Ενότητα 29

2. (α) $f(x) = e^x + x^2 - 1$

$f'(x) = e^x + 2x$

$f''(x) = e^x + 2 > 0$ f κυρτή

(β) $f(x) = x^3 - \sqrt{x}$, $x \geq 0$

$f'(x) = 3x^2 - \frac{1}{2\sqrt{x}}$

$f''(x) = 6x - \frac{-\cancel{2}\frac{1}{\cancel{2}\sqrt{x}}}{4x}$

$f''(x) = 6x + \frac{1}{4x\sqrt{x}} > 0$ f κυρτή

(ε) $f(x) = 2^x + x^2 - 2x - 1$

$f'(x) = 2^x \ln 2 + 2x - 2$

$f''(x) = 2^x \ln^2 2 + 2 > 2$

f κυρτή. $\boxed{(a^x)' = a^x \ln a}$

3. (a) $f(x) = e^{x-1} - x \quad , x \in \mathbb{R}$

$f'(x) = e^{x-1} - 1$

$\to f'(x) = 0 \quad \to e^{x-1} - 1 = 0$

$e^{x-1} = 1$

$x - 1 = 0$

$\boxed{x = 1}$

x		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$f(x) \geq 0$

$f''(x) = e^{x-1} > 0 \quad f$ κυρτή

(β) $f(x) = x - x\ln x + 1 \quad , x > 0$

$f'(x) = 1 - \ln x - 1 = -\ln x$

$\to f'(x) = 0 \Rightarrow -\ln x = 0 \Rightarrow x = 1$

x	0		1	
f'		$+$	0	$-$
f		$\nearrow$		$\searrow$

$f''(x) = -\frac{1}{x} < 0$

f κοίλη.

4. $g(x) = f(x)\, e^{-x}$

$$g'(x) = f'(x)\, e^{-x} - f(x)\, e^{-x}$$

$$\boxed{g'(x) = e^{-x}\left(f'(x) - f(x)\right)}$$

$$g''(x) = -e^{-x}\left(f'(x) - f(x)\right) + e^{-x}\left(f''(x) - f'(x)\right)$$

$$g''(x) = e^{-x}\left(-f'(x) + f(x) + f''(x) - f'(x)\right)$$

$$\boxed{g''(x) = e^{-x}\left(f''(x) - 2f'(x) + f(x)\right)}$$

$g''(x) > 0$

g κυρτη

ομως $\quad f'(x) < \dfrac{f''(x) + f(x)}{2}$

$$2f'(x) < f''(x) + f(x)$$

$$0 < f''(x) - 2f'(x) + f(x)$$

7. ⓐ $f(x)=\sqrt{1+x^2}\ ,\ x\in \mathbb{R}$

$$f'(x)=\frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}$$

$$f''(x)=\frac{\sqrt{x^2+1}-x\,\frac{2x}{2\sqrt{x^2+1}}}{x^2+1}$$

$$f''(x)=\frac{\sqrt{x^2+1}-\frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$$

$$f''(x)=\frac{\cancel{x^2}+1-\cancel{x^2}}{(x^2+1)\sqrt{x^2+1}}=\frac{1}{(x^2+1)\sqrt{x^2+1}}>0$$

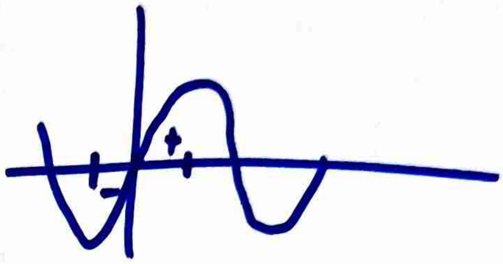
f κυρτή.

γ) $f(x) = \varepsilon\varphi x - x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

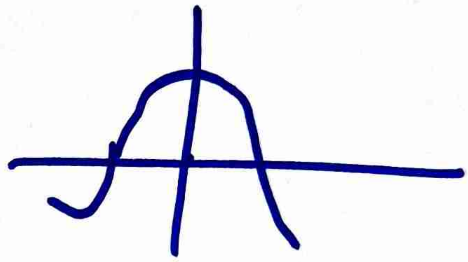
$$f'(x) = \frac{1}{\sigma\upsilon\nu^2 x} - 1$$

$$f''(x) = \frac{-2\sigma\upsilon\nu x(-\eta\mu x)}{\sigma\upsilon\nu^4 x} = \frac{2\eta\mu x}{\sigma\upsilon\nu^3 x}$$

$$f''(x) = \frac{2\eta\mu x}{\underset{+}{\sigma\upsilon\nu^3 x}}$$



x		0	
f''	−		+
f	∩		∪



6. ⓐ $f(x) = \frac{1}{x^2+4} \quad x \in \mathbb{R}$

$$f'(x) = \frac{-2x}{(x^2+4)^2}$$

$$f''(x) = \frac{-2(x^2+4)^2 - (-2x)\, 2(x^2+4)\, 2x}{(x^2+4)^4}$$

$$f''(x) = \frac{-2(x^2+4) + 8x^2}{(x^2+4)^3} = \frac{6x^2-8}{(x^2+4)^3}$$

$$f''(x) = 2\,\frac{3x^2-4}{(x^2+4)^3}$$

$\rightarrow f''(x) = 0 \quad \Rightarrow 3x^2 - 4 = 0 \qquad x^2 = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{2}$

x		$-\frac{2\sqrt{3}}{2}$		$\frac{2\sqrt{3}}{2}$	
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

① $f(x) = \dfrac{x-1}{x-2} \quad , x \neq 2$

$$f'(x) = \frac{(x-2) - (x-1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

$$f''(x) = \frac{-(-1)\,2(x-2)}{(x-2)^4} = \frac{2}{(x-2)^3}$$

$$\boxed{f''(x) = \frac{2}{(x-2)^3}}$$

x		2	
f''	$-$		$+$
f	$\cap$		$\cup$

ε) $f(x) = \dfrac{x}{x^2-1}$, $x \neq 1$ $x \neq -1$

$$f'(x) = \frac{x^2-1 - x\,2x}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$$

$$f''(x) = \frac{-2x\,(x^2-1)^2 - (-x^2-1)\,2\,(x^2-1)\,2x}{(x^2-1)^4}$$

$$f''(x) = \frac{-2x\,(x^2-1) - (-x^2-1)\,4x}{(x^2-1)^3}$$

$$f''(x) = \frac{-2x^3 + 2x - 4x\,(-x^2-1)}{(x^2-1)^3}$$

$$f''(x) = \frac{-2x^3 + 2x + 4x^3 + 4x}{(x^2-1)^3}$$

$$f''(x) = \frac{2x^3 + 6x}{(x^2-1)^3} = \frac{2x\,\overset{+}{(x^2+3)}}{(x^2-1)^3}$$

x		-1		0		1	
$2x$	$-$		$-$		$+$		$+$
x^2-1	$+$		$-$		$-$		$+$
f''	$-$		$+$		$-$		$+$
f	$\cap$		$\cup$		$\cap$		$\cup$

5. ⓐ $f(x) = x^3 - 6x^2$

$f'(x) = 3x^2 - 12x$

$f''(x) = 6x - 12$

$\longrightarrow f''(x) = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$

x		2	
f''	$-$		$+$
f	$\cap$		$\cup$

ⓑ $f(x) = x^6 - 3x^5$

$f'(x) = 6x^5 - 15x^4$

$f''(x) = 30x^4 - 60x^3 = 30x^3(x-2)$

$\longrightarrow f''(x) = 0$

$30x^3(x-2) = 0$

$x = 0 \quad x = 2$

x		0		2	
x^3	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

8. ⓐ $f(x) = 2e^{x-1} - x^2$

$f'(x) = 2e^{x-1} - 2x$

$f''(x) = 2e^{x-1} - 2 = 2(e^{x-1} - 1)$

$\rightarrow e^{x-1} - 1 = 0$

$e^{x-1} = 1 \qquad \underline{\underline{x = 1}}$

x		1	
f''	$-$		$+$
f	$\cap$		$\cup$

ⓨ $f(x) = (x^2+1)e^{-x}$

$f'(x) = 2xe^{-x} - (x^2+1)e^{-x} = e^{-x}(2x - x^2 - 1)$

$f''(x) = -e^{-x}(2x - x^2 - 1) + e^{-x}(2 - 2x)$

$f''(x) = -e^{-x}(2x - x^2 - 1 - 2 + 2x)$

$f''(x) = -e^{-x}(-x^2 + 4x - 3)$

$f''(x) = e^{-x}(x^2 - 4x + 3)$

x		1		3	
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

10. (α) $f(x) = \text{συν}x - \frac{x^3}{6} + \frac{x^2}{2} - 1$

$f'(x) = -\text{ημ}x - \frac{3}{6}x^2 + \frac{2}{2}x$

$f''(x) = -\text{συν}x - \frac{6}{6}x + 1$

$f''(x) = -\text{συν}x - x + 1$ $\qquad f''(0) = 0$

$f'''(x) = \text{ημ}x - 1 \leq 0$

x		0	
f'''	$-$		$-$
f''	$\searrow$ $+$	0	$\searrow$ $-$
f	$\cup$		$\cap$

1. (γ) $f(x) = 2x + \ln(x^2+1)$

$$f'(x) = 2 + \frac{2x}{x^2+1}$$

$$f''(x) = \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2-2x^2}{(x^2+1)^2} = 2\,\frac{1-x^2}{(x^2+1)^2}$$

x		-1		1	
f''	$-$		$+$		$-$
f	$\cap$		$\cup$		$\cap$

(ε) $f(x) = x^x$, $x > 0$

$$f(x) = e^{\ln x^x} = e^{x\ln x}$$

$$f'(x) = e^{x\ln x}(\ln x + 1)$$

$$f''(x) = e^{x\ln x}(\ln x+1)^2 + e^{x\ln x}\frac{1}{x}$$

$$f''(x) = e^{x\ln x}\left((\ln x+1)^2 + \frac{1}{x}\right) > 0$$

f κυρτ.

9. (a) $f(x) = \dfrac{\ln x}{x}, \; x > 0$

$$f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$f''(x) = \frac{-\frac{1}{x}x^2 - (1-\ln x)2x}{x^4} = \frac{-x-(1-\ln x)2x}{x^4}$$

$$f''(x) = \frac{-1-2(1-\ln x)}{x^3} = \frac{-1-2+2\ln x}{x^3}$$

$$f''(x) = \frac{2\ln x - 3}{x^3}$$

$\rightarrow 2\ln x - 3 = 0 \Rightarrow \ln x = \frac{3}{2} \Rightarrow x = e^{\frac{3}{2}}$

$x = \sqrt{e^3} = e\sqrt{e}$

x		$e\sqrt{e}$	
f''	$-$		$+$
f	$\cap$		$\cup$

Ενότητα 26

16. ⓐ $f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$f'(x) = \frac{e^x x - (e^x - 1)}{x^2} = \frac{xe^x - e^x + 1}{x^2}$$

$$\boxed{\varphi(x) = xe^x - e^x + 1} \qquad \varphi(0) = 0$$

$$\varphi'(x) = e^x + xe^x - e^x = xe^x$$

$$\to \varphi'(x) = 0 \Rightarrow xe^x = 0 \Rightarrow \underline{\underline{x = 0}}$$

x		0	
φ'	$-$		$+$
φ	$\searrow +$	0	$+ \nearrow$
f'	$+$		$+$
f	$\nearrow$		$\nearrow$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1 \quad \Bigg\} \quad f(0) = \lim_{x \to 0} f(x)$$

Συνεχής στο 0!

$$f(0) = 1$$

(B) $f(x) = \begin{cases} x + \dfrac{ημx}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$\left.\begin{array}{l} \lim\limits_{x \to 0} f(x) = 1 \\ f(0) = 1 \end{array}\right\} \checkmark$$

$$f'(x) = 1 + \frac{x\,συνx - ημx}{x^2} = \frac{x^2 + x\,συνx - ημx}{x^2}$$

$\boxed{φ(x) = x^2 + x\,συνx - ημx}$ $\quad φ(0) = 0$

$φ'(x) = 2x + συνx - x\,ημx - συνx$

$φ'(x) = 2x - x\,ημx$

$φ'(x) = x(2 - ημx)$
$\quad\quad\quad\quad\quad (+)$

$\to φ'(x) = 0 \Rightarrow x = 0$

x		0	
$φ'$	$-$		$+$
$φ$	↘ $+$	0	$+$ ↗
f'	$+$		$+$
f	↗		↗

17. (a) $e^x > 2x$

$e^x - 2x > 0$

$\boxed{\varphi(x) = e^x - 2x}$

$\varphi'(x) = e^x - 2$

$\to e^x - 2 = 0$

$e^x = 2$

$\underline{x = \ln 2}$

x		$\ln 2$	
φ'	$-$		$+$
φ	$\searrow$		$\nearrow$

$\varphi(x) \geq \varphi(\ln 2)$

$e^x - 2x \geq e^{\ln 2} - 2\ln 2$

$e^x - 2x \geq 2 - \ln 4$

$e^x - 2x \geq \ln e^2 - \ln 4$

$e^x - 2x \geq \ln \frac{e^2}{4}$

$e^x - 2x > 0$

$e^x > 2x$.

(γ) $x^2 \geq 2\ln x + 1 \quad \forall x > 0$

$$\underbrace{x^2 - 2\ln x - 1}_{f(x)} \geq 0$$

$$f'(x) = 2x - \frac{2}{x} = \frac{2x^2 - 2}{x} = 2\,\frac{x^2 - 1}{x}$$

x	0	1
f'	$-$	$+$
f	$\searrow$	$\nearrow$

$$f(x) \geq f(1)$$

$$x^2 - 2\ln x - 1 \geq 2$$

$$x^2 - 2\ln x - 1 > 0$$

19. (a) $x^{e} = e^{x}$ $\quad (0, +\infty)$

$\ln x^{e} = \ln e^{x}$

$e \ln x = x \ln e$

$\frac{\ln x}{x} = \frac{\ln e}{e}$

$f(x) = f(e)$

$\underline{\underline{x = e}}$

$f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

$\to f'(x) = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow 1 = \ln x$

$\underline{\underline{x = e}}$

x	0	e
f'	$+$	$-$
f	$\nearrow$	$\searrow$

$f(x) \leq f(e)$

Το "="

$\underline{x = e}$

(13) $\cos x = 1 - \frac{x^2}{2}$

$\cos x - 1 + \frac{x^2}{2} = 0 \implies f(x) = 0$

$\underline{\underline{x = 0}}$

$f(x) = \cos x - 1 + \frac{x^2}{2}$

$f'(x) = -\sin x + \frac{1}{2} 2x$

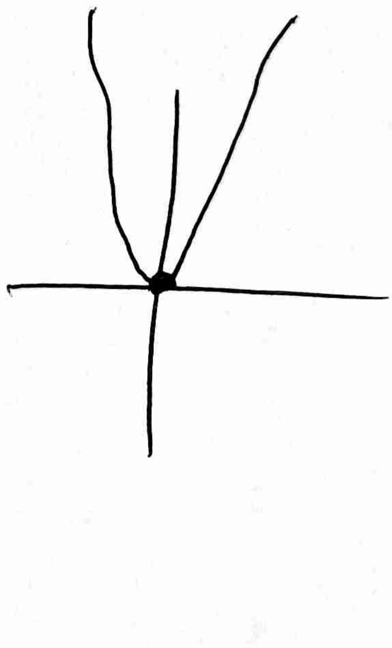
$f'(x) = x - \sin x \quad f'(0) = 0$

$f''(x) = 1 - \cos x \geq 0$

x		0	
f''	+		+
f'	$\nearrow$ −	0	+ $\nearrow$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

$f(x) \geq 0$



Ενοτητα 27

1. (α) $f(x) = x^3 - x^2 + x - 1$

$$f'(x) = 3x^2 - 2x + 1 > 0$$

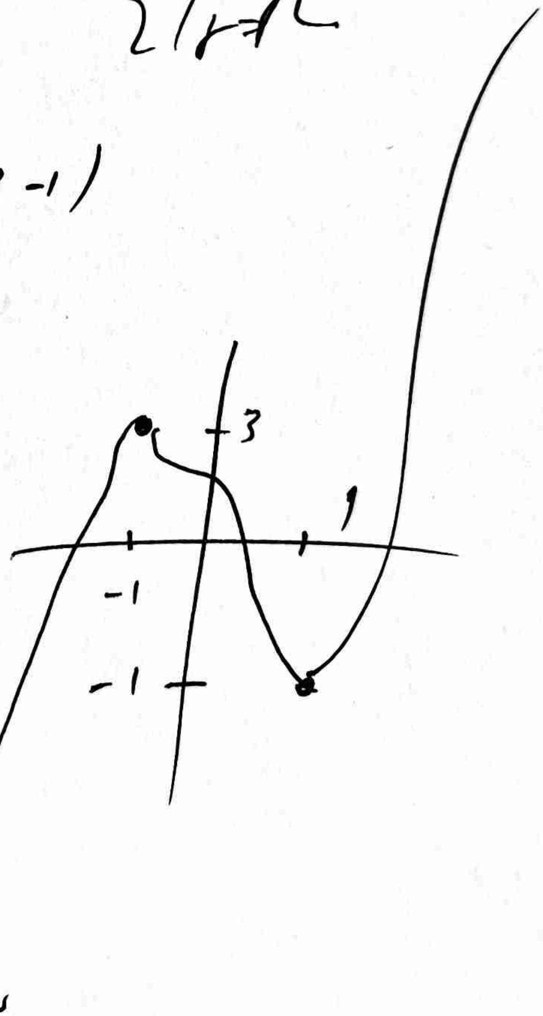
$f \nearrow$

$$\left.\begin{array}{l} \lim\limits_{x \to -\infty} f(x) = -\infty \\ \lim\limits_{x \to +\infty} f(x) = +\infty \end{array}\right\} \mathbb{R}.$$

(β) $f(x) = x^3 - 3x + 1$ $\qquad \Sigma T_f = \mathbb{R}$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

x		-1		1	
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$



$$\lim_{x \to -\infty} f(x) = -\infty$$

$f(-1) = 3$

$f(1) = -1$ $\qquad \lim\limits_{x \to +\infty} f(x) = +\infty$

Επομενο Μαθημα

26

23

24

31

32

33