129

Ⓑ $f(x) = \begin{cases} \dfrac{x\ln x}{x-1}, & 0 < x \neq 1 \\ 1, & x = 1 \end{cases}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x\ln x}{x-1} = \lim_{x\to 1} \frac{\ln x + 1}{1} = 1$$

$f(1) = 1$

Συνεχής στο 1!

$$f'(x) = \frac{(x\ln x)'(x-1) - (x\ln x)\cdot(x-1)'}{(x-1)^2}$$

$$f'(x) = \frac{\left(\ln x + x\frac{1}{x}\right)(x-1) - x\ln x}{(x-1)^2}$$

$$f'(x) = \frac{(\ln x + 1)(x-1) - x\ln x}{(x-1)^2}$$

$$f'(x) = \frac{\cancel{x \ln x} - \ln x + x - 1 + \cancel{x \ln x}}{(x-1)^2}$$

$$f'(x) = \frac{-(\ln x \overset{-}{-} x + 1)}{(x-1)^2} \geq 0$$

$f \nearrow$

- $\ln x \leq x - 1$

$\ln x - x + 1 \leq 0$

29.

(a) $f(x) = \begin{cases} \dfrac{xe^x}{e^x - 1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{xe^x}{e^x - 1} = \lim_{x \to 0} \frac{e^x + xe^x}{e^x}$$

$$= \lim_{x \to 0} 1 + x = 1.$$

$f(0) = 1$ Συνεχής στο 0!

$$f'(x) = \frac{(xe^x)'(e^x - 1) - xe^x(e^x - 1)'}{(e^x - 1)^2}$$

$$f'(x) = \frac{(e^x + xe^x)(e^x - 1) - xe^x e^x}{(e^x - 1)^2}$$

$$f'(x) = \frac{e^{2x} - e^x + \cancel{xe^{2x}} - xe^x - \cancel{xe^{2x}}}{(e^x - 1)^2}$$

$$\boxed{f'(x) = \frac{e^x(e^x - 1 - x)}{(e^x - 1)^2} \quad f \nearrow}$$

• $e^x \geq x + 1$

$e^x - x - 1 \geq 0$

28. (δ) $f(x) = \begin{cases} x^2 e^{1-x} + 2, & x<1 \\ x - 2\sqrt{x-1}, & x \geq 1 \end{cases}$

$$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = 3 \\ \lim\limits_{x \to 1^+} f(x) = 1 \end{array}\right\} \text{όχι συνεχής.}$$

$x<1$

$f_1(x) = x^2 e^{1-x} + 2$

$f_1'(x) = 2x e^{1-x} - x^2 e^{1-x}$

$f_1'(x) = e^{1-x}(2x - x^2)$

0   2

$x>1$

$f_2(x) = x - 2\sqrt{x-1}$

$f_2'(x) = 1 - \frac{\cancel{2}}{\cancel{2}\sqrt{x-1}}$

$f_2'(x) = \frac{\sqrt{x-1} - 1}{\sqrt{x-1}}$

$f_2'(x) = 0 \rightarrow x = 2$

x		0		1		2	
f_1'	$-$	0	$+$		$+$	0	$-$
f_2'					$-$	0	$+$
f'	$-$		$+$		$-$		$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

28. (γ) $f(x) = \begin{cases} \ln(e^x - x), & x \leq 0 \\ \ln x - x, & x > 0 \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \ln(e^x - x) = 0$$

ΟΧΙ Συνεχής!

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\ln x - x) = -\infty$$

$\underline{x < 0}$

$f_1(x) = \ln(e^x - x)$

$f_1'(x) = \dfrac{e^x - 1}{e^x - x}$

$f_1'(x) = 0 \Rightarrow \dfrac{e^x - 1}{e^x - x} = 0$

$e^x - 1 = 0$

$e^x = 1$

$x = 0$

$\underline{x > 0}$

$f_2(x) = \ln x - x$

$f_2'(x) = \dfrac{1}{x} - 1$

$f_2'(x) = \dfrac{1 - x}{x}$

1

x		0		1	
f_1'	$-$	0	//////		//////
f_2'	//////		$+$	0	$-$
f'	$-$		$+$		$-$
f	↘		↗		↘

28. (B) $f(x) = \begin{cases} ημx, & -π < x \le 0 \\ x^2 \ln x, & x > 0 \end{cases}$

• $\lim\limits_{x \to 0^-} (ημx) = 0$

• $\lim\limits_{x \to 0} (x^2 \ln x) = \lim\limits_{x \to 0} \dfrac{\ln x}{\frac{1}{x^2}} = \lim\limits_{x \to 0} \dfrac{\frac{1}{x}}{\frac{-2x}{x^4}} =$

$= \lim\limits_{x \to 0} -\dfrac{x^4}{2x^2} = \lim\limits_{x \to 0} -\dfrac{x^2}{2} = 0$

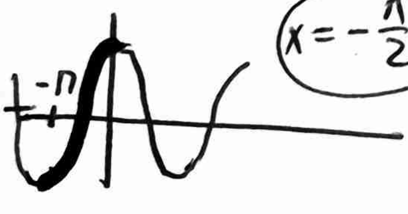
Άρα $\lim\limits_{x \to 0} f(x) = 0$ και $f(0) = 0$.

Συνεχής στο 0!.

$-π < x < 0$

$f_1(x) = ημx$

$f_1'(x) = συνx$



$x = -\frac{π}{2}$

$x > 0$

$f_2(x) = x^2 \ln x$

$f_2'(x) = 2x \ln x + x^2 \frac{1}{x}$

$f_2'(x) = 2x \ln x + x$

$f_2'(x) = x(2\ln x + 1)$

$\rightarrow x = 0$ ή $2\ln x + 1 = 0$

$\ln x = -\frac{1}{2}$

$x = \sqrt{e}/e$

x	$-\pi$		$-\frac{1}{2}$		0		$\frac{\sqrt{e}}{e}$	
f_1'		$-$	0	$+$				
f_2'						$-$	0	$+$
f'		$-$		$+$		$-$		$+$
f		↘		↗		↘		↗

28. ⓐ $f(x) = \begin{cases} (x-2)e^{x-1}+2, & x<1 \\ -x^2+2x, & x \geq 1 \end{cases}$

• $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \left[(x-2)e^{x-1}+2\right] = -1+2 = 1$

• $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \left(-x^2+2x\right) = 1$

Άρα $\lim\limits_{x\to 1} f(x) = 1$ και $f(1) = 1$.

Συνεχής στο 1!

$x<1$

$f_1(x) = (x-2)e^{x-1}+2$

$f_1'(x) = e^{x-1} + (x-2)e^{x-1}$

$f_1'(x) = e^{x-1}(x-1)$ ①

$x>1$

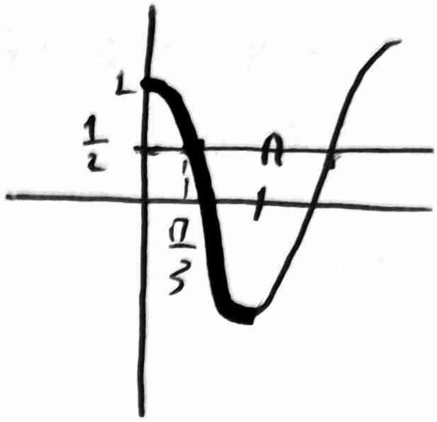
$f_2(x) = -x^2+2x$

$f_2'(x) = -2x+2$ ①

x		1	
f_1'	$-$		$\cancel{\ }$
f_2'	$\cancel{\ }$	0	$-$
f'	$-$		$-$
f	$\searrow$		$\searrow$

25. ⓐ $f(x) = 2\eta\mu x - x + 3 \qquad x \in [0, \pi]$.

$f'(x) = 2\sigma\upsilon\nu x - 1$



$\to f'(x) = 0 \Rightarrow 2\sigma\upsilon\nu x - 1 = 0$

$\sigma\upsilon\nu x = \frac{1}{2}$

$x = \frac{\pi}{3}$.

x	0	$\frac{\pi}{3}$	π
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$f(x) \le f\left(\frac{\pi}{3}\right)$

$f(x) \le \frac{2\sqrt{3}}{2} - \frac{\pi}{3} + 3$.

$f'(0) = 1$

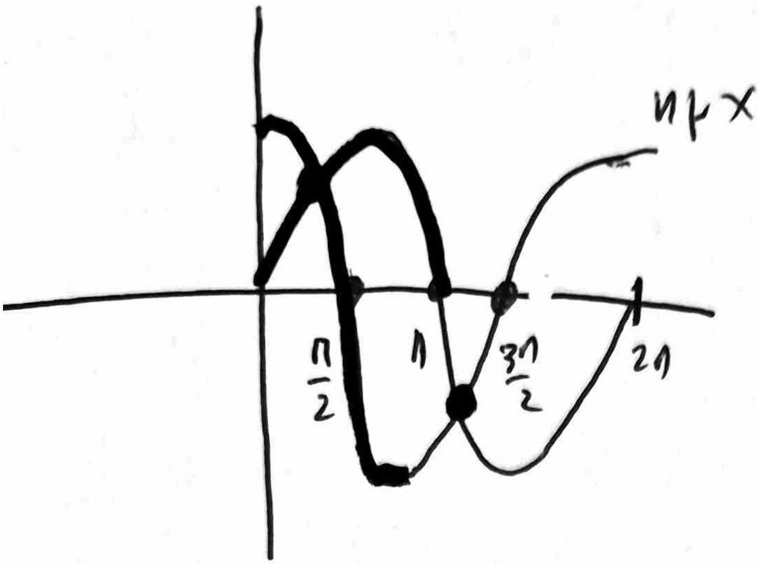
$f'(\pi) = -3$

ⓑ $f(x) = e^{-x}\eta\mu x \;, \; x \in [0, \pi]$

$f'(x) = -e^{-x}\eta\mu x + e^{-x}\sigma\upsilon\nu x$

$$\boxed{f'(x) = e^{-x}(\sigma\upsilon\nu x - \eta\mu x)}$$

$f'(x) = 0 \Rightarrow \sigma\upsilon\nu x - \eta\mu x = 0 \Rightarrow \underline{\eta\mu x = \sigma\upsilon\nu x}$



Όταν $x \in [0, \pi]$

το $συνx = ημx$

αν $x = \frac{\pi}{4}$

x	0		$\pi/4$		π
f'		$+$	0	$-$	
f		$\nearrow$		$\searrow$	

$f(x) \leq f\left(\frac{\pi}{4}\right)$

$$f(x) \leq e^{-\frac{\pi}{4}} \frac{\sqrt{2}}{2}$$

$f'(0) = 1$

$f'(\pi) = -1$

24. α) $f(x) = \ln(e^x - x)$, $x \in \mathbb{R}$.

$e^x - x > 0$ που ισχύει.

$$\boxed{f'(x) = \frac{e^x - 1}{e^x - x}}$$

$f'(x) = 0 \Rightarrow e^x - 1 = 0 \Rightarrow e^x = 1$

$x = 0$.

x		0	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

$f(x) \geq 0$

γ) $f(x) = \ln(x - \ln x)$ πρέπει $x > 0$

$x - \ln x > 0$

$0 > \ln x - x$ ✓

$$f'(x) = \frac{1 - \frac{1}{x}}{x - \ln x}$$

$$\boxed{f'(x) = \frac{x-1}{x(x - \ln x)}}$$

$f'(x) = 0 \Rightarrow x - 1 = 0$

$x = 1$

x		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

23. ⓐ $f(x) = 2e^x - x^2$

$f'(x) = 2e^x - 2x$

$f'(x) = 2(e^x - x) > 0$ $\quad f \nearrow$

$\quad\quad\quad\quad (+)$

$\cdot\ e^x \geq x+1 \quad \Rightarrow e^x - x \geq 1 \quad \Rightarrow e^x - x > 0$

ⓑ $f(x) = x\ln x - x^2 - x$

$f'(x) = \ln x + x\,\frac{1}{x} - 2x - 1$

$f'(x) = \ln x + 1 - 2x - 1$

$f'(x) = \ln x - 2x < 0$

$\cdot\ \ln x \leq x - 1$

$\ln x \leq 2x - 2$

$\ln x - 2x \leq -2$

$f \searrow$

ε) $f(x) = \ln x + e^{-x} + 1$, $x>0$.

$f'(x) = \frac{1}{x} - e^{-x}$

$f'(x) = \frac{1}{x} - \frac{1}{e^x}$

$f'(x) = \frac{\overset{(+)}{e^x - x}}{xe^x} \geq 0$

• $e^x \geq x+1$ $\Rightarrow$ $\underline{\underline{e^x - x \geq 1}}$

22. ⓐ. $f(x) = 2e^x - x^2 - 2x$

$$f'(x) = 2e^x - 2x - 2$$

$$f'(x) = 2(e^x - x - 1) \geq 0$$

$\oplus$ $\qquad f \nearrow$,

• $e^x \geq x + 1$

$e^x - x - 1 \geq 0$

ⓑ $f(x) = \ln^2 x - 2x + 2 \quad , x > 0$.

$$f'(x) = 2\ln x \, \frac{1}{x} - 2$$

$$f'(x) = \frac{2\ln x - 2x}{x}$$

$$\boxed{f'(x) = 2\,\frac{\ln x - x}{x} < 0 \quad f \searrow}$$

$\ominus$

• $\ln x \leq x - 1$

$\ln x - x \leq -1$

21. (γ) $f(x) = 2\sqrt{x} - \dfrac{\ln x}{\sqrt{x}}$, $x>0$

$$f'(x) = 2 \cdot \frac{1}{2\sqrt{x}} - \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x}$$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{2x\sqrt{x}\,\frac{\sqrt{x}}{x} - \frac{2x\sqrt{x}\ln x}{2\sqrt{x}}}{2x\sqrt{x}\; x}$$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{2x - x\ln x}{2x^2\sqrt{x}}$$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{2-\ln x}{2x\sqrt{x}} = \frac{2x-2+\ln x}{2x\sqrt{x}}$$

$$\boxed{f'(x) = \frac{2x-2+\ln x}{2x\sqrt{x}}.}$$

$\varphi(x) = 2x - 2 + \ln x$ $\qquad\qquad$ $\varphi(1) = 0$

$\varphi'(x) = 2 + \frac{1}{x} > 0$

x	0		1	
φ'		$+$	ϕ	$+$
φ		$\nearrow$ $-$	0	$+$ $\nearrow$
$2x\sqrt{x}$		$+$		$+$
f'		$-$		$+$
f		$\searrow$		$\nearrow$

$x < 1 \Rightarrow \varphi(x) < \varphi(1) \Rightarrow \varphi(x) < 0$

$x > 1 \Rightarrow \varphi(x) > \varphi(1) \Rightarrow \varphi(x) > 0$

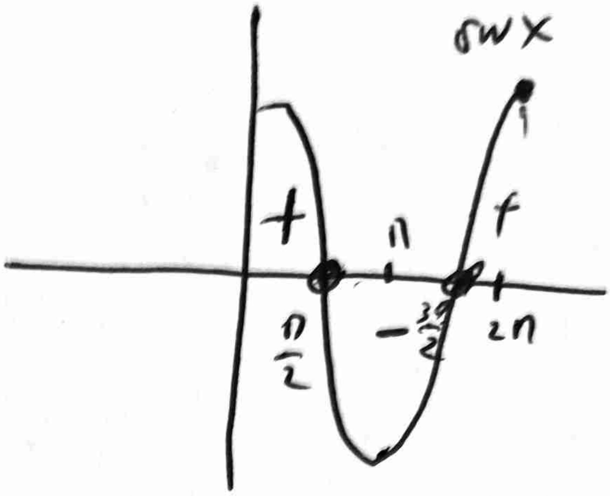
$f(x) \geq f(1)$

$f(x) \geq 2$

14. α) $f(x) = e^{ημx}, \quad x \in [0, 2π]$

$f'(x) = e^{ημx}(ημx)' = e^{ημx} συνx$

$$\boxed{f'(x) = e^{ημx} συνx}$$



x	0		$π/2$		$3π/2$		$2π$
f'		+	0	−	0	+	
f		↗		↘		↗	

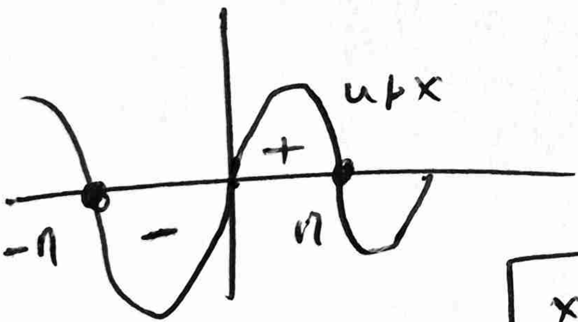
γ) $f(x) = ημ^2x + 4συνx, \quad x \in [-π, π]$

$f'(x) = 2ημx\, συνx - 4ημx$

$$\boxed{f'(x) = 2ημx(\underbrace{συνx - 2}_{-})}$$

$-1 \le συνx \le 1$

$-3 \le συνx - 2 \le -1$



$f(x) \le f(0)$

$f(x) \le 4$

x	$-π$		0		$π$
$2ημx$		−		+	
$συνx-2$		−		−	
f'		+		−	
f		↗		↘	

13. ⓐ $f(x) = (x-2)e^x - \frac{1}{2}\cdot x^2 + x - 1$

$f'(x) = 1\cdot e^x + (x-2)\cdot e^x - \frac{1}{2}2x + 1 - 0$

$f'(x) = e^x + (x-2)e^x - x + 1$

$f'(x) = e^x(1+x-2) - (x-1)$

$f'(x) = e^x(x-1) - (x-1)$

$f'(x) = (e^x-1)(x-1)$

$\longrightarrow f'(x) = 0 \Rightarrow e^x - 1 = 0$ ή $x-1=0$

$e^x = 1$

$x=0$ $\qquad$ $x=1$

x		0			
e^x-1	$-$	0	$+$		$+$
$x-1$	$-$		$-$	0	$+$
f'	$+$		$-$		$+$
f	↗		↘		↗

(γ) $f(x) = (x^2-4x)\ln x - \frac{x^2}{2} + 4x$

$$f'(x) = (2x-4)\ln x + (x^2-4x)\frac{1}{x} - \frac{1}{2}2x + 4$$

$$f'(x) = (2x-4)\ln x + \cancel{x} - \cancel{4} - \cancel{x} + \cancel{4}$$

$$\boxed{f'(x) = (2x-4)\ln x}$$

→ $f'(x) = 0 \Rightarrow 2x-4=0$ ή $\ln x = 0$

$x=2$ $\quad$ $x=1$

x	0	1	2
$2x-4$	−	−	0 +
$\ln x$	− 0	+	+
f'	+	−	+
f	↗	↘	↗

ε) $f(x) = \dfrac{e^x}{x^2}$, $x \neq 0$.

$$f'(x) = \frac{(e^x)' \cdot x^2 - e^x \cdot (x^2)'}{(x^2)^2}$$

$$f'(x) = \frac{e^x \cdot x^2 - e^x \cdot 2x}{x^4}$$

$$f'(x) = \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}$$

$\rightarrow f'(x) = 0 \Rightarrow e^x(x-2) = 0$

$x = 2$.

x		0		2	
$x-2$	$-$	‖	$-$		$+$
x^3	$-$	‖	$+$		$+$
f'	$+$	‖	$-$		$+$
f	$\nearrow$	‖	$\searrow$		$\nearrow$

Ενότητα 27

12. ⓐ $f(x) = x\ln x - 2x \quad , x>0$

$$f'(x) = \ln x + x \cdot \frac{1}{x} - 2$$

$$\boxed{f'(x) = \ln x - 1.}$$

$\rightarrow \quad f'(x)=0 \Rightarrow \ln x - 1 = 0$

$\ln x = 1$

$e^{\ln x} = e^{1}$

$x = e$

x		e	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(e)$

$f(x) \geq -e$

(γ) $f(x) = \dfrac{\ln x}{x}$, $x > 0$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\boxed{f'(x) = \frac{1 - \ln x}{x^2}}$$

$\rightarrow$ $f'(x) = 0 \Rightarrow \dfrac{1 - \ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0$

$1 = \ln x$

$e^1 = x$

x		e	
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$f(x) \leq f(e)$

$f(x) \leq \dfrac{1}{e}$

(ε). $f(x) = x - e\ln x \quad , x>0$

$$f'(x) = 1 - \frac{e}{x}$$

$$f'(x) = \frac{x-e}{x}$$

x		e	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(e)$

$f(x) \geq 0$

$$f'(x) = 0 \Rightarrow \frac{x-e}{x} = 0 \Rightarrow x - e = 0 \Rightarrow \underline{\underline{x = e}}$$

(J) $f(x) = x\ln^2 x - 3x + 2$

$$f'(x) = \ln^2 x + \cancel{x} \cdot 2\ln x \frac{1}{\cancel{x}} - 3$$

$$\boxed{f'(x) = \ln^2 x + 2\ln x - 3}$$

$\rightarrow f'(x) = 0 \Rightarrow \ln^2 x + 2\ln x - 3 = 0$

$\Delta = 16$

$$\ln x = \frac{-2 \pm 4}{2} \begin{cases} \rightarrow 1 \\ \rightarrow -3 \end{cases}$$

$\ln x = 1 \qquad \ln x = -3$

$\underline{\underline{x = e}} \qquad x = e^{-3}.$

x	0	e^{-3}	e	
$\ln x - 1$	$-$	$-$	0	$+$
$\ln x + 3$	$-$	0	$+$	$+$
f'	$+$	$-$	$+$	
f	$\nearrow$	$\searrow$	$\nearrow$	

Ενότητα

24

$f(1)=0$

8. $f(x)=e^{1-x}+\ln x-1 \ , \ x>0$

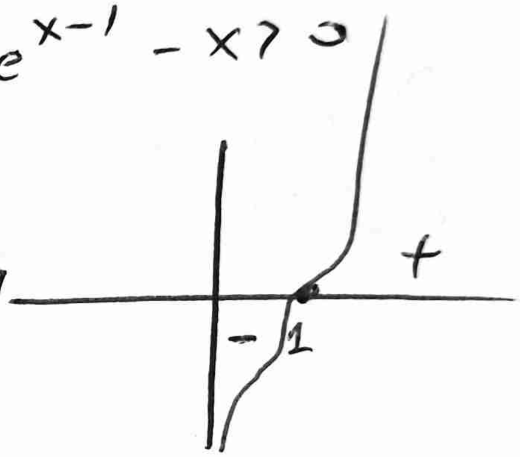
ⓐ $f'(x)=-e^{1-x}+\frac{1}{x}$

$f'(x)=-\frac{1}{e^{x-1}}+\frac{1}{x}=\frac{e^{x-1}-x}{xe^{x-1}}>0$ ⊕ ⊕ $f\nearrow$

• $e^{x}\geq x+1$

$e^{x-1}\geq x-1+1$

$e^{x-1}-x\geq 2 \Rightarrow e^{x-1}-x>0$



x	0	1	
$f(x)$	$\nearrow$ −	0	+ $\nearrow$

$x<1 \Rightarrow f(x)<f(1) \Rightarrow f(x)<0$

$x>1 \Rightarrow f(x)>f(1) \Rightarrow f(x)>0$,

(B) i) $e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -\ln \varepsilon\varphi x$

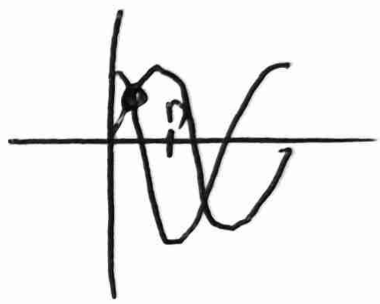
$e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -\ln \dfrac{\eta\mu x}{\sigma\upsilon\nu x}$ $\quad (0, \frac{\pi}{2})$

$e^{1-\eta\mu x} - e^{1-\sigma\upsilon\nu x} = -(\ln \eta\mu x - \ln \sigma\upsilon\nu x)$

$e^{1-\eta\mu x} + \ln \eta\mu x - 1 = e^{1-\sigma\upsilon\nu x} + \ln \sigma\upsilon\nu x - 1$

$f(\eta\mu x) = f(\sigma\upsilon\nu x)$

f 1-1



$\eta\mu x = \sigma\upsilon\nu x$

$x = \frac{\pi}{4}$

11). $e^{1-x} - e^{1-x^2} > \ln x \quad , x > 0$

$e^{1-x} - e^{1-x^2} > \ln \frac{x^2}{x}$

$e^{1-x} - e^{1-x^2} > \ln x^2 - \ln x$

$e^{1-x} + \ln x - 1 > e^{1-x^2} + \ln x^2 - 1$

$f(x) > f(x^2)$

$f \nearrow$

$x > x^2$

$\underline{1 > x > 0}$

(8) $\lim_{x\to1^+} \dfrac{\ln x}{(x-1)\,f(x)} =$

$= \lim_{x\to1^+} \dfrac{\frac{1}{x}}{f(x) + (x-1)f'(x)} =$

$= \lim_{x\to1^+} \dfrac{1}{x f(x) + x(x-1)f'(x)}$

$\lim_{x\to1^+} \dfrac{\ln x}{x-1} \cdot \dfrac{1}{\underset{(+)}{f(x)}} = 1\cdot(+\infty)$

$= +\infty$

$\to \lim_{x\to1^+} \dfrac{\ln x}{x-1} = \lim_{x\to1^+} \dfrac{1}{x} = 1$

Επομενο Μαθημα

24

1
2
4
5.