21.

Ενότητα 35

(a) $\int_2^3 \frac{1}{x^2-1}\,dx \stackrel{(*)}{=}$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-1)$$

$$1 = Ax + A + Bx - B$$

$$0x + 1 = (A+B)x + A - B$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \oplus \quad 2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

$$B = -\frac{1}{2}.$$

$$\stackrel{(*)}{=} \int_2^3 \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\,dx$$

$$= \frac{1}{2}\int_2^3 \frac{1}{x-1}\,dx - \frac{1}{2}\int_2^3 \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\left(\ln|x-1|\right)_2^3 - \frac{1}{2}\left(\ln|x+1|\right)_2^3 = \frac{1}{2}\ln 2 - \frac{1}{2}\left(\ln\frac{4}{3}\right.$$

(B) $\displaystyle\int_0^{-1} \frac{2x-3}{x^2-4x+3}\,dx$ (*)

$$\frac{2x-3}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

$$2x-3 = A(x-1) + B(x-3)$$

$$2x-3 = Ax - A + Bx - 3B$$

$$2x-3 = (A+B)x - A - 3B$$

$$\begin{cases} 2 = A+B \\ -3 = -A-3B \end{cases}$$

(+) $-1 = -2B \Rightarrow B = \frac{1}{2}$

$2 = A + \frac{1}{2}$ $A = \frac{3}{2}$

$$\int_0^{-1} \frac{\frac{3}{2}}{x-3} + \frac{\frac{1}{2}}{x-1}\,dx =$$

$$= \frac{3}{2}\left(\ln|x-3|\right)_0^{-1} + \frac{1}{2}\left(\ln|x-1|\right)_0^{-1} =$$

$$= \frac{3}{2}\ln\frac{4}{3} + \frac{1}{2}\ln 2.$$

22. (a) $\int_0^1 \frac{x}{x+1}\,dx = \int_0^1 \frac{(x+1)\cdot 1 - 1}{x+1}\,dx$

$$\begin{array}{r|l} x & x+1 \\ \hline -(x+1) & 1 \\ \hline -1 & \end{array}$$

$\boxed{x = (x+1)\cdot 1 - 1}$

$= \int_0^1 \frac{x+1}{x+1} - \frac{1}{x+1}\,dx$

$= \int_0^1 1\,dx - \int_0^1 \frac{1}{x+1}\,dx$

$= (x)_0^1 - (\ln|x+1|)_0^1$

$= 1 - (\ln 2)$

(B) $\int_0^1 \frac{3x+2}{2x+1}\,dx = \int_0^1 \frac{\frac{3}{2}(2x+1) + \frac{1}{2}}{2x+1}\,dx$

$$\begin{array}{r|l} 3x+2 & 2x+1 \\ \hline -\left(3x+\frac{3}{2}\right) & \frac{3}{2} \\ \hline \frac{1}{2} & \end{array}$$

$= \int_0^1 \frac{\frac{3}{2}\cancel{(2x+1)}}{\cancel{2x+1}} + \frac{\frac{1}{2}}{2x+1}\,dx$

$= \int_0^1 \frac{3}{2}\,dx + \frac{1}{2}\int_0^1 \frac{1}{2x+1}\,dx$

$= \frac{3}{2}(x)_0^1 + \frac{1}{2}\frac{1}{2}(\ln|2x+1|)_0^1 = \frac{3}{2} + \frac{1}{4}\ln 3.$

23. (a) $\displaystyle\int_2^3 \frac{x^2+1}{x^2-x}\,dx = \int_2^3 \frac{x^2-x+x+1}{x^2-x}\,dx$

$$\begin{array}{r|l} x^2+1 & x^2-x \\ \cline{2-2} -(x^2-x) & 1 \\ \hline x+1 & \end{array}$$

$x^2+1 = (x^2-x)\cdot 1 + x+1$

$$= \int_2^3 \frac{x^2-x}{x^2-x}\,dx + \int_2^3 \frac{x+1}{x^2-x}\,dx$$

$$= \int_2^3 1\,dx + \int_2^3 \frac{x+1}{x^2-x}\,dx$$

$$= (x)_2^3 + \underline{I}$$

$$= 1 + \underline{I}.$$

$$\underline{I} = \int_2^3 \frac{x+1}{x^2-x}\,dx \xrightarrow{\quad A,\,B \quad} \text{ΚΤΔ},$$

(B) $\int_0^1 \frac{x^3}{x^2-4}\,dx = \int_0^1 \frac{(x^2-4)\cdot x + 4x}{x^2-4}\,dx$

$$\begin{array}{l|l} x^3 & x^2-4 \\ \hline -(x^3-4x) & x \\ \hline 4x & \end{array}$$

$\boxed{x^3 = (x^2-4)x + 4x}$

$= \int_0^1 \frac{\cancel{(x^2-4)}x}{\cancel{x^2-4}}\,dx + \int_0^1 \frac{4x}{x^2-4}\,dx$

$= \int_0^1 x\,dx + 2\int_0^1 \frac{2x}{x^2-4}\,dx$

$= \frac{1}{2}\left(x^2\right)_0^1 + 2\left(\ln|x^2-4|\right)_0^1$

$= \frac{1}{2} + 2\left(\ln 3 - \ln 4\right)$

24. ⓐ $\int_0^1 x^2 \ln(x+1)\,dx =$

$$= \int_0^1 \left(\frac{x^3}{3}\right)' \ln(x+1)\,dx$$

$$= \left(\frac{x^3}{3}\ln(x+1)\right)_0^1 - \int_0^1 \frac{x^3}{3}\,\frac{1}{x+1}\,dx$$

$$= \frac{1}{3}(\ln 2) - \frac{1}{3}\int_0^1 \frac{x^3}{x+1}\,dx$$

$$\begin{array}{l|l} x^3 & x+1 \\ \hline -(x^3+x^2) & x^2-x+1 \\ \hline -x^2 & \\ -(-x^2-x) & \\ \hline x & \\ -(x+1) & \\ \hline -1 & \end{array}$$

$$= \frac{1}{3}\ln 2 - \frac{1}{3}\int_0^1 \frac{(x+1)(x^2-x+1)-1}{x+1}\,dx$$

$$= \frac{1}{3}\ln 2 - \frac{1}{3}\int_0^1 x^2-x+1 - \frac{1}{x+1}\,dx$$

$$= \frac{1}{3}\ln 2 - \frac{1}{3}\left(\frac{1}{3}(x^3)_0^1 - \frac{1}{2}(x^2)_0^1 + (x)_0^1 - (\ln(x+1))_0^1\right)$$

$$= \frac{1}{3}\ln 2 - \frac{1}{3}\left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2\right) = \frac{2}{3}\ln 2 + \text{κτλ.}$$

(B) $\int_0^1 \ln(4-x^2)\,dx = \int_0^1 (x)' \ln(4-x^2)\,dx$

$= \left(x \ln(4-x^2)\right)_0^1 - \int_0^1 x \cdot \frac{-2x}{4-x^2}\,dx$

$= \ln 3 + 2\int_0^1 \frac{x^2}{4-x^2}\,dx$

$$\begin{array}{r|l} x^2 & 4-x^2 \\ \hline -(x^2-4) & -1 \\ \hline 4 & \end{array}$$

$= \ln 3 + 2\int_0^1 \frac{-(4-x^2)+4}{4-x^2}\,dx$

$= \ln 3 + 2\int_0^1 -1\,dx + 8\int_0^1 \frac{1}{4-x^2}\,dx$

$= \ln 3 - 2 + 32\int_0^1 \frac{1}{4-x^2}\,dx$

$\longrightarrow \int_0^1 \frac{1}{4-x^2}\,dx \underline{\quad A, B \quad}$ κτλ.

5. $f(x) = \ln \frac{1+e^x}{2}$

ⓐ $f'(x) = \frac{1}{\frac{1+e^x}{2}} \cdot \left(\frac{1+e^x}{2}\right)' = \frac{\cancel{2}}{1+e^x} \cdot \frac{1}{\cancel{2}} e^x$

$f'(x) = \frac{e^x}{1+e^x} > 0 \quad f \nearrow$

$f''(x) = \frac{e^x(1+e^x) - e^x e^x}{(1+e^x)^2} = \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2}$

$f''(x) = \frac{e^x}{(1+e^x)^2} > 0 \quad f$ κυρτή.

ⓑ $y - f(0) = f'(0)(x-0)$

$y - 0 = \frac{1}{2}x \qquad y = \frac{1}{2}x$

ⓒ i) $\int_0^1 f(x)\,dx > \frac{1}{4}$

Αφού f κυρτή $\quad f(x) \geq \frac{1}{2}x$

$\int_0^1 f(x)\,dx > \int_0^1 \frac{1}{2}x\,dx \Rightarrow \int_0^1 f(x)\,dx > \frac{1}{4}$.

11). $\int_1^2 \sqrt{x}\, f(x)\, dx > \frac{1}{5}(4\sqrt{2}-1)$

$$f(x) \geq \frac{1}{2} x$$

$$\sqrt{x}\, f(x) \geq \frac{1}{2} x \sqrt{x}$$

$$\int_1^2 \sqrt{x}\, f(x)\, dx > \int_1^2 \frac{1}{2} x \sqrt{x}\, dx$$

$$\rightarrow \int_1^2 \frac{1}{2} x \sqrt{x}\, dx \xrightarrow[dx = 2t\,dt]{\substack{\sqrt{x} = t \\ x = t^2}} \frac{1}{2} \int_1^{\sqrt{2}} t^2\, t\; 2t\, dt$$

$$= \int_1^{\sqrt{2}} t^4\, dt = \frac{1}{5} \left(t^5\right)\Big|_1^{\sqrt{2}} = \frac{1}{5}\left(\sqrt{2}^5 - 1\right)$$

$$= \frac{1}{5}\left(\sqrt{2}^2 \sqrt{2}^2 \sqrt{2} - 1\right) =$$

$$= \frac{1}{5}\left(4\sqrt{2} - 1\right)$$

8. $f \uparrow$

$f(2)=2$

(δ) Ν.δ.ο $\quad 0<\int_0^1 x f(x^2+1)\,dx<1 \quad f(1)=0.$

$x^2+1=t$

$2x\,dx=dt$

$x\,dx=\frac{1}{2}dt$

$$0<\frac{1}{2}\int_1^2 f(t)\,dt<1$$

$$0<\int_1^2 f(t)\,dt<2$$

αρκει ν.δ.ο.

$1<x<2$

$f \uparrow$

$f(1)<f(x)<f(2)$

$0<f(x)<2$

$\int_1^2 0\,dx<\int_1^2 f(x)\,dx<\int_1^2 2\,dx$

$0<\int_1^2 f(x)\,dx<2$ ✓

(γ) $\int_1^2 \frac{f(x)}{x^2}dx < -1$ $f(2) = -2$

$1 < x < 2$

$f \uparrow$

$f(1) < f(x) < f(2)$

$f(1) < f(x) < -2$

$$\frac{f(x)}{x^2} < \frac{-2}{x^2}$$

$$\int_1^2 \frac{f(x)}{x^2}dx < \int_1^2 \frac{-2}{x^2}dx$$

$$\int_1^2 \frac{f(x)}{x^2}dx < +2\left(\frac{1}{x}\right)_1^2$$

$$\int_1^2 \frac{f(x)}{x^2}dx < -1$$ ✓

3. $f(x)=\dfrac{x}{\ln x},\ x>1$

Ενότητα 37

(α) $f'(x)=\dfrac{\ln x-1}{\ln^2 x}$

$f'(x)=0 \Rightarrow \ln x-1=0$

$\ln x=1$

$x=e$

x	1		e	
f'		$-$	0	$+$
f		$\searrow$		$\nearrow$

$f(x)\geq f(e)$

$f(x)\geq e$

(β) Νδο $\int_2^3 f(x)\,dx>e$

$f(x)\geq e$

$\int_2^3 f(x)\,dx>\int_2^3 e\,dx$

$\int_2^3 f(x)\,dx> e\,(x)_2^3$

$\int_2^3 f(x)\,dx>e$ ✓

6. $f(x) = \sqrt{x^2+1}$

(a) $\lim_{x\to+\infty}(f(x) - x) = \lim_{x\to+\infty}\left(\sqrt{x^2+1} - x\right) =$

$= \lim_{x\to+\infty} \dfrac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x} =$

$= \lim_{x\to+\infty} \dfrac{1}{\sqrt{x^2+1} + x} = 0$ ✓

Αρκεί $x < \sqrt{x^2+1}$

Αν $x \geq 0 \Rightarrow x^2 < \cancel{\sqrt{x^2+1}}$

$\cancel{x^2} < \cancel{x^2} + 1$

$0 < 1$ ✓

Αν $x < 0$ ✓

Άρα $\sqrt{x^2+1} > x \quad \forall x \in \mathbb{R}$.

(B) $\int_0^1 f(x)\,dx > \frac{1}{2}$

$f(x) > x$

$\int_0^1 f(x)\,dx > \int_0^1 x\,dx$

$\int_0^1 f(x)\,dx > \frac{1}{2}(x^2)\Big|_0^1$

$\int_0^1 f(x)\,dx > \frac{1}{2}$

(Γ) $\int_1^2 \frac{f(x)}{\sqrt{x}}\,dx > \frac{2}{3}(2\sqrt{2}-1)$.

$f(x) > x$

$\frac{f(x)}{\sqrt{x}} > \frac{x}{\sqrt{x}}$

$\int_1^2 \frac{f(x)}{\sqrt{x}}\,dx > \int_1^2 \frac{x}{\sqrt{x}}\,dx$

$$\to \int_1^2 \frac{x}{\sqrt{x}}\,dx \quad \xrightarrow[\substack{x=t^2 \\ dx=2t\,dt}]{\sqrt{x}=t}$$

$$= \int_1^{\sqrt{2}} \frac{t^{\cancel{2}}}{\cancel{t}}\, 2t\,dt = 2\int_1^{\sqrt{2}} t^2\,dt$$

$$= \frac{2}{3}\left(t^3\right)_1^{\sqrt{2}} = \frac{2}{3}\left(\sqrt{2}^3 - 1\right) =$$

$$= \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

7. $f(x) = e^{x^2}$

(a) Νδο $f(x) \geq x^2+1$

$e^{x^2} \geq x^2+1$

• $e^x \geq x+1 \implies e^{x^2} \geq x^2+1$ ✓

(β) $\int_0^2 f(x)\,dx > \frac{14}{3}$

$f(x) \geq x^2+1$

$\int_0^2 f(x)\,dx > \int_0^2 x^2+1\,dx$

$\int_0^2 f(x)\,dx > \frac{1}{3}(x^3)_0^2 + (x)_0^2$

$\int_0^2 f(x)\,dx > \frac{8}{3} + 2$

$\int_0^2 f(x)\,dx > \frac{14}{3}$

Ενοτητα 36

2. (α) $E: C_f,\ x'x,\ x=0,\ x=1$

$$E=\int_0^1 |f(x)|\,dx$$

$$E=\int_0^1 |\overset{\oplus}{x^2+1}|\,dx=\int_0^1 x^2+1\,dx$$

$$E=\frac{1}{3}\left(x^3\right)_0^1+\left(x\right)_0^1$$

$$E=\frac{1}{3}+1=\frac{4}{3},$$

(β) $E: C_f,\ x'x,\ y'y,\ x=1$

$$E=\int_0^1 |f(x)|\,dx=\int_0^1 |\overset{\ominus}{-\sqrt{x}}|\,dx$$

$$E=\int_0^1 \sqrt{x}\,dx \ \underset{dx=2t\,dt}{\overset{\sqrt{x}=t,\ x=t^2}{=\!=\!=\!=}}\ \int_0^1 t\,2t\,dt$$

$$E=2\int_0^1 t^2\,dt=\frac{2}{3}\left(t^3\right)_0^1=\frac{2}{3}.$$

4. ⓐ $\mathcal{E}: (f, x'x, x=0, x=2$

$$\mathcal{E} = \int_0^2 |f(x)|\,dx = \int_0^2 \overset{\ominus}{|x^2-3x|}\,dx$$

x		0		3	
x^2-3x	+		−		+

$$= \int_0^2 -x^2+3x\,dx =$$

$$= -\frac{1}{3}(x^3)_0^2 + \frac{3}{2}(x^2)_0^2$$

$$= -\frac{8}{3} + 6$$

ⓑ $\mathcal{E}: (f, x'x, x=-1, x=0.$

$$\mathcal{E} = \int_{-1}^0 |f(x)|\,dx = \int_{-1}^0 \overset{\oplus}{|x^2-3x|}\,dx$$

$$= \frac{1}{3}(x^3)_{-1}^0 - \frac{3}{2}(x^2)_{-1}^0$$

$$= \frac{1}{3}\,1 + \frac{3}{2} \qquad = \frac{1}{3} + \frac{3}{2}.$$

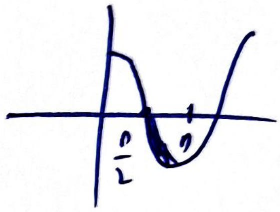
3. α) $E : C_f, x'x, \; x=1, \; x=2$

$$E = \int_1^2 |f(x)|\,dx = \int_1^2 \left|x+\frac{1}{x}\right|^{\oplus} dx$$

$$E = \int_1^2 x+\frac{1}{x}\,dx = \frac{1}{2}\left(x^2\right)_1^2 + \left(\ln x\right)_1^2$$

$$E = \frac{3}{2} + \ln 2 .$$

β) $E : C_f, x'x, \; x=\frac{\pi}{2}, \; x=\pi$



$$E = \int_{\frac{\pi}{2}}^{\pi} |f(x)|\,dx = \int_{\frac{\pi}{2}}^{\pi} |\sigma\upsilon\nu x|^{\ominus}\,dx$$

$$= -\int_{\frac{\pi}{2}}^{\pi} \sigma\upsilon\nu x\,dx = -\left(\eta\mu x\right)_{\frac{\pi}{2}}^{\pi} =$$

$$= -(0-1) = 1 .$$

7. B $E: C_f, x'x, x=2$

$f(x)=0 \Rightarrow \frac{1}{x}-e^{x-1}=0$

$f(x)=f(1)$

f 1-1

$x=1$

$f'(x)=-\frac{1}{x^2}-e^{x-1}<0$

$f\downarrow$

$E=\int_1^2 |f(x)|\,dx = \int_1^2 \left|\frac{1}{x}-e^{x-1}\right| dx$

$-$

$1<x<2$

$f\downarrow$

$f(1)>f(x)>f(2)$

$0>f(x)$

$= \int_1^2 -\frac{1}{x}+e^{x-1}\,dx = -\left(\ln|x|\right)_1^2+\left(e^{x-1}\right)_1^2$

$= -\ln 2 + e - 1$

6. $f(x) = \frac{x}{x^2+1}$

$E: (C_f, x'x, \underline{\underline{x=1}}$

$\longrightarrow f(x)=0 \Rightarrow \frac{x}{x^2+1}=0 \Rightarrow \underline{\underline{x=0}}$

$$E = \int_0^1 |f(x)|\,dx = \int_0^1 \left|\overset{+}{\frac{x}{x^2+1}}\right| dx$$

$$E = \int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx =$$

$$= \frac{1}{2}\left(\ln(x^2+1)\right)_0^1 =$$

$$= \frac{1}{2}\ln 2 .$$

8. $f(x) = e^x - \ln(x+1)$.

(a) ΝΔΟ $f(x) \geq 1 \quad \forall x > -1$.

$f'(x) = e^x - \dfrac{1}{x+1} \qquad f'(0) = 0$

$f''(x) = e^x + \dfrac{1}{(x+1)^2} > 0$

x	-1	0	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

$\underline{\underline{f(x) \geq 1}}$

$E : C_g \,,\ x'x \,,\ x = 1 .$ $\qquad g(x) = 2x(f(x)-1)$

$$E = \int_0^1 |g(x)|\,dx = \int_0^1 |2x(f(x)-1)|\,dx \ \underline{\ \circledast\ }$$

$\oplus \quad \oplus$

$g(x) = 0 \Rightarrow 2x(f(x)-1) = 0$

$2x = 0$ ή $f(x) - 1 = 0$

$x = 0$ $\qquad f(x) = 1$

$\qquad x = 0$.

$$= \int_0^1 2x(e^x - \ln(x+1) - 1)\,dx$$

$$= \int_0^1 2xe^x\,dx - 2x\ln(x+1)\,dx - 2x\,dx$$

$$= \underbrace{\int_0^1 2xe^x\,dx}_{I} - \underbrace{\int_0^1 2x\ln(x+1)\,dx}_{J} - \underbrace{\int_0^1 2x\,dx}_{K}.$$

$$I = \int_0^1 2x e^x \, dx = \int_0^1 2x (e^x)' \, dx$$

$$= \left(2x e^x\right)_0^1 - \int_0^1 2 \cdot e^x \, dx$$

$$= 2e - 2 \left(e^x\right)_0^1 = 2e - 2(e-1)$$

$$= 2.$$

$$J = \int_0^1 2x \ln(x+1) \, dx = \int_0^1 (x^2)' \ln(x+1) \, dx$$

$$= \left(x^2 \ln(x+1)\right)_0^1 - \int_0^1 x^2 \frac{1}{x+1} \, dx$$

$$= \ln 2 - \int_0^1 \frac{x^2}{x+1} \, dx = \ln 2 - \int_0^1 \frac{\cancel{(x+1)}(1-x)+1}{\cancel{x+1}}$$

$$\begin{array}{r|l} x^2 & x+1 \\ \cline{2-2} -(x^2+x) & x - x \\ \hline -x & \\ -(-x-1) & \\ \hline 1 & \end{array}$$

$$= \ln 2 - \int_0^1 1 - x \, dx - \int_0^1 \frac{1}{x+1} \, dx$$

$$= \ln 2 - (x)_0^1 + \frac{1}{2}(x^2)_0^1 - \left(\ln(x+1)\right)_0^1$$

$$= \cancel{\ln 2} - 1 + \frac{1}{2} - \cancel{\ln 2} = -\frac{1}{2}.$$

Άρα $E = 2 - \left(-\frac{1}{2}\right) + 1$

$E = 3 + \frac{1}{2} = \frac{7}{2}.$

Για Τριτη 3/3

Θεματα

Σελ 422.

Μπαρλα.

1 2 3 4

5 6 7 8

9 10