Ενότητα 24

1. $f(x) = x^7 + 5x - 6$

ⓐ $f'(x) = 7x^6 + 5 > 0$

$f \nearrow$

ⓑ $\pi > 3$

$f \nearrow$

$f(\pi) > f(3)$

ⓒ i) $x > 1 \Rightarrow f(x) > f(1) \Rightarrow f(x) > 0$.

ii). Αν $\lambda > \mu \Rightarrow f(\lambda) > f(\mu)$

$\lambda^7 + 5\lambda \cancel{-6} > \mu^7 + 5\mu \cancel{-6}$

$\lambda^7 - \mu^7 > 5\mu - 5\lambda$

$\lambda^7 - \mu^7 > 5(\mu - \lambda)$

$\lambda^7 - \mu^7 > -5(\lambda - \mu) \Rightarrow \frac{\lambda^7 - \mu^7}{\lambda - \mu} > -5.$

(δ) $\lim\limits_{x \to 1^+} \dfrac{1}{f(x)} = +\infty$

x		1	
$f(x)$	$\nearrow$	0	$\nearrow$

$x > 1$

$f(x) > f(1)$

$f(x) > 0$

2. $f(x) = x^7+2x-3$

(a) $x^7+2x=3$

$f'(x) = 7x^6+2>0$

$f \nearrow$

$\Rightarrow f : 1-1$.

$x^7+2x-3=0$

$f(x)=0$

$f(x)=f(1)$

$f : 1-1$

$x=1$

(B) i) $x < \dfrac{3}{x^6+2}$

$x(x^6+2) < 3$

$x^7+2x-3<0$

$f(x)<0$

$f(x)<f(1)$

$f \nearrow$

$x<1$.

11). $e^{7x}+2e^{x}>3$

$$e^{7x}+2e^{x}-3>0$$

$$f(e^{x})>0$$

$$f(e^{x})>f(1)$$

$$f\nearrow$$

$$e^{x}>1$$

$$x>0$$

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8 i) $f(e^{x}-1)-2x=x^{7}-3$

$$f(e^{x}-1)=x^{7}+2x-3$$

$$f(e^{x}-1)=f(x)$$

f 1-1

$$e^{x}-1=x$$

$$e^{x}-x-1=0$$

$$x=0$$

• $e^{x}\geq x+1$

$e^{x}-x-1\geq 0$

Το "=" $x=0$

11). $f(x^7+x-3) + x^7 = -2x-3$

$f(x^7+x-3) + x^7+2x-3 = -6$

$$\underbrace{f(x^7+x-3) + f(x) + 6}_{g(x)} = 0.$$

$g(x) = 0$

$g(x) = 0$

$g \nearrow$

$g(x) = g(1)$

$g \text{ 1-1}$

$x = 1$

$g(1) = f(-1) + f(1) + 6$

$g(1) = -\cancel{6} + 0 + \cancel{6} = 0$

(δ) $\lim\limits_{x \to 1} \dfrac{1}{f(x)}$

x		1	
$f(x)$	$\nearrow$ $-$		$+$ $\nearrow$

$$\lim_{x \to 1^-} \frac{1}{f(x)} = -\infty$$

$$\lim_{x \to 1^+} \frac{1}{f(x)} = +\infty$$

Το οριο δεν υπαρχει

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4. $f(x) = x^2 + \ln x - 1 \quad , x > 0$

ⓐ Παρατηρω οτι $f(1) = 0$.

$f'(x) = 2x + \frac{1}{x} > 0$

$f \nearrow$

x		1	
$f(x)$	$-$	0	$+$

$x < 1 \Rightarrow f(x) < f(1) \Rightarrow f(x) < 0$

$x > 1 \Rightarrow f(x) > f(1)$
$\Rightarrow f(x) > 0$

ⓑ. $(x^2+1)^2 + \ln(x^2+1) = 1$

$(x^2+1)^2 + \ln(x^2+1) - 1 = 0$

$f(x^2+1) = f(1)$

f 1-1

$x^2+1 = 1$

$x^2 = 0$

$x = 0$

① $xe^{x^2} > e \qquad (0, +\infty)$

$\ln xe^{x^2} > \ln e$

$\ln x + \ln e^{x^2} > \ln e.$

$\ln x + x^2 > 1.$

$\underbrace{\ln x + x^2 - 1}_{f(x)} > 0$

$f(x) > f(1)$

$f \nearrow$

$x > 1.$

(5) $\ln \dfrac{x^2+1}{x+1} > (x+1)^2 - (x^2+1)^2$

$$\ln(x^2+1) - \ln(x+1) > (x+1)^2 - (x^2+1)^2$$

$$\ln(x^2+1) + (x^2+1)^2 - 1 > (x+1)^2 + \ln(x+1) - 1$$

$$f(x^2+1) > f(x+1).$$

$f \nearrow$

$\underline{\underline{x+1>0}}$

$\underline{\underline{x>-1}}$

$$x^2 + \cancel{1} > x + \cancel{1}$$

$$x^2 - x > 0$$

x		0		1	
x^2-x	+	|	−	|	+

$$x \in (-1, 0) \cup (1, +\infty).$$

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$$f(x) = \frac{x^3}{x^2+1}$$

(a) $$f'(x) = \frac{3x^2(x^2+1) - x^3\,2x}{(x^2+1)^2}$$

$$f'(x) = \frac{3x^4+3x^2-2x^4}{(x^2+1)^2}$$

$$f'(x) = \frac{x^4+3x^2}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2} > 0 \quad f \nearrow$$

(B) i) $f(5x^3) > f(8x^2+8)$.

$f \nearrow$

$$5x^3 > 8x^2+8$$

$$5x^3 > 8(x^2+1)$$

$$\frac{5x^3}{x^2+1} > 8$$

$$\frac{x^3}{x^2+1} > \frac{8}{5} \implies f(x) > f(2)$$

$$\underline{\underline{x > 2}}$$

11). $(x^2+1)f(e^x-1)-x^3=0$

$$(x^2+1)f(e^x-1)=x^3$$

$$f(e^x-1)=\frac{x^3}{x^2+1}$$

$$f(e^x-1)=f(x)$$

f : 1-1

$$e^x-1=x$$

$$e^x-x-1=0$$

$$x=0$$

- $e^x\geq x+1$

$$e^x-x-1\geq 0$$

Το "=" για $x=0$.

(1) $$\lim_{x \to 0} \frac{\ln x}{f(2x+1) - f(x+1)}$$

$$= \lim_{x \to 0^+} \ln x \quad \frac{1}{\underbrace{f(2x+1) - f(x+1)}_{+}}$$

$$= -\infty \cdot (+\infty) = -\infty$$

- $2x > x$

$2x+1 > x+1$

$f \nearrow$

$f(2x+1) > f(x+1)$

$f(2x+1) - f(x+1) > 0.$

11. (a) $x\ln x = 1 - x^3$, $x > 0$.

$$\underbrace{x\ln x - 1 + x^3}_{f(x)} = 0 \quad \Rightarrow \quad f(x) = 0$$

$\underline{\underline{x = 1}}$

$$f'(x) = \ln x + x \cdot \frac{1}{x} + 3x^2$$

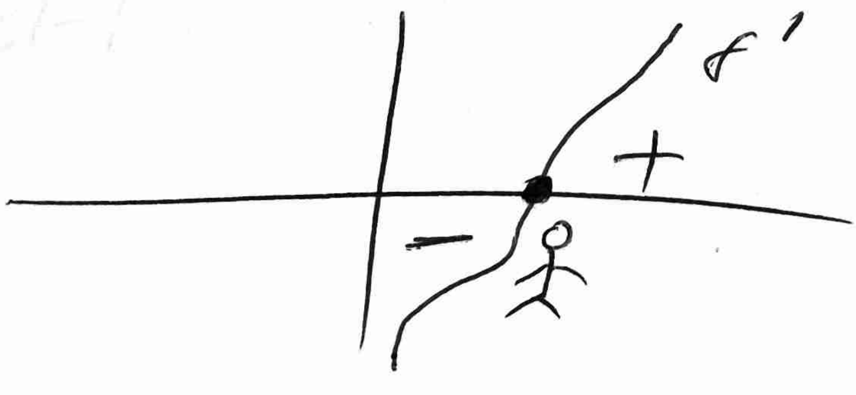
$$f'(x) = \ln x + 1 + 3x^2$$

$$f''(x) = \frac{1}{x} + 6x > 0$$

x	0	1	
f''	$+$		$+$
f'	$-\infty \nearrow$		$\nearrow +\infty$
f			

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \ln x + 1 + 3x^2 = -\infty$$

$$\lim_{x\to +\infty} f'(x) = +\infty$$



Εναλλακτικά

$$x \ln x = 1 - x^3 \quad , \; x > 0$$

$$\ln x = \frac{1}{x} - x^2$$

$$\underbrace{\ln x - \frac{1}{x} + x^2}_{f(x)} = 0 \Rightarrow f(x) = 0$$

$$f(x) = f(1)$$

f 1-1

$$x = 1$$

$$f'(x) = \frac{1}{x} + \frac{1}{x^2} + 2x > 0$$

$f \nearrow$

f 1-1

(13) $e^{x^3-1}\cdot x^x>1$ στο $(0,+\infty)$.

$$\ln e^{x^3-1}\cdot x^x>\ln 1$$

$$\ln e^{x^3-1}+\ln x^x>0$$

$$x^3-1+x\ln x>0$$

$$x^2-\frac{1}{x}+\ln x>0$$

$$f(x)>0$$

$$f(x)>f(1)$$

$$f\nearrow$$

$$x>1$$

14. (B) $f(x) = x^2 + 7 + \ln x$

$g(x) = 8x - 7\ln x$

Έστω $f(x) < g(x)$

$x^2 + 7 + \ln x < 8x - 7\ln x$

$\underbrace{x^2 - 8x + 8\ln x + 7}_{h(x)} < 0$

$h(x) < 0$

$h(x) < h(1)$

$h \nearrow$

$x < 1$

$h'(x) = 2x - 8 + \frac{8}{x}$

$h'(x) = \frac{2x^2 - 8x + 8}{x}$

$h'(x) = 2\,\frac{x^2 - 4x + 4}{x}$

$h'(x) = 2\,\frac{(x-2)^2}{x} \geq 0,$

$h \nearrow$

Συμπέρασμα

Αν $x < 1$ $f(x) < g(x)$

Αν $x > 1$ $f(x) > g(x)$

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14.

$f(x) = \dfrac{e^x - 1}{x}$ $\qquad g(x) = \dfrac{x}{2}$

Σ-οτω $\quad f(x) < g(x)$ $\qquad \underline{\underline{x \neq 0}}$

$$\frac{e^x - 1}{x} < \frac{x}{2}$$

$$\frac{e^x - 1}{x} - \frac{x}{2} < 0$$

$$\frac{2e^x - 2 - x^2}{2x} < 0$$

Άτοπο!

$$\boxed{\varphi(x) = 2e^x - 2 - x^2}$$

$\varphi'(x) = 2e^x - 2x$

$\varphi'(x) = 2(e^x - x) > 0$ $\qquad \cdot\, e^x \geq x + 1$

$e^x - x \geq 1$

$e^x - x > 0$

x		0	
φ'	$+$		$+$
φ	$\nearrow -$	0	$\nearrow +$
$2x$	$-$		$+$
$\frac{\varphi(x)}{2x}$	$+$		$+$

$$f(x)-g(x)=\frac{2e^x-2-x^2}{2x}>0$$

$$f(x)-g(x)>0$$

$$f(x)>g(x)$$

15. (B) $\ln(e^x - x) = \sigma\upsilon\nu x \quad x \in (0, \frac{\pi}{2})$

$$\underbrace{\ln(e^x - x) - \sigma\upsilon\nu x}_{f(x)} = 0$$

Η $f(x)$ συνεχής στο $[0, \frac{\pi}{2}]$ ως π.σ.σ.

$f(0) = -1$

$f(\frac{\pi}{2}) = \ln(e^{\frac{\pi}{2}} - \frac{\pi}{2}) > 0$

$\left.\right\} f(0) f(\frac{\pi}{2}) < 0$ Bolzano

$\exists \, \xi \in (0, \frac{\pi}{2})$

τ.ω $f(\xi) = 0$

$e^x \geq x + 1$

$e^{\frac{\pi}{2}} \geq \frac{\pi}{2} + 1$

$e^{\frac{\pi}{2}} - \frac{\pi}{2} \geq 1$

$\ln(e^{\frac{\pi}{2}} - \frac{\pi}{2}) \geq \ln 1 \qquad \ln(e^{\frac{\pi}{2}} - \frac{\pi}{2}) \geq 0.$

$$f(x) = \ln(e^x - x) - \sigma\upsilon\nu x$$

$$f'(x) = \frac{\overset{+}{e^x - 1}}{\underset{+}{e^x - x}} + \overset{+}{\eta\mu x}$$

$x \in (0, \frac{\pi}{2})$

$x > 0$

$\Rightarrow e^x > e^0$

$e^x > 1$

$e^x - 1 > 0$

$e^x \geq x + 1$

$e^x - x \geq 1$

$f \nearrow$

Επομενο Μαθημα

24	26
9	2
10 .	3
14 α γ	4
15 α γ	5
	6
	7 .
	28
	29