Ενότητα 27

1. Ⓑ $f(x) = x + \frac{2\ln x}{x}$, $x > 0$

$$f'(x) = 1 + \frac{2 \cdot \frac{1}{x} x - 2\ln x}{x^2}$$

$$f'(x) = 1 + 2\,\frac{1 - \ln x}{x^2}$$

$$\boxed{f'(x) = \frac{x^2 + 2 - 2\ln x}{x^2}}$$

$\varphi(x) = x^2 + 2 - 2\ln x$

$\varphi'(x) = 2x - \frac{2}{x} = \frac{2x^2 - 2}{x} = 2\,\frac{x^2 - 1}{x}$

$\rightarrow \varphi'(x) = 0 \Rightarrow 2\,\frac{x^2 - 1}{x} = 0 \Rightarrow x^2 - 1 = 0$

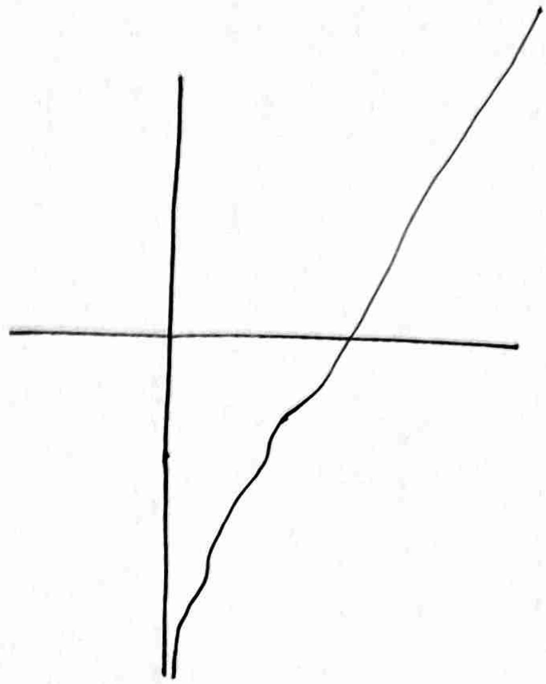
$x = 1$ $\quad$ $x = -1$

x	0	1	
φ'	$-$	0	$+$
φ	$\searrow +$		$+ \nearrow$
x^2	$+$		$+$
f'	$+$		$+$
f	$\nearrow$		$\nearrow$

$\varphi(x) \geq \varphi(1)$

$\varphi(x) \geq 3$

x	0 $\qquad\qquad$ $+\infty$
$f(x)$	$-\infty \nearrow +\infty$



$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x+\frac{2\ln x}{x}\right)$$

$$= \lim_{x\to 0^+}\left(x+2\ln x \; \frac{1}{x}\right) = 0+(-\infty)(+\infty) = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x+\frac{2\ln x}{x}\right) = +\infty$$

$$\to \lim_{x\to+\infty}\frac{2\ln x}{x} = \lim_{x\to+\infty}\frac{2\,\frac{1}{x}}{1} = 0$$

$$\Sigma\, T_f = R.$$

13. (Β) Νδο η εξίσωση $(x-1)\ln x = xe^{-1} - x + 2$

έχει μοναδική ρίζα στο $(2, e)$.

$$\underbrace{(x-1)\ln x - xe^{-1} + x - 2}_{f(x)} = 0$$

$$f(2) = \ln 2 - 2e^{-1} = \ln 2 - \frac{2}{e} < 0$$

$$f(e) = (e-1) - e\cdot e^{-1} + e - 2 =$$

$$= e - 1 - e^{0} + e - 2 =$$

$$= e - 1 - 1 + e - 2 = 2e - 4 = 2(e-2) > 0$$

$f(2)f(e) < 0$

Bolzano.

$\longrightarrow$ $\ln x \le x - 1$

$\ln \frac{2}{e} \le \frac{2}{e} - 1$

$\ln 2 - \cancel{\ln e} \le \frac{2}{e} - \cancel{1}$

$\ln 2 \le \frac{2}{e}$

$$f'(x) = \ln x + \frac{x-1}{x} - \frac{1}{e} + 1$$

$$f'(x) = \ln x + 1 - \frac{1}{x} - \frac{1}{e} + 1$$

$$\boxed{f'(x) = \ln x + 2 - \frac{1}{e} - \frac{1}{x}}$$

$$f''(x) = \frac{1}{x} + \frac{1}{x^2} > 0 \quad \Rightarrow \quad f' \nearrow$$

x	2	e
f''	$+$	
f'	$\nearrow$ $+$	
f	$\nearrow$	

$x > 2$

$f'(x) > f'(2)$

$f'(x) > \ln 2 + 2 - \frac{1}{e} - \frac{1}{2}$

$f'(x) > 0$

Το x_0

μοναδικό.

7. ⓖ $f(x) = \begin{cases} -x^2+1, & x<0 \\ e^x - x, & x \geq 0 \end{cases}$

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= 1 \\ \lim_{x\to 0^+} f(x) &= 1 \end{aligned}\right\} \lim_{x\to 0} f(x) = 1 \quad \text{και } f(0)=1$$

Συνεχής στο 0.

$\underline{x<0}$

$f_1(x) = -x^2+1$

$f_1'(x) = -2x > 0$

$\underline{x \geq 0}$

$f_2(x) = e^x - x$

$f_2'(x) = e^x - 1$

$\to e^x - 1 = 0$

$e^x = 1$

$x = 0$

x	$-\infty$	0	$+\infty$
f_1'	$+$		//////
f_2'	//////	0	$+$
f'	$+$		$+$
f	$-\infty \nearrow$		$\nearrow +\infty$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} -x^2+1 = -\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (e^x - x) = \lim_{x \to +\infty} x\left(\frac{e^x}{x} - 1\right) = +\infty.$$

$$\to \lim_{x \to +\infty} \frac{e^x}{x} = \lim_{x \to +\infty} \frac{e^x}{1} = +\infty$$

$$\sum T_f = \mathbb{R}.$$

4. $f(x) = x - \frac{1}{2}\ln^2 x \quad , x>0$

ⓐ $f'(x) = 1 - \frac{1}{2} 2\ln x \frac{1}{x}$

$f'(x) = 1 - \frac{\ln x}{x}$

$\cdot \ln x \leq x-1$

$f'(x) = \frac{\overset{+}{x - \ln x}}{x} > 0 \quad f \nearrow$

$1 \leq x - \ln x$

$0 < x - \ln x$

ⓑ $\cdot$ Αφού $f \nearrow \Rightarrow f$ 1-1

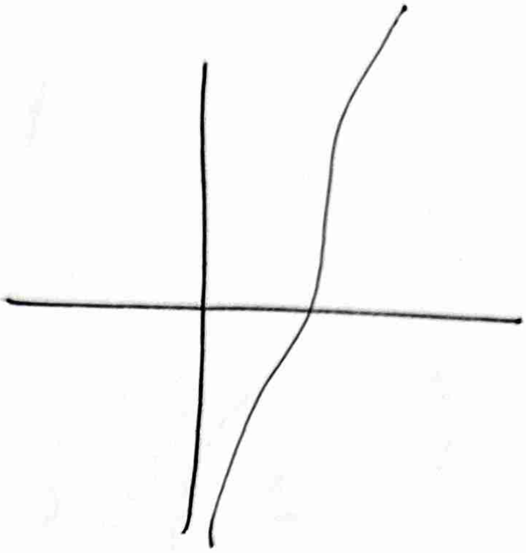
$D_{f^{-1}} = \Sigma T_f$.

Το πεδίο ορισμού της αντίστροφης

ΕΙΝΑΙ το σύνολο τιμών

της $f(x)$.

x	0 $+\infty$
$f(x)$	$-\infty$ $\nearrow$ $+\infty$



$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x-\frac{1}{2}\ln^2 x\right) =$$

$$= \lim_{x\to+\infty} x\left(1-\frac{1}{2}\overbrace{\cancel{\frac{\ln^2 x}{x}}}^{0}\right) = +\infty.$$

$$\rightarrow \lim_{x\to+\infty}\frac{\ln^2 x}{x} = \lim_{x\to+\infty}\frac{2\ln x\,\frac{1}{x}}{1} = \lim_{x\to+\infty}\frac{2\ln x}{x}$$

$$= \lim_{x\to+\infty}\frac{2\frac{1}{x}}{1} = 0$$

$$\lim_{x\to0^+} f(x) = -\infty$$

$$\Sigma T_f = R.$$

$$D_{f^{-1}} = R.$$

(γ) $f^{-1}\left(f(x) - e + \frac{3}{2}\right) > 1$

$f \nearrow$

$f\left(f^{-1}\left(f(x) - e + \frac{3}{2}\right)\right) > f(1)$

$f(x) - e + \frac{3}{2} > 1$

$f(x) > 1 - \frac{3}{2} + e$

$f(x) > e - \frac{1}{2}$

$f(x) > f(e)$

$f \nearrow$

$x > e$.

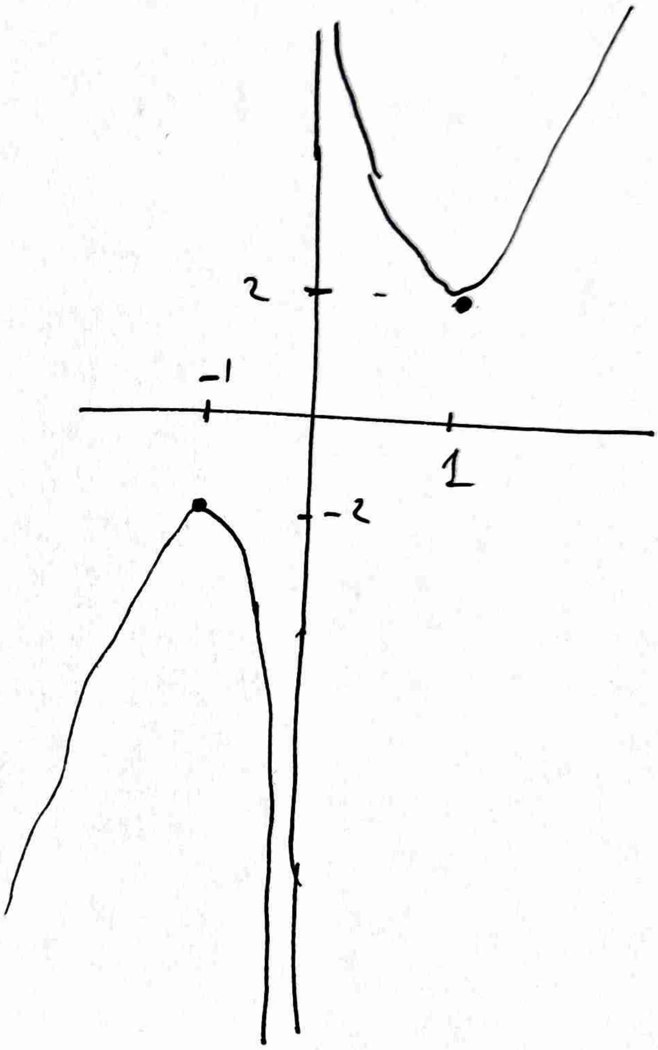
1. (δ) $f(x) = x + \frac{1}{x}, \; x \neq 0$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

$\longrightarrow f'(x) = 0 \Rightarrow \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0$

$x = 1$ $\quad$ $x = -1$

x	$-\infty$		-1		0		1		
f'		$+$	0	$-$	‖	$-$	0	$+$	
f	$-\infty$	$\nearrow$	-2	$\searrow$ $-\infty$	‖	$+\infty$ $\searrow$	2	$\nearrow$	$+\infty$



$\lim_{x \to -\infty} f(x) = -\infty$

$f(-1) = -2$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = +\infty$

$f(1) = 2$

$\lim_{x \to +\infty} f(x) = +\infty$

$\Sigma T_f = (-\infty, -2] \cup [2, +\infty)$

Αποδειξας

Διαγωνισματι

3.5 Νδο αν η f παρ/μη στο x_0 τοτε ειναι και συνεχης στο x_0

Αποδειξη

$$f(x) - f(x_0) = f(x) - f(x_0)$$

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)$$

$$\lim_{x \to x_0}[f(x) - f(x_0)] = \lim_{x \to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)\right)$$

$$\lim_{x \to x_0}[f(x) - f(x_0)] = f'(x_0) \cdot 0$$

$$\lim_{x \to x_0}[f(x) - f(x_0)] = 0$$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Συνεχης στο x_0

3.6 Ndo $(c)' = 0$

Απόδειξη

Έστω $f(x) = c$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c - c}{x - x_0} = 0$$

3.7 Ndo $(x)' = 1$

Απόδειξη

Έστω $f(x) = x$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{x - x_0}}{\cancel{x - x_0}} = 1$$

3.9 Ndo $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Απόδειξη

Έστω $f(x) = \sqrt{x}$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{\sqrt{x}^2 - \sqrt{x_0}^2}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{\cancel{x - x_0}}{(\cancel{x - x_0})(\sqrt{x} + \sqrt{x_0})} = \frac{1}{2\sqrt{x_0}}$$

3.10 Ndo $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

Απόδειξη

Έστω $h(x) = f(x) + g(x)$

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) + g'(x_0)$$

3.16

Έστω f ορισμένη στο Δ

Αν η f συνεχής στο Δ και αν

$f'(x)=0$ $\forall x$ εσωτερικό του Δ.

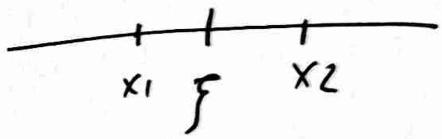
Τότε νδο $f(x)$ σταθερή.

Απόδειξη

Αρκεί νδο $\forall x_1, x_2 \in \Delta$ η $f(x_1)=f(x_2)$

→ $x_1 = x_2 \Rightarrow f(x_1)=f(x_2)$

→ $x_1 < x_2$



$$f'(\xi)=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

όμως $f'(x)=0$ άρα $f'(\xi)=0$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=0 \Rightarrow f(x_2)-f(x_1)=0 \Rightarrow \underline{f(x_1)=f(x_2)}$$

→ $x_1 > x_2$

ομοίως.

3.17

Εστω f, g συνεχεί/ στο Δ.

Αν $f'(x) = g'(x)$ $\forall x$ εσωτερικο του Δ.

Τοτε $f(x) = g(x) + c$.

Αποδειξη

Γνωριζω οτι $f'(x) = g'(x)$

$$f'(x) - g'(x) = 0$$

$$\left[(f-g)(x)\right]' = 0$$

$$(f-g)(x) = c$$

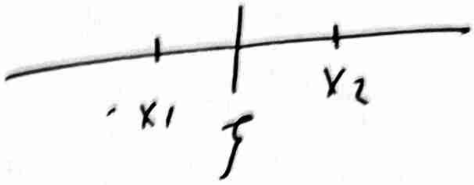
$$f(x) - g(x) = c$$

$$f(x) = g(x) + c.$$

3.18 Αν $f'(x) > 0$ νδο $f \nearrow$

Αποδειξη

Εστω $x_1 < x_2$



$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

ομως $f'(x) > 0$

$$f'(\xi) > 0$$

$$\frac{f(x_2) - f(x_1)}{\underset{+}{x_2 - x_1}} > 0$$

$$\Rightarrow f(x_2) - f(x_1) > 0$$

$$f(x_2) > f(x_1)$$

$$f \nearrow$$