1.

(α) Αν $\alpha < -3$ και $\beta < 2$

νδο $\alpha\beta - 6 > 2\alpha - 3\beta$.

$$\alpha\beta - 6 - 2\alpha + 3\beta > 0$$

$$\alpha(\beta - 2) + 3(\beta - 2) > 0$$

$$(\beta - 2)(\alpha + 3) > 0$$

(−) (−) ✓

- $\alpha < -3 \Rightarrow \alpha + 3 < 0$
- $\beta < 2 \Rightarrow \beta - 2 < 0$

(β) Νδο $\beta^2 \geq \frac{\alpha\beta}{2} - \frac{(\alpha-\beta)^2}{8}$

$$8\beta^2 \geq 4\alpha\beta - (\alpha-\beta)^2$$

$$8\beta^2 \geq 4\alpha\beta - (\alpha^2 - 2\alpha\beta + \beta^2)$$

$$8\beta^2 \geq 4\alpha\beta - \alpha^2 + 2\alpha\beta - \beta^2$$

$$9\beta^2 - 6\alpha\beta + \alpha^2 \geq 0 \qquad (3\beta - \alpha)^2 \geq 0 \quad ✓$$

(γ) Νδο $x^2+6x+2026>0$

$$x^2+6x+9+2017>0$$

$$(x+3)^2+2017>0$$

(δ). Νδο αν $a+\beta\geq 0$ τότε $a^3+\beta^3\geq a\beta(a+\beta)$

$$(a+\beta)(a^2-a\beta+\beta^2)-a\beta(a+\beta)\geq 0$$

$$(a+\beta)(a^2-a\beta+\beta^2-a\beta)\geq 0$$

$$(a+\beta)(a^2-2a\beta+\beta^2)\geq 0$$

$$\underset{(+)}{(a+\beta)}\ \underset{(+)}{(a-\beta)^2}\geq 0$$

(ε) $(a^2+\beta^2)(x^2+y^2) \geq (ax+\beta y)^2$

$a^2x^2 + a^2y^2 + \beta^2x^2 + \beta^2y^2 \geq a^2x^2 + 2ax\beta y + \beta^2y^2$

$a^2y^2 - 2ax\beta y + \beta^2x^2 \geq 0$

$(ay - \beta x)^2 \geq 0$

⑦ Av $x \geq -1$ vdo $x^3 - x^2 \geq x - 1$

$x^3 - x^2 - x + 1 \geq 0$

$x^2(x-1) - (x-1) \geq 0$

$(x-1)(x^2-1) \geq 0$

$(x-1)(x-1)(x+1) \geq 0$

$(x-1)^2 (x+1) \geq 0$

(+) (+) ✓

• $x \geq -1$ $\quad -1 x + 1 \geq 0$

(n) Αν $\alpha < \beta < -4$ να δ.ο $4\beta - \alpha(4-\beta) < \alpha^2$

$$4\beta - \alpha(4-\beta) - \alpha^2 < 0$$

$$4\beta - 4\alpha + \alpha\beta - \alpha^2 < 0$$

$$4(\beta-\alpha) + \alpha(\beta-\alpha) < 0$$

$$(\beta-\alpha)(4+\alpha) < 0$$

(+) (−) ✓

- $\alpha < \beta \Rightarrow 0 < \beta - \alpha$
- $\alpha < -4 \Rightarrow \alpha + 4 < 0$

⑧ Αν α, β ομόσημοι και $\alpha \le \beta$.

Να δ.ο $\alpha - \frac{1}{\alpha} \le \beta - \frac{1}{\beta}$

- $\alpha \le \beta$ (+)
- $\alpha \le \beta \to \frac{1}{\alpha} \ge \frac{1}{\beta} \Rightarrow -\frac{1}{\alpha} \le -\frac{1}{\beta}$

$$\alpha - \frac{1}{\alpha} \le \beta - \frac{1}{\beta}$$

2ος τρόπος

Αφού α, β ομόσημοι

$\Rightarrow \alpha \cdot \beta > 0$

$$\alpha^2\beta - \alpha\beta\frac{1}{\alpha} \le \alpha\beta^2 - \alpha\beta\frac{1}{\beta}$$

$$\alpha^2\beta - \beta \le \alpha\beta^2 - \alpha$$

$$\alpha^2\beta - \beta - \alpha\beta^2 + \alpha \le 0$$

$$\alpha\beta(\alpha - \beta) + \alpha - \beta \le 0$$

$$(\alpha - \beta)(\alpha\beta + 1) \le 0 \quad \checkmark$$

(−) (+)

- $\alpha \le \beta \Rightarrow \alpha - \beta \le 0$
- $\alpha\beta > 0$

$\alpha\beta + 1 > 1$

2. (a) NdS $x^2+x+1>0$

$$2x^2+2x+2>0$$

$$x^2+x^2+2x+1+1>0$$

$$x^2+(x+1)^2+1>0$$ ✓

(B). NdS $x^2+\frac{1}{x^2} \geq x+\frac{1}{x}$

$$x^4+1 \geq x^3+x$$

$$x^4-x^3-x+1 \geq 0$$

$$x^3(x-1)-(x-1) \geq 0$$

$$(x-1)(x^3-1) \geq 0$$

$$(x-1)(x-1)(x^2+x+1) \geq 0$$

$$(x-1)^2(x^2+x+1) \geq 0.$$

(+) (+) ✓

3. $a^2+\beta^2+\gamma^2 \geq \alpha\beta+\beta\gamma+\gamma\alpha$.

$2a^2+2\beta^2+2\gamma^2-2\alpha\beta-2\beta\gamma-2\gamma\alpha \geq 0$

$a^2-2a\beta+\beta^2+\beta^2-2\beta\gamma+\gamma^2+a^2-2a\gamma+\gamma^2 \geq 0$

$(a-\beta)^2+(\beta-\gamma)^2+(a-\gamma)^2 \geq 0$ ✓

4. (a) Av $x,y>0$ vδο $\frac{x}{y}+\frac{y}{x} \geq 2$

$\Rightarrow xy\frac{x}{y}+xy\frac{y}{x} \geq 2xy$

$x^2+y^2 \geq 2xy$

$x^2+y^2-2xy \geq 0$

$(x-y)^2 \geq 0$ ✓

(B) Να δ.ο $\frac{\alpha+\beta}{\gamma} + \frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} \geq 6$

$$\frac{\alpha}{\gamma} + \frac{\beta}{\gamma} + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} + \frac{\gamma}{\beta} + \frac{\alpha}{\beta} \geq 6$$

$$\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} \geq 6$$

Απόδειξη πριν ότι $\frac{x}{y} + \frac{y}{x} \geq 2$

όμοιοι τρόποι ότι

$\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} \geq 2$

$\frac{\beta}{\gamma} + \frac{\gamma}{\beta} \geq 2$

$\frac{\beta}{\alpha} + \frac{\alpha}{\beta} \geq 2$

} ① $\frac{\alpha+\beta}{\gamma} + \frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} \geq 6$

5. Έστω $0 < x < y$

Συγκρίνω $A = (x-2y)^2$

$B = y(4y-3x)$.

Έστω $A < B$

$$(x-2y)^2 < y(4y-3x)$$

$$x^2 - 4xy + 4y^2 < 4y^2 - 3xy$$

$$x^2 - xy < 0$$

$$x(x-y) < 0$$

$\oplus$ $\ominus$ ✓

- $x > 0$
- $x < y \Rightarrow x - y < 0$

ΟΥΤΩ $A < B$.

6. (a) $x^2+y^2+10=2(x-3y)$

$x^2+y^2+10=2x-6y$

$x^2+y^2+10-2x+6y=0$

$x^2-2x+1+y^2+6y+9=0.$

$$(x-1)^2+(y+3)^2=0$$

$x-1=0$ and $y+3=0$

$x=1$ $\quad$ $y=-3$

(B) $(x+5)^2+(y-2)^2=4(x+y+1)$

$x^2+10x+25+y^2-4y+4=4x+4y+4$

$x^2+6x+9+y^2-8y+16=0$

$(x+3)^2+(y-4)^2=0$

$x=-3$ $\quad$ $y=4$

7. Έστω $1<x<2$ και $2<y<3$

(α) $2x-4y$

- $1<x<2 \Rightarrow 2<2x<4 \Rightarrow 4>2x>2$
- $2<y<3 \Rightarrow -8>-4y>-12$ (Θ)

$$-4 > 2x-4y > -10$$

(β) $x^2-\frac{1}{x}$

- $1<x<2 \Rightarrow 1<x^2<4$ (Θ)
- $1<x<2 \Rightarrow 1>\frac{1}{x}>\frac{1}{2} \Rightarrow -1<-\frac{1}{x}<-\frac{1}{2}$

$$0<x^2-\frac{1}{x}<4-\frac{1}{2}$$

$$0<x^2-\frac{1}{x}<\frac{7}{2}$$

⑧ $y^2 - \frac{x}{y}$

- $2 < y < 3 \Rightarrow 4 < y^2 < 9$ ④
- $1 < x < 2 \Rightarrow 2 > x > 1$
- $2 < y < 3 \Rightarrow \frac{1}{2} > \frac{1}{y} > \frac{1}{3}$

⊙ $1 > \frac{x}{y} > \frac{1}{3}$

$-1 < -\frac{x}{y} < -\frac{1}{3}$

$3 < y^2 - \frac{x}{y} < 9 - \frac{1}{3}$

$3 < y^2 - \frac{x}{y} < \frac{26}{3}$

⑥ $(x+y)^2 - 2x$

- $1 < x < 2$
- $2 < y < 3$

$\Rightarrow 3 < x+y < 5 \Rightarrow 9 < (x+y)^2 < 25$

- $1 < x < 2 \Rightarrow -2 > -2x > -4$

$-4 < -2x < -2$ (+)

$5 < (x+y)^2 - 2x < 23$

(ε) · $\frac{x-y}{x+y}$

- $1<x<2 \Rightarrow 2>x>1$
- $2<y<3 \Rightarrow -2>-y>-3$

$\Big\}\oplus$

$0>x-y>-2$

$-2<x-y<0$

- $1<x<2$
- $2<y<3$

$\Big\}\oplus$ $\quad 3<x+y<5$

$\Big\}\odot$

$\frac{1}{3} > \frac{1}{x+y} > \frac{1}{5}$

$-\frac{1}{3} < -\frac{1}{x+y} < -\frac{1}{2}$

$\frac{2}{3} > -\frac{x-y}{x+y} > 0$

$-\frac{2}{3} < \frac{x-y}{x+y} < 0.$

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7. (γ)

$3 < x < 5$

$\Gamma = |x-2|^{(+)} + |x-6|^{(-)} + |x|^{(+)}$

- $3 < x < 5 \Rightarrow 1 < x-2 < 3$
- $3 < x < 5 \Rightarrow -3 < x-6 < -1$
- $3 < x < 5$

$\Gamma = \cancel{x}-2 - \cancel{x} + 6 + x$

$\Gamma = x + 4.$

8. (γ) $x > 3$

$A = |x-2|^{(+)} - |x-3|^{(+)} = x-2-(x-3) = \underline{1}.$

- $x > 3 \Rightarrow x-2 > 3-2 \Rightarrow x-2 > 1$
- $x > 3 \Rightarrow x-3 > 0$

9. (B) $B = |6 - |x-5|| \quad (-)$

• $-1 < x < 3$

$-1-5 < x-5 < 3-5$

$-6 < x-5 < -2$

$B = |6 - (-x+5)|$

$B = |6+x-5|$

$B = |1+x| \quad (+) \quad = x+1$

• $-1 < x < 3$

$0 < x+1 < 4$

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Ιδιοτητα

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

$|x| = |-x|$

4. Αν $\alpha \neq \beta$

$$A = \left|\frac{\alpha-\beta}{\beta-\alpha}\right| = \frac{|\alpha-\beta|}{|\beta-\alpha|} = \frac{|\alpha-\beta|}{|\alpha-\beta|} = 1.$$

5. Αν $x \neq 0$, $y \neq 0$

$$A = \frac{|x|}{x} + \frac{|y|}{y}.$$

1. Αν $x>0, y>0 \Rightarrow A = \frac{x}{x} + \frac{y}{y} = 1+1 = 2$
2. Αν $x>0, y<0 \Rightarrow A = \frac{x}{x} - \frac{y}{y} = 1-1 = 0$
3. Αν $x<0, y>0 \Rightarrow A = -\frac{x}{x} + \frac{y}{y} = -1+1 = 0$
4. Αν $x<0, y<0 \Rightarrow A = -\frac{x}{y} - \frac{y}{y} = -1-1 = -2$.

10. ⑤ $B = \left| \overset{(-)}{|a-\beta|} + \overset{(-)}{|\beta-\gamma|} \right|$

• $\alpha < \beta < \gamma$

$\alpha < \beta \Rightarrow \alpha - \beta < 0$

$\beta < \gamma \Rightarrow \beta - \gamma < 0$

$B = | -a+\beta -\beta+\gamma |$

$B = | \overset{(+)}{\gamma - a} | = \gamma - a.$

$\gamma > a \Rightarrow \gamma - a > 0$

Μεγάλη Προσοχή

1. $|x| = \theta \Rightarrow x = \theta$ ή $x = -\theta$

2. $|x| = |y| \Rightarrow x = y$ ή $x = -y$

3. $|x| \leq \theta \Rightarrow -\theta \leq x \leq \theta$

$x \in [-\theta, \theta]$

4. $|x| \geq \theta \Rightarrow x \geq \theta$ ή $x \leq -\theta$

$x \in (-\infty, -\theta] \cup [\theta, +\infty)$