(0)=2$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(2x\ln x - x^2 + 2\right) =$$

$$= \lim_{x\to+\infty} x^2\left(2\underbrace{\cancel{\frac{\ln x}{x}}}_{\to 0} - 1 + \underbrace{\cancel{\frac{2}{x^2}}}_{\to 0}\right) = -\infty$$

$$\to \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$$



$\Sigma T_f = (-\infty, 2]$

(δ) $f(|x|) - f(|\eta\mu x|) = 0$

$f(|x|) = f(|\eta\mu x|)$

f: 1-1 αφού $f \downarrow$

$|x| = |\eta\mu x|$

όμως $|\eta\mu x| \le |x|$ και το "="

επιτυγχάνεται για $x=0$.

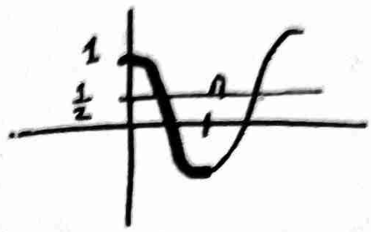
Θεμα 32

- $f : [0,\pi] \longrightarrow R$
- $f(x)=2\eta\mu x - x$

(α) $f'(x)=2\sigma\upsilon\nu x - 1$

$\longrightarrow 2\sigma\upsilon\nu x - 1 = 0 \iff \sigma\upsilon\nu x = \frac{1}{2}$

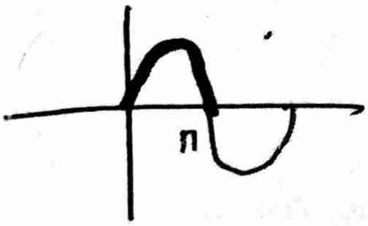
$\boxed{x = \frac{\pi}{3}}$



x	0	$\frac{\pi}{3}$	π
f'	$+$	0	$-$
f	↗		↘

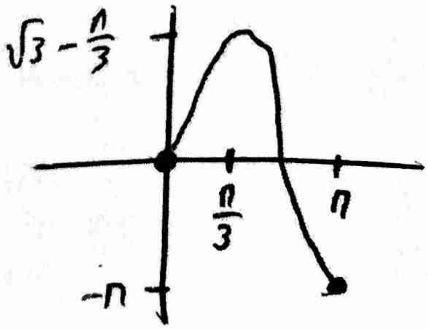
$f(x) \leq f(\frac{\pi}{3}) \iff f(x) \leq \sqrt{3} - \frac{\pi}{3}$

$f''(x) = -2\eta\mu x < 0$ $\quad$ f κοίλη



(β) εξίσωση $\eta\mu x = \frac{x}{2} \iff \iff 2\eta\mu x - x = 0$

$f(x)=0$ $\quad (0,\pi)$.



$f(0)=0$

$f(\pi) = -\pi$

$0 < x \le \frac{\pi}{3}$

- $f \uparrow$
- f συνεχ.
- $\Sigma T f = (0, \sqrt{3} - \frac{\pi}{3})$

Το $0 \notin \Sigma T f$.

$\frac{\pi}{3} \le x < \pi$

- $f \downarrow$
- f συνεχ.
- $\Sigma T f = (-\pi, \sqrt{3} - \frac{\pi}{3}]$

Το $0 \in \Sigma T f$

άρα $\exists x_0$ μοναδικό

τ.ω $f(x_0) = 0$.

(γ). $\lim_{x \to 0} f(x) \ln x = \lim_{x \to 0} \left[\frac{f(x) - f(0)}{x - 0}\right] x \ln x = f'(0) \cdot 0 = 0$

$\to \lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} =$

$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$

(δ). $\int_0^\pi f(x) \sigma \upsilon \nu x \, dx = \int_0^\pi (2\eta\mu x - x) \sigma \upsilon \nu x \, dx =$

$= 2\int_0^\pi \eta\mu x \, \sigma \upsilon \nu x \, dx - \int_0^\pi x \, \sigma \upsilon \nu x \, dx = 2.$

$\to \int_0^\pi \eta\mu x \, \sigma \upsilon \nu x \, dx \xrightarrow[\sigma \upsilon \nu x dx = dt]{\eta\mu x = t} \int_0^0 t \, dt = 0$

$\to \int_0^\pi x \sigma \upsilon \nu x dx = \int_0^\pi x (\eta\mu x)' dx = \cancel{(x \eta\mu x)_0^\pi} - \int_0^\pi \eta\mu x \, dx = (\sigma \upsilon \nu x)_0^\pi = -2$

Θεμα 33

• $f(x) = x^4 - x^3 + \lambda x^2 - \lambda x$

ⓐ Η $f(x)$ ειναι συνεχης στο $[0,1]$ ως πολυωνυμο.
Η $f(x)$ ειναι παρ/μη στο $(0,1)$ ως πολυωνυμο.

$$\left.\begin{array}{l} f(0)=0 \\ f(1)=0 \end{array}\right\} \text{Rolle} \quad \exists \xi \in (0,1) \text{ τ.ω } f'(\xi)=0.$$

ⓑ. Νδο η $4x^3 - 3x^2 + 2\lambda x - \lambda = 0$ εχει ριζα στο $(0,1)$.

$f'(x) = 4x^3 - 3x^2 + 2\lambda x - \lambda$

$f'(\xi) = \boxed{4\xi^3 - 3\xi^2 + 2\lambda\xi - \lambda = 0}$

ⓒ $f''(x) = 12x^2 - 6x + 2\lambda = 2(6x^2 - 3x + \lambda)$

$\Delta = 9 - 24\lambda = 3(3 - 8\lambda)$ (+)

ομως $\lambda < \frac{3}{8} \Rightarrow 8\lambda < 3$

$3 - 8\lambda > 0$

$\Delta > 0$

x		ρ_1		ρ_2	
f''	+	0	−	0	+
f	∪		∩		∪

αρα η $f''(x) = 0$ εχει 2 ριζες ανισες.

⑧ $\int_0^1 x f(x^2)\, dx = 0 \iff \int_0^1 \frac{1}{2} f(t)\, dt = 0$

$$\boxed{\begin{aligned} x^2 &= t \\ 2x\, dx &= dt \\ x\, dx &= \tfrac{1}{2}\, dt \end{aligned}}$$

$\frac{1}{2} \int_0^1 f(t)\, dt = 0$

$\int_0^1 f(t)\, dt = 0$

$\iff \int_0^1 t^4 - t^3 + \lambda t^2 - \lambda t \; dt = 0$

$\frac{1}{5}(t^5)_0^1 - \frac{1}{4}(t^4)_0^1 + \frac{\lambda}{3}(t^3)_0^1 - \frac{\lambda}{2}(t^2)_0^1 = 0$

$\frac{1}{5} - \frac{1}{4} + \frac{\lambda}{3} - \frac{\lambda}{2} = 0$

$-\frac{1}{20} + \frac{-\lambda}{6} = 0 \iff \frac{\lambda}{6} = -\frac{1}{20} \iff \lambda = -\frac{6}{20}$

$\lambda = -\frac{3}{10}$

Θεμα 37

- $f: (0,+\infty) \longrightarrow \mathbb{R}$
- $f(1) = 1$
- $f'(1) = 0$
- $x^2 f'(x) + 1 = x \quad \forall x > 0$

ⓐ $f'(x) = \dfrac{x-1}{x^2} = \dfrac{1}{x} - \dfrac{1}{x^2}$

$f'(x) = \left(\ln x + \dfrac{1}{x}\right)' \Leftrightarrow f(x) = \ln x + \dfrac{1}{x} + c$

$\underline{x=1}$

$f(1) = 1 + c$

$1 = 1 + c \Leftrightarrow c = 0$

$$\boxed{f(x) = \ln x + \frac{1}{x}}$$

ⓑ $f'(x) = \dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x-1}{x^2}$

x	0		1	
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$f(x) \geq 1$.

$\lim\limits_{x\to 1} (f(x)-1)\ln(x-1) =$

$= \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1} (x-1)\ln(x-1) = f'(1) \cdot 0 = 0,$

$\rightarrow \lim\limits_{x\to 1} (x-1)\ln(x-1) = \lim\limits_{x\to 1} \dfrac{\ln(x-1)}{\frac{1}{x-1}} = \lim\limits_{x\to 1} \dfrac{\frac{1}{x-1}}{-\frac{1}{(x-1)^2}} = 0$

(γ) $f''(x) = \dfrac{x^2-(x-1)2x}{x^4} = \dfrac{x-2(x-1)}{x^3} = \dfrac{x-2x+2}{x^3}$

$f''(x) = \dfrac{2-x}{x^3}$

x	0		2	
f''		+	0	−
f		∪		∩

(δ) $f'\left(x+\frac{1}{x}\right) = \frac{1}{4}$

$f'\left(x+\frac{1}{x}\right) = f'(2)$

x	0		2	
f''		+		−
f'		↗		↘

Ισχύει ότι όταν $x>0$ τότε $x+\frac{1}{x}>2$

γιατί $x^2+1>2x \Rightarrow x^2-2x+1>0 \Rightarrow (x-1)^2>0$

Άρα $x+\frac{1}{x}>2$

f' ↓ στο $(2,+\infty)$ άρα 1-1.

$f'\left(x+\frac{1}{x}\right) = f'(2)$

f' 1-1

$x+\frac{1}{x}=2 \Leftrightarrow x^2+1=2x$

$(x-1)^2=0$

$x=1$.

Θεμα 38

- $f'(x) = \begin{cases} -x, & x<1 \\ -\frac{1}{x^2}, & x>1 \end{cases}$

- $f(0)=\frac{1}{2}$

- f συνεχής.

(α) $f(x)= \begin{cases} -\frac{1}{2}x^2+c_1, & x<1 \\ \frac{1}{x}+c_2, & x\geq 1. \end{cases}$

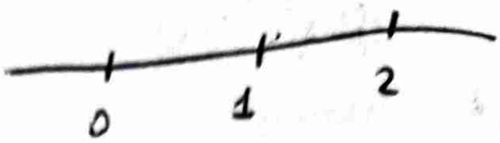
$f(0)= \boxed{c_1=\frac{1}{2}}$

$\lim_{x\to 1^-} f(x)= \lim_{x\to 1^+} f(x) \Leftrightarrow \lim_{x\to 1^-}\left(-\frac{1}{2}x^2+\frac{1}{2}\right)=\lim_{x\to 1^+}\left(\frac{1}{x}+c_2\right)$

$\Leftrightarrow 0 = 1 + c_2 \Leftrightarrow \boxed{c_2=-1}$

$$\boxed{f(x)= \begin{cases} -\frac{1}{2}x^2+\frac{1}{2}, & x<1 \\ \frac{1}{x}-1, & x\geq 1. \end{cases}}$$

(B) Η f(x) είναι συνεχής στο [0,2] ως πράξεις συνεχών συναρτήσεων.



Η f(x) είναι παρ/μη στο (0,2) ως πράξεις παραγωγίσιμων συναρτήσεων.

$$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{\frac{1-x^2}{2}-0}{x-1}=\lim_{x\to1^-}\frac{1-x^2}{2x-2}=$$

$$=\lim_{x\to1^-}\frac{-\cancel{2}x}{\cancel{2}}=-1.$$

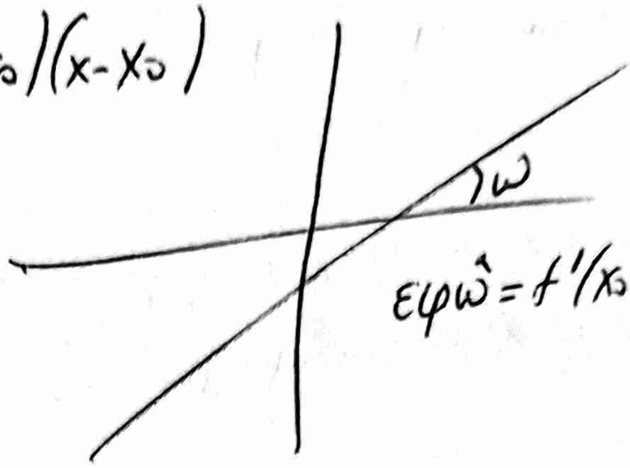
$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{\frac{1}{x}-1-0}{x-1}=\lim_{x\to1^+}\frac{1-x}{x^2-x}$$

$$=\lim_{x\to1^+}\frac{-1}{2x-1}=\frac{-1}{1}=-1$$

Η f παρ/μη στο 1.

Ικανοποιούνται οι προϋποθέσεις του ΘΜΤ.

(γ). $\varepsilon: y - f(x_0) = f'(x_0)(x - x_0)$



$\varepsilon\varphi\hat{\omega} = f'(x_0)$

$f'(x_0) = \varepsilon\varphi \frac{3\pi}{4}$

$f'(x_0) = \varepsilon\varphi\, 135$

$f'(x_0) = -1$.

$x \le 1$

$f'(x) = -1$

$-x = -1$

$\boxed{x = 1}$

$x > 1$

$f'(x) = -1$

$-\frac{1}{x^2} = -1$

$1 = x^2$

$\cancel{x = 1}$ ή $\cancel{x = -1}$

$\varepsilon: y - f(1) = f'(1)(x-1)$

$y - 0 = -1(x-1)$

$\boxed{y = -x + 1}$

(δ). Αφού $D_f = \mathbb{R}$ δεν έχω κατακόρυφη.

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1 - x^2}{2} = -\infty$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x} - 1 = -1$

$\varepsilon: y = -1$

$+\infty$

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$$\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{\frac{1-x^2}{2}}{x}=\lim_{x\to-\infty}\frac{1-x^2}{2x}=$$

$$=\lim_{x\to-\infty}\frac{-\cancel{2}x}{\cancel{2}}=+\infty$$ Δεν έχει πλάγια.

$x<1$

$f_1'(x)=-x$

$\rightarrow -x=0$

$x=0$

$x\geq 1$

$f_2'(x)=-\frac{1}{x^2}<0$

x		0		1	
f_1'	$+$	0	$-$		//////
f_2'	//////		//////		$-$
f'	$+$		$-$		$-$
f	$\nearrow$		$\searrow$		$\searrow$

$f(x)\leq f(0)$

$f(x)\leq \frac{1}{2}$

$x<1$

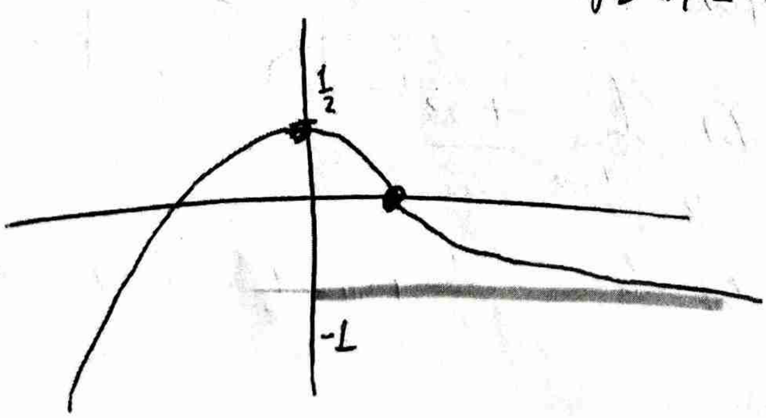
$f_1''(x)=-1<0$

x		1	
f_1''	$-$		//////
f_2''	//////		$+$
f''	$-$		$+$
f	$\cap$		$\cup$

$x>1$

$f_2''(x)=-\frac{-2x}{x^4}$

$f_2''(x)=\frac{2}{x^3}>0$



Επομενο Μαθημα

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