Ενότητα 32

2. (δ) $f(x) = x+2 + \frac{1}{x-2}$, $x \neq 2$

Σύνολο Τιμών

$$f'(x) = 1 + \frac{-1}{(x-2)^2} = \frac{(x-2)^2-1}{(x-2)^2} = \frac{x^2-4x+4-1}{(x-2)^2}$$

$$\boxed{f'(x) = \frac{x^2-4x+3}{(x-2)^2}}$$

$\rightarrow f'(x)=0 \Rightarrow x^2-4x+3=0$

$x=1$ $x=3$

x	$-\infty$		1		2		3	
f'		$+$	0	$-$	‖	$-$	0	$+$
f	$-\infty$	$\nearrow$	2	$\searrow$ $-\infty$	‖	$+\infty$ $\searrow$	6	$\nearrow$ $+\infty$

- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x+2+\frac{1}{x-2} = -\infty$

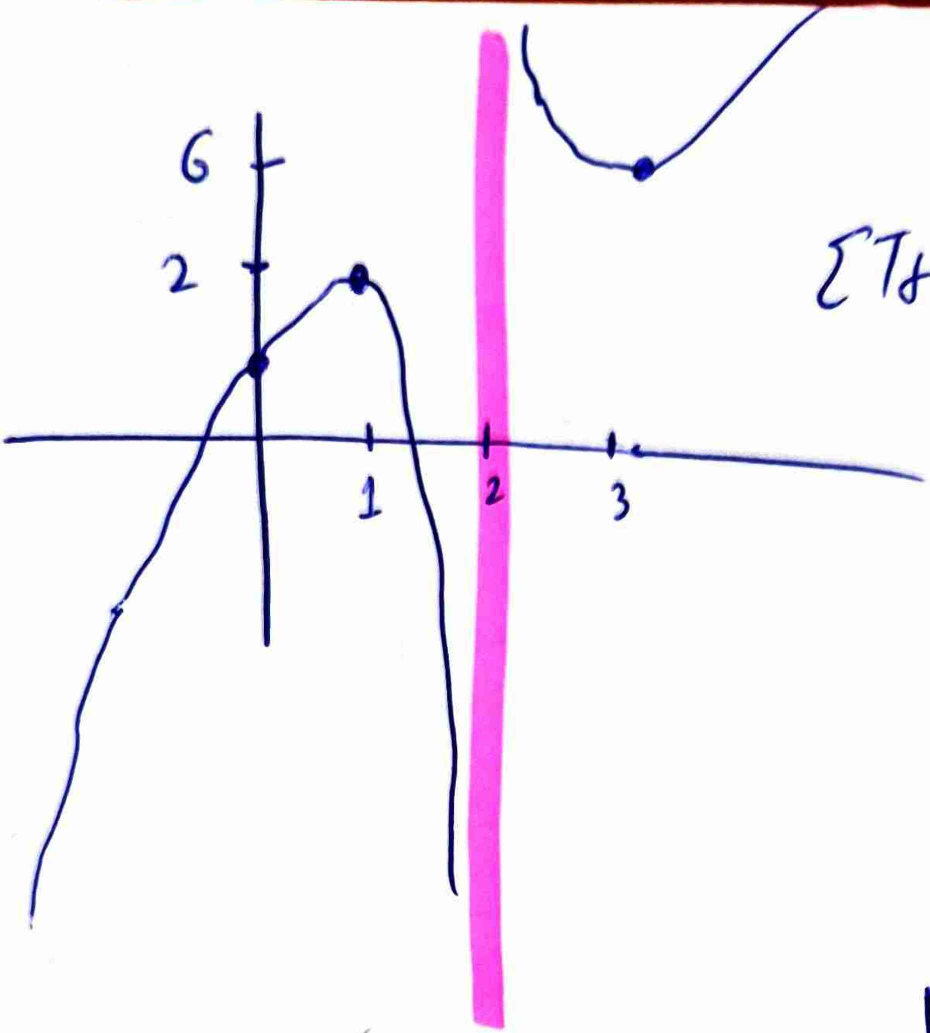
- $f(3)=6$

- $f(1)=2$

- $\lim\limits_{x \to 2^-} f(x) = 4-\infty = -\infty$

- $\lim\limits_{x \to \infty} f(x) = +\infty$

- $\lim\limits_{x \to 2^+} f(x) = +\infty$



$[T_f = (-\infty, 2] \cup [6, +\infty)$.

Κυρτότητα

x		2	
f''	$-$		$+$
f	$\frown$		$\smile$

$$f'(x) = 1 - \frac{1}{(x-2)^2}$$

$$f''(x) = -\frac{-2\cancel{(x-2)}}{(x-2)^{\cancel{4}}} = \frac{2}{(x-2)^3}$$

Ασύμπτωτη

$$\lim_{x \to 2^-} f(x) = -\infty$$

$\varepsilon_1 : x = 2$ κατακόρυφη ασύμπτ.

$$\left.\begin{aligned} \lim_{x \to -\infty} f(x) &= -\infty \\ \lim_{x \to +\infty} f(x) &= +\infty \end{aligned}\right\}$$ Δεν έχει οριζόντια ασύμπτωτη.

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x+2+\frac{1}{x-2}}{x}=\lim_{x\to+\infty}\frac{1-\frac{1}{(x-2)^2}}{1}=1$$

$$\lim_{x\to+\infty} f(x)-1\cdot x=\lim_{x\to+\infty} \cancel{x}+2+\frac{1}{x-2}-\cancel{x}=2$$

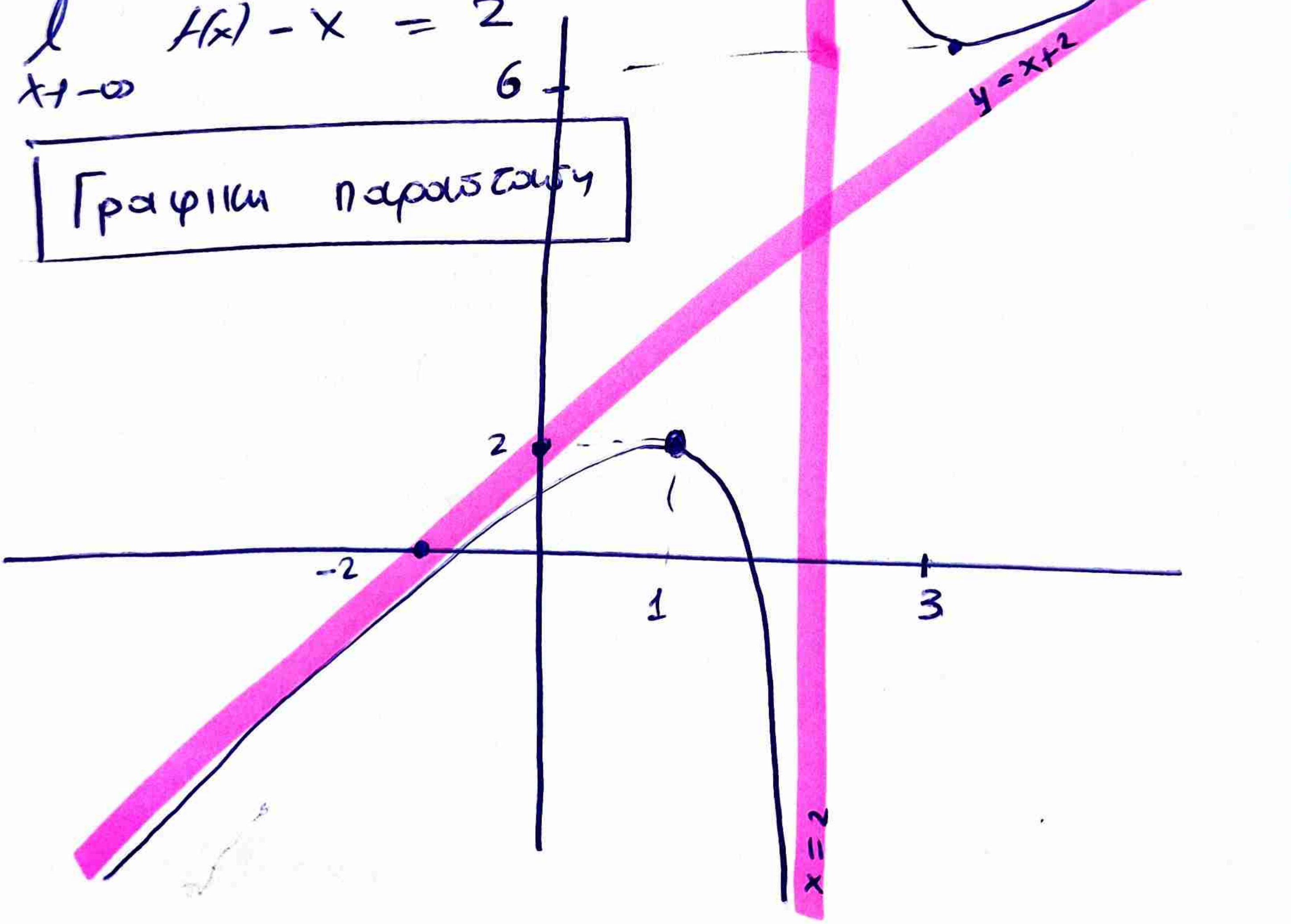
Ειδ $y=1\cdot x+2$ πλαγια στο $+\infty$

$$\lim_{x\to-\infty}\frac{f(x)}{x}=1$$

Ειδ $y=x+2$ πλαγια $-\infty$

$$\lim_{x\to-\infty} f(x)-x=2$$

Γραφικη παρασταση



4. ⓐ $f(x) = \frac{1+2\ln x}{x}$, $x>0$

$$f'(x) = \frac{2\frac{1}{x}x - (1+2\ln x)}{x^2} = \frac{2-1-2\ln x}{x^2} = \frac{1-2\ln x}{x^2}$$

$$f'(x) = \frac{1-2\ln x}{x^2} = 0 \Leftrightarrow 1-2\ln x = 0 \Rightarrow 1 = 2\ln x$$

$$1 = \ln x^2$$

$$e = x^2 \Rightarrow x = \sqrt{e}$$

x	0		$\sqrt{e}$	
f'		$+$	0	$-$
f	$-\infty$	$\nearrow$	$\frac{2\sqrt{e}}{e}$	$\searrow$ 0

$f(x) \leq f(e^{1/2})$

$f(x) \leq \frac{2}{e^{1/2}}$

$f(x) \leq \frac{2}{\sqrt{e}}$

$f(x) \leq \frac{2\sqrt{e}}{e}$

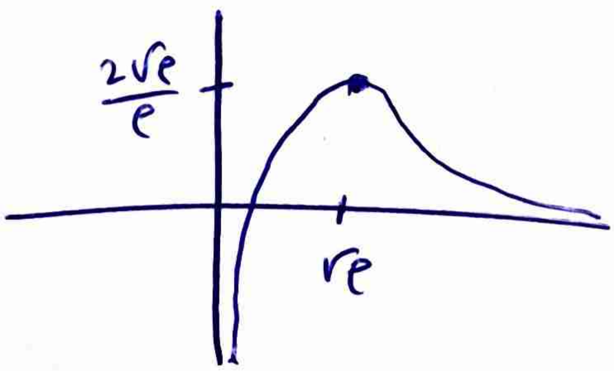
- $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{1+2\ln x}{x} =$

$$= \lim_{x\to 0^+}(1+2\ln x)\frac{1}{x} = (-\infty)(+\infty)$$

$$= -\infty$$

- $f(\sqrt{e}) = \frac{2\sqrt{e}}{e}$

- $\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} \frac{1+2\ln x}{x} = \lim\limits_{x\to+\infty} \frac{2\frac{1}{x}}{1} = 0$



$\Sigma T_f = \left(-\infty, \frac{2\sqrt{e}}{2}\right].$

$$f'(x) = \frac{1-2\ln x}{x^2}$$

$$f''(x) = \frac{-\frac{2}{x}x^2 - (1-2\ln x)\,2x}{x^4} = \frac{-2x-2x(1-2\ln x)}{x^4}$$

$$f''(x) = \frac{-2x(1+1-2\ln x)}{x^4} = \frac{-2(2-2\ln x)}{x^3}$$

$$f''(x) = \frac{-4(1-\ln x)}{x^3} = \frac{4(\ln x-1)}{x^3}$$

$\longrightarrow f''(x)=0 \Rightarrow \ln x-1=0 \Rightarrow \ln x=1 \Rightarrow \underline{\underline{x=e}}$

x	0	e	
f''	$-$		$+$
f	$\frown$		$\smile$

$\forall x\leq e$ f κοίλη

$$f(x) \leq -\frac{1}{e^2}x+\frac{4}{e}$$

$\forall x\geq e$ f κυρτή

$$f(x) \geq -\frac{1}{e^2}x+\frac{4}{e}$$

$$y-f(e)=f'(e)(x-e)$$

$$y-\frac{3}{e} = -\frac{1}{e^2}(x-e)$$

$$y-\frac{3}{e} = -\frac{1}{e^2}x+\frac{1}{e}$$

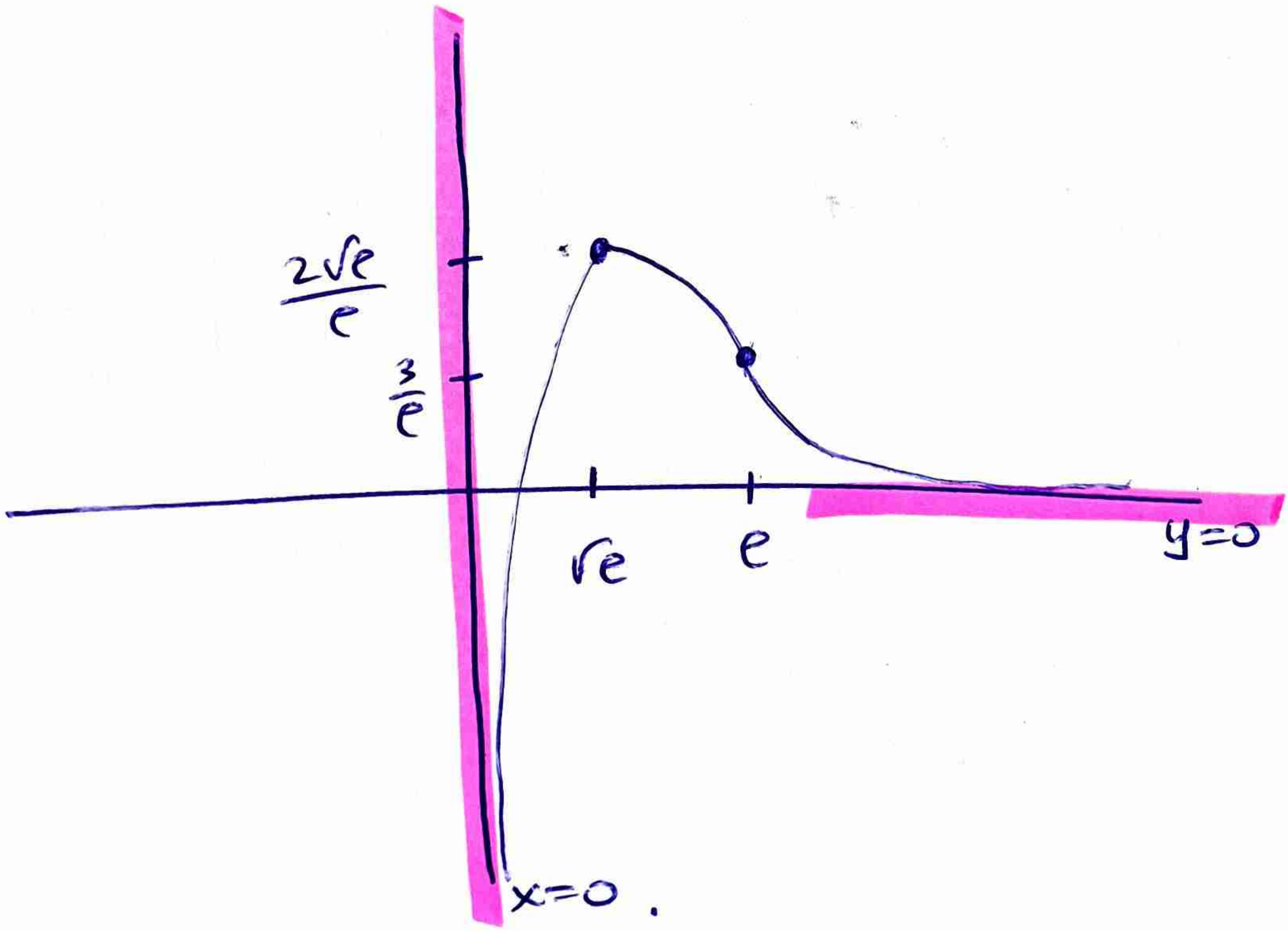
$$\boxed{y=-\frac{1}{e^2}x+\frac{4}{e}}$$

$\lim_{x \to 0^+} f(x) = -\infty$ $\qquad$ $\varepsilon_1 : x=0$ κατακόρυφη

$\lim_{x \to +\infty} f(x) = 0$ $\qquad$ $\varepsilon_2 : y=0$ οριζόντια $+\infty$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\frac{1+2\ln x}{x}}{x} = \lim_{x \to +\infty} \frac{1+2\ln x}{x^2}$$

$$= \lim_{x \to +\infty} \frac{\frac{2}{x}}{2x} = 0$$ δεν έχει πλάγια.



2. B $f(x) = \frac{1}{x^2+1}$, $x \in \mathbb{R}$.

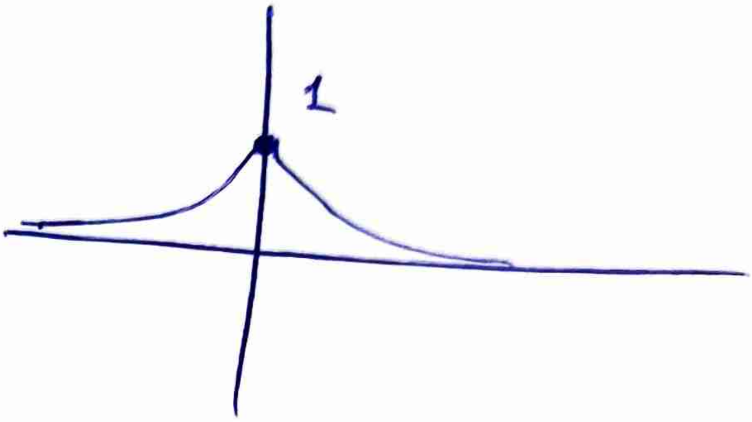
$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$f'(x) = 0$

$-2x = 0$

$x = 0$

x	$-\infty$	0	$+\infty$
f'	$+$		$-$
f	$0 \nearrow$	1	$\searrow 0$



$$\lim_{x \to -\infty} f(x) = 0$$

$$\lim_{x \to +\infty} f(x) = 0$$

$f(0) = 1$

$$f''(x) = \frac{-2 \cdot (x^2+1)^2 - (-2x)\, 2(x^2+1) \cdot 2x}{(x^2+1)^4}$$

$$f''(x) = \frac{-2(x^2+1) + 8x^2}{(x^2+1)^3}$$

$$f''(x) = \frac{-2x^2 - 2 + 8x^2}{(x^2+1)^3} = \frac{6x^2 - 2}{(x^2+1)^3} = 2\,\frac{(3x^2-1)}{(x^2+1)^3}$$

$f''(x)=0 \Rightarrow 3x^2-1=0$

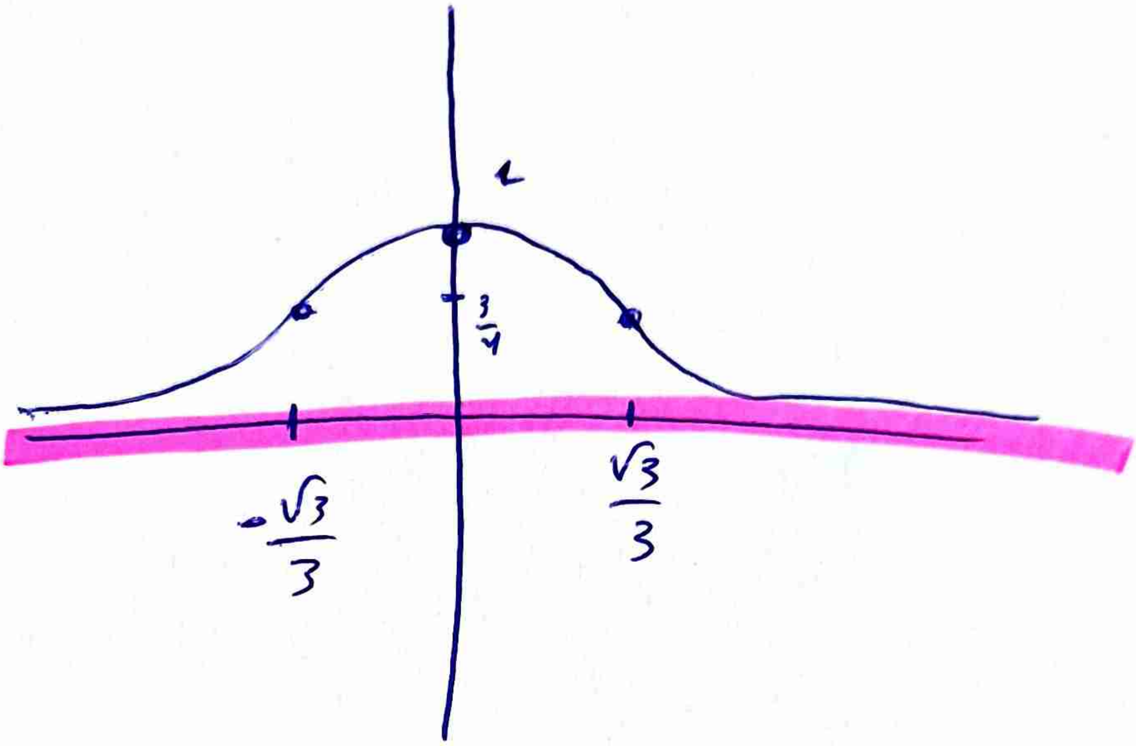
$x^2=\frac{1}{3}$ $\quad x=\pm\frac{\sqrt{3}}{3}$

x		$-\sqrt{3}/3$		$\sqrt{3}/3$	
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

Αφού $D_f=\mathbb{R}$ δεν έχω κατακόρυφη

Αφού $\lim\limits_{x\to-\infty} f(x)=0$ και $\lim\limits_{x\to+\infty} f(x)=0$ η $\varepsilon: y=0$ οριζόντια $\pm\infty$

Δεν έχω πλάγια.


1
3/4
-√3/3
√3/3

4. (B) $f(x) = x + \ln(x^2+1) \qquad x \in \mathbb{R}.$

$$f'(x) = 1 + \frac{2x}{x^2+1} = \frac{x^2+1+2x}{x^2+1} = \frac{(x+1)^2}{x^2+1} \geq 0$$

$f \nearrow$

x	$-\infty$	$+\infty$
f	$-\infty$ $\nearrow$	$+\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left[x + \ln(x^2+1)\right] = \lim_{x\to-\infty} x\left(1 + \overset{0}{\cancel{\frac{\ln(x^2+1)}{x}}}\right)$$

$$= -\infty$$

$$\rightarrow \lim_{x\to-\infty} \frac{\ln(x^2+1)}{x} = \lim_{x\to-\infty} \frac{2x}{x^2+1} = \lim_{x\to-\infty} \frac{2}{2x} = 0$$

$\Sigma T_f = \mathbb{R}.$

$$\lim_{x\to+\infty} f(x) = +\infty$$

$$f'(x) = 1 + \frac{2x}{x^2+1}.$$

$$f''(x) = \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

$$f''(x) = 2\,\frac{1-x^2}{(x^2+1)^2}$$

$f''(x) = 0 \Rightarrow 1 - x^2 = 0 \qquad x = 1 \qquad x = -1$

x		-1		1	
f''	$-$		$+$		$-$
f	$\cap$		$\cup$		$\cap$

$$\left.\begin{array}{l}\lim\limits_{x\to-\infty} f(x) = -\infty \\ \lim\limits_{x\to+\infty} f(x) = +\infty\end{array}\right\}$$ Δεν έχει οριζόντια

Αφού $D_f = \mathbb{R}$ δεν έχει

κατακόρυφη.

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x+\ln(x^2+1)}{x} =$$

$$= \lim_{x\to+\infty} 1 + \cancel{\frac{2x}{x^2+1}}^{\,0} = 1$$

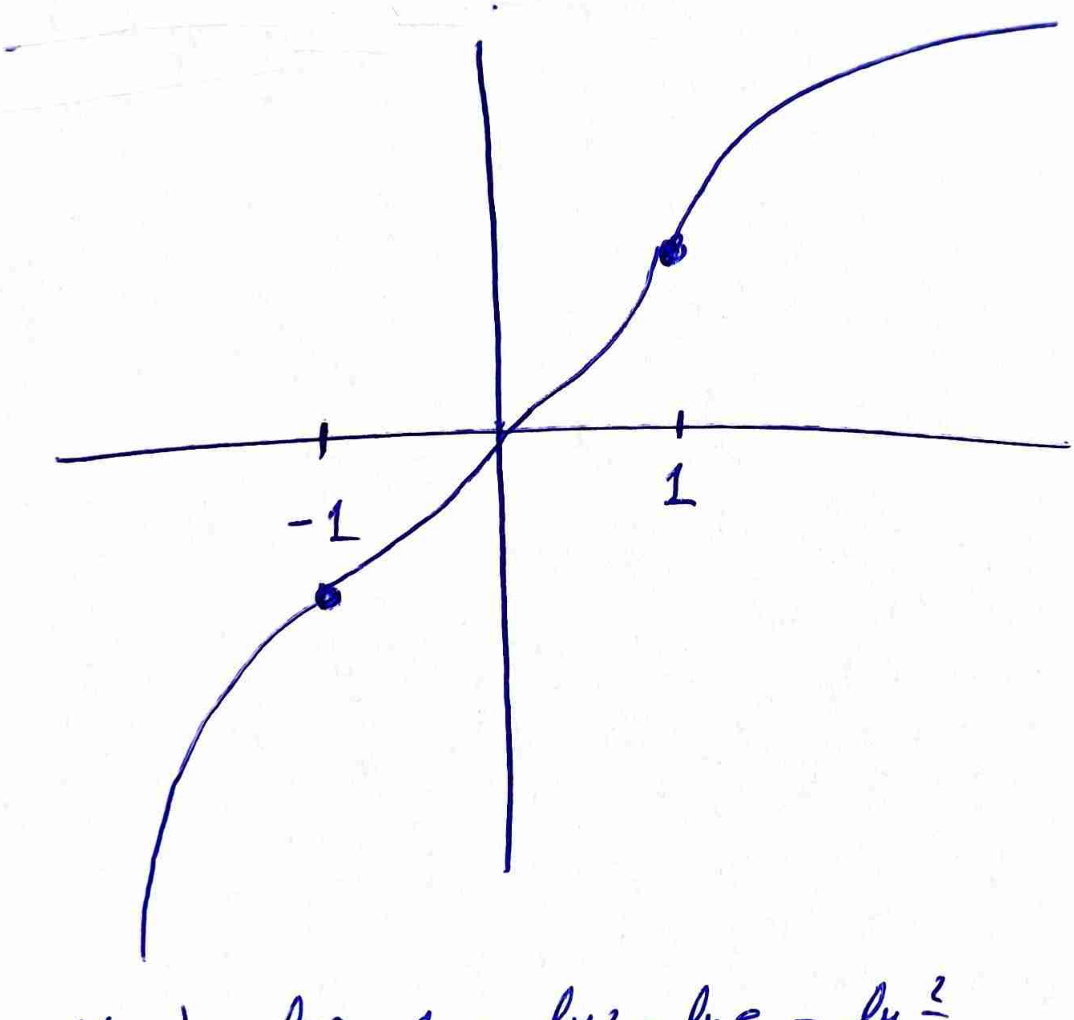
$$\longrightarrow \lim_{x\to+\infty} \frac{2x}{x^2+1} = \lim_{x\to+\infty} \frac{2}{2x} = 0$$

$$\lim_{x\to+\infty} f(x)-x = \lim_{x\to+\infty} \cancel{x}+\ln(x^2+1)-\cancel{x}$$

$$= +\infty$$

Δεν έχει πλάγια στο $+\infty$, $-\infty$

Ομοίως στο $-\infty$

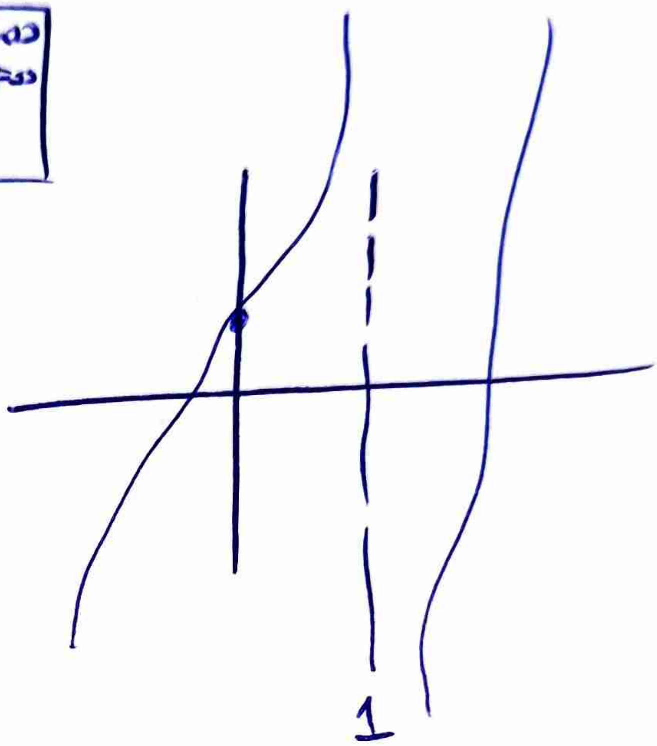


$$f(-1) = \ln 2 - 1 = \ln 2 - \ln e = \ln\frac{2}{e}$$

2. (i) $f(x) = x - \dfrac{1}{x-1}, \quad x \neq 1.$

$$f'(x) = 1 - \frac{-1}{(x-1)^2} = 1 + \frac{1}{(x-1)^2} > 0$$

x	$-\infty$		1		$+\infty$
f	$-\infty$	$\nearrow$	$+\infty \;\Vert\; -\infty$	$\nearrow$	$+\infty$



$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to 1^-} f(x) = +\infty$$

$$\lim_{x \to 1^+} f(x) = -\infty$$

$\Sigma T_f = \mathbb{R}$.

$$f''(x) = \frac{-2(x-1)}{(x-1)^4} = \frac{-2}{(x-1)^3}$$

x		1	
f''	$+$	$\Vert$	$-$
f	$\cup$	$\Vert$	$\cap$

$$\lim_{x\to 1^-} f(x) = +\infty$$ ε: $x=1$ κατακορυφη

$$\left.\begin{aligned} &\lim_{x\to -\infty} f(x) = -\infty \\ &\lim_{x\to +\infty} f(x) = +\infty \end{aligned}\right\}$$ Δεν εχει οριζοντια

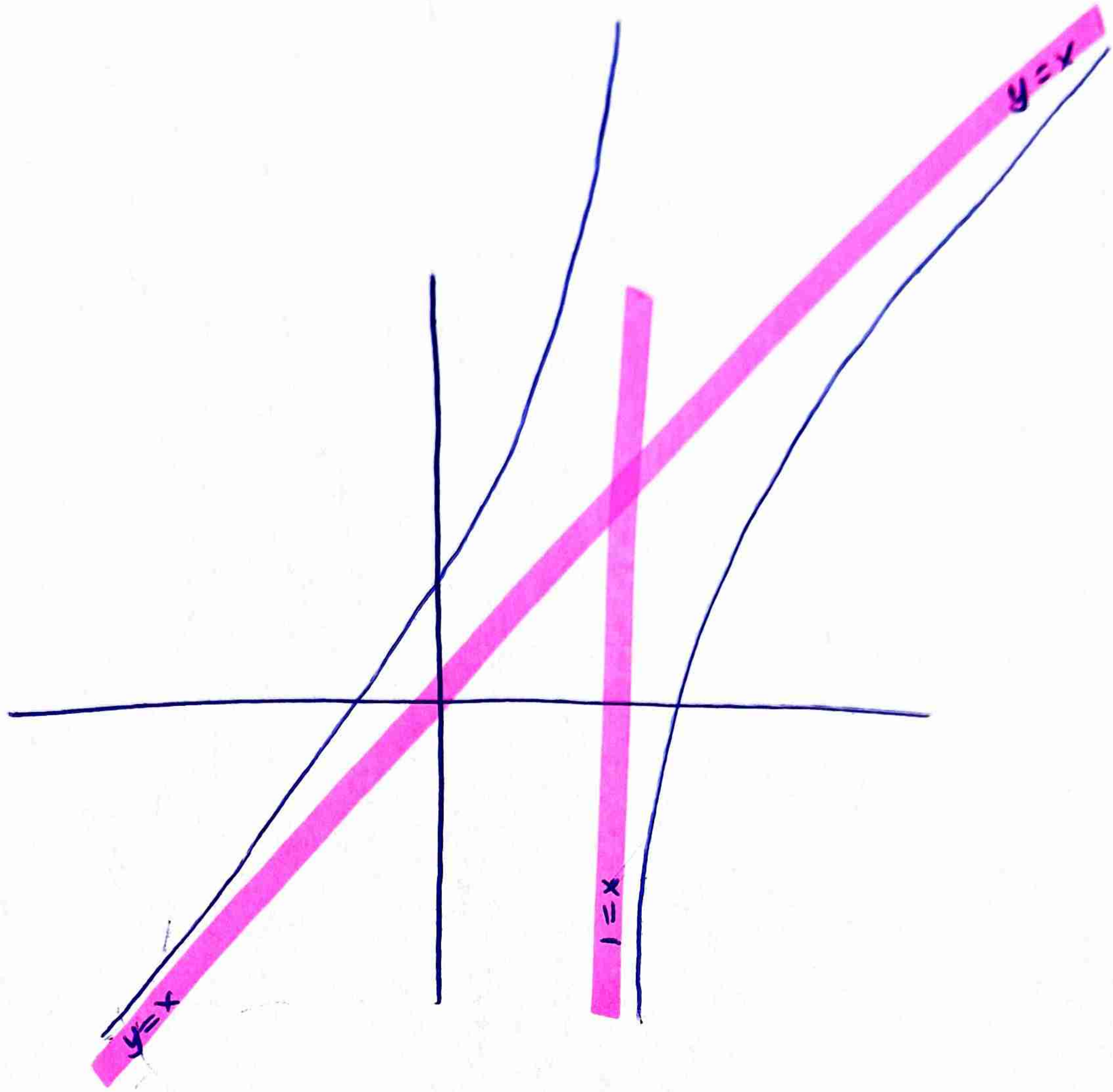
$$\lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to +\infty} \frac{x - \frac{1}{x-1}}{x} =$$

$$= \lim_{x\to +\infty} 1 + \frac{1}{(x-1)^2} = \underline{1}$$

$$\lim_{x\to +\infty} \left(f(x) - x\right) = \lim_{x\to +\infty} \cancel{x} - \frac{1}{x-1} - \cancel{x} = 0$$

$$\varepsilon: y = x$$ πλαγια στο $\pm\infty$

Στο $-\infty$ ομοια.



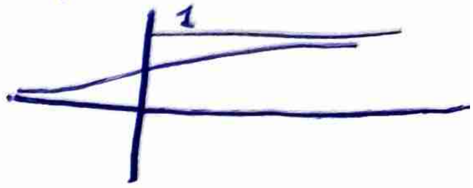
y=x
y=x
x=1

3. (a) $f(x) = \dfrac{e^x}{1+e^x}, \; x \in \mathbb{R}.$

$$f'(x) = \frac{e^x(1+e^x) - e^x e^x}{(1+e^x)^2} = \frac{e^x + \cancel{e^x e^x} - \cancel{e^x e^x}}{(1+e^x)^2}$$

$$f'(x) = \frac{e^x}{(1+e^x)^2} > 0 \qquad f \nearrow$$

x	$-\infty$	$+\infty$
$f(x)$	$0 \nearrow$	1



$$\lim_{x \to -\infty} f(x) = 0$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^x}{1+e^x} = \lim_{x \to +\infty} \frac{e^x}{e^x} = 1$$

$$f''(x) = \frac{e^x(1+e^x)^2 - e^x\, 2(1+e^x)e^x}{(1+e^x)^4}$$

$$f''(x) = \frac{e^x(1+e^x) - 2e^x e^x}{(1+e^x)^3} = \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3}$$

$$f''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3} = \frac{e^x(1-e^x)}{(1+e^x)^3}$$

$\to f''(x) = 0 \Rightarrow 1 - e^x = 0$

$1 = e^x \qquad x = 0$

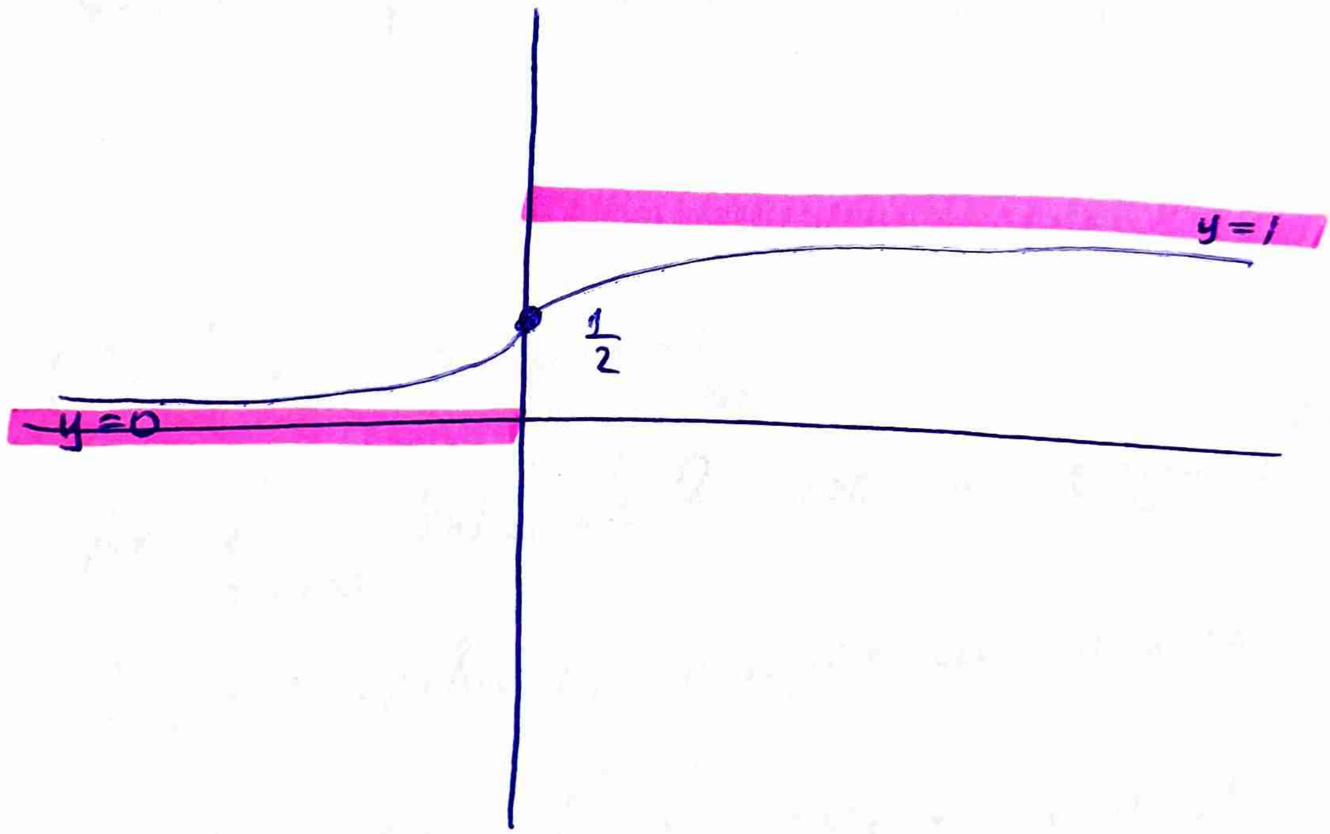
x		0	
f''	$+$		$-$
f	$\cup$		$\cap$

Αφού $D_f = \mathbb{R}$ δεν έχω κατακόρυφη.

$\lim_{x \to -\infty} f(x) = 0$ $\quad$ $\varepsilon_1 : y = 0$ οριζόντια στο $-\infty$

$\lim_{x \to +\infty} f(x) = 1$ $\quad$ $\varepsilon_2 : y = 1$ οριζόντια στο $+\infty$.

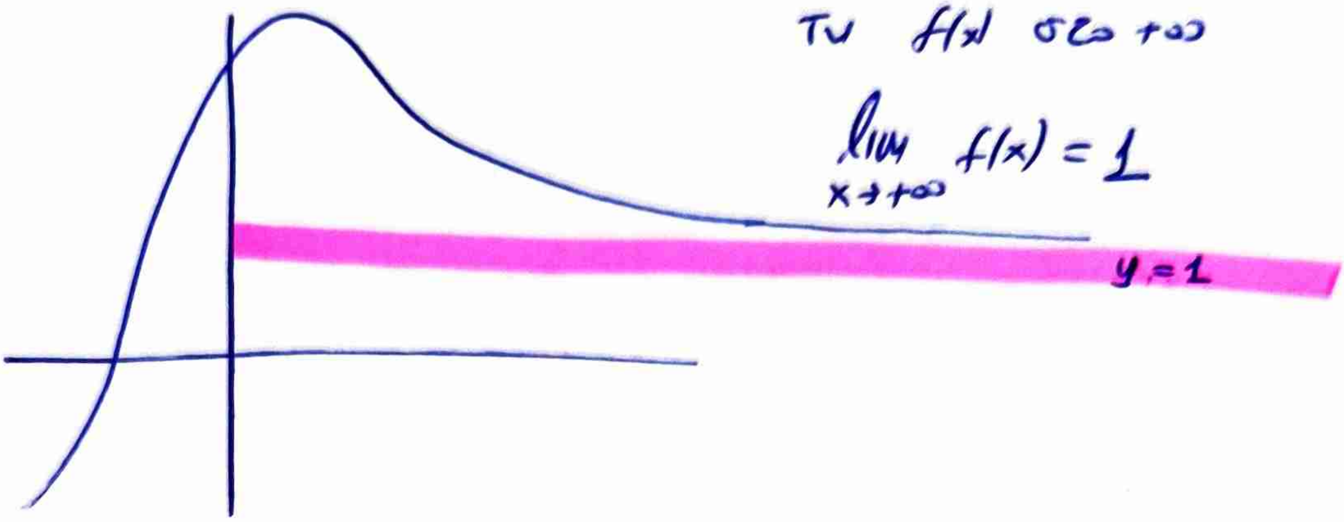
Δεν έχει πλάγια.



Ασυμπτωτες

Η εξ $y=1$ λεγεται οριζοντια ασυμπτωτη της $f(x)$ στο $+\infty$

$$\lim_{x\to+\infty} f(x) = 1$$

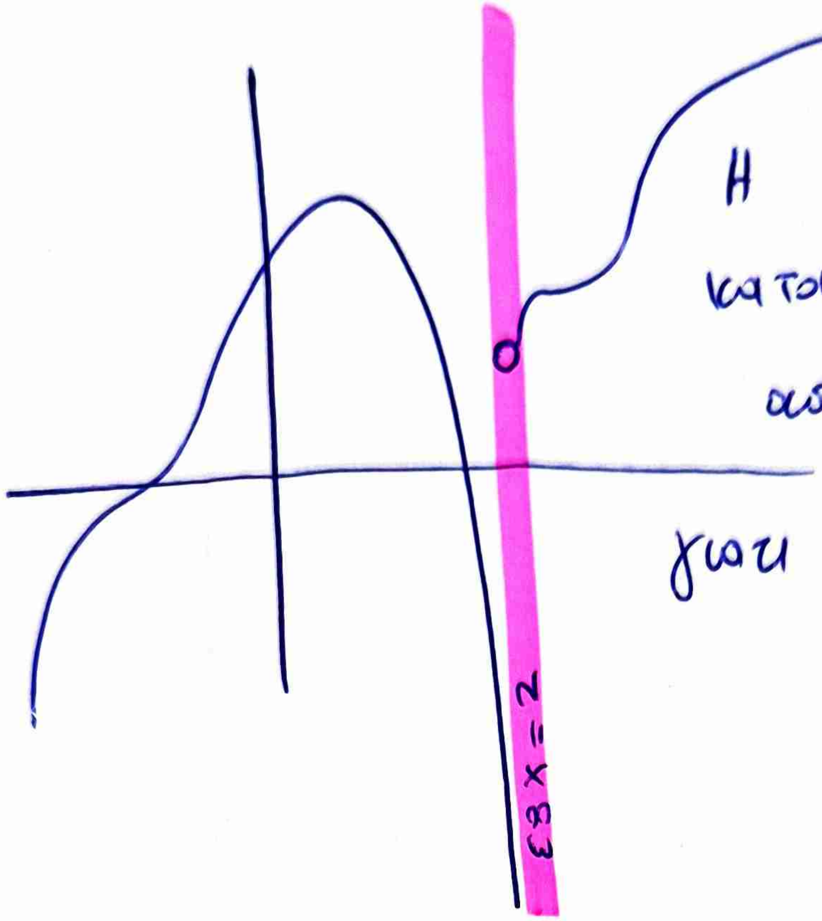


Τις οριζοντιες ασυμπτωτες τις ψαχνω μονο στα απειρα.

Οριζοντια ασυμπτωτη

Αν $\lim_{x\to+\infty} f(x) = l \in \mathbb{R}$ τοτε η εξ $y=l$ λεγεται οριζοντια ασυμπτωτη στο $+\infty$

Αν $\lim_{x\to-\infty} f(x) = l \in \mathbb{R}$ τοτε η εξ $y=l$ λεγεται οριζοντια ασυμπτωτη στο $-\infty$.



Η ε: x=2 λέγεται κατακόρυφη ασύμπτωτη.

γιατί $\lim_{x \to 2^-} f(x) = -\infty$

Αν τουλάχιστον ένα από τα όρια

$\lim_{x \to x_0^-} f(x) = \infty$ ή $\lim_{x \to x_0^+} f(x) = \infty$

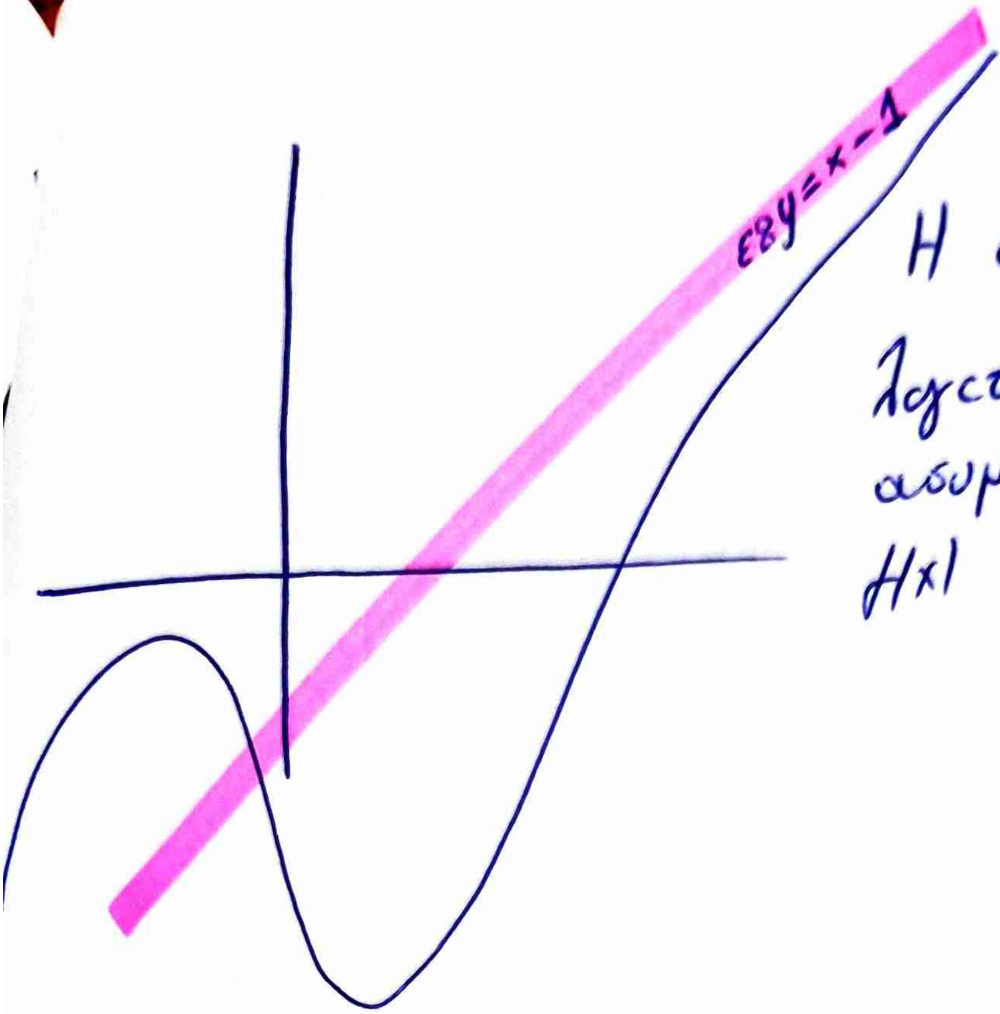
τότε λέμε ότι η ευθεία ε: $x = x_0$

είναι κατακόρυφη ασύμπτωτη.

Τις κατακόρυφες ασύμπτωτες τις

ψάχνω σε ανοιχτά άκρα του D_f

ή σε σημεία ασυνέχειας!



Η ε: y = x − 1 λέγεται πλάγια ασύμπτωτη της f(x) στο +∞.

Αν έχει οριζόντια ασύμπτωτη στο +∞ τότε δεν έχει πλάγια.

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \alpha \in \mathbb{R}^*$$

$$\lim_{x \to +\infty} f(x) - \alpha x = b \in \mathbb{R}$$

$\varepsilon: y = \alpha x + b$ πλάγια ασύμπτωτη

Ενοτητα 27

$D_{f^{-1}} = \Sigma T_f$

7. (B) $f(x) = \begin{cases} \dfrac{e^x}{\ln x - x}, & x>0 \\ 0, & x=0 \end{cases}$

$$f'(x) = \frac{e^x(\ln x - x) - e^x\left(\frac{1}{x} - 1\right)}{(\ln x - x)^2} = \frac{\overset{(+)}{e^x}\overset{(-)}{\left(\ln x - x - \frac{1}{x} + 1\right)}}{\underset{(+)}{(\ln x - x)^2}} < 0$$

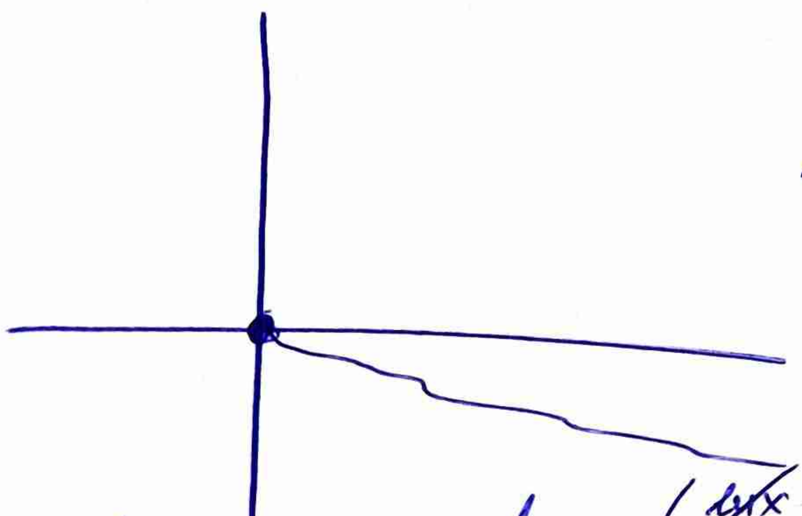
$f \downarrow$

• $\ln x \le x - 1$

$\ln x - x + 1 \le 0$

$\ln x - x + 1 - \frac{1}{x} < 0$

$$\lim_{x\to+\infty} \frac{e^x}{\ln x - x} = \lim_{x\to+\infty} \frac{e^x}{\frac{1}{x} - 1} = -\infty$$



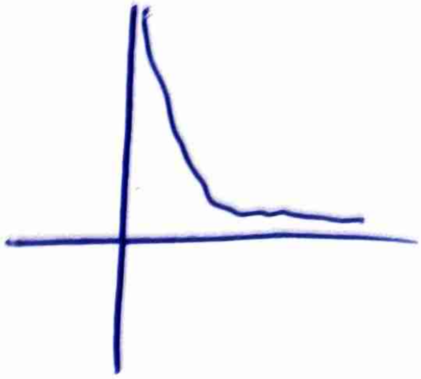
$\Sigma_T f = (-\infty, 0]$.

$\rightarrow \lim_{x\to+\infty} \ln x - x = \lim_{x\to+\infty} x\left(\frac{\ln x}{x} - 1\right) = -\infty$

$\rightarrow \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$

ⓐ $f(x) = \begin{cases} \dfrac{x-1}{x\ln x}, & 0<x\neq 1 \\ 1, & x=1 \end{cases}$

$D_f = (0, +\infty)$



$$f'(x) = \frac{x\ln x - (x-1)(\ln x + 1)}{x^2 \ln^2 x}$$

$$f'(x) = \frac{\cancel{x\ln x} - \cancel{x\ln x} - x + \ln x + 1}{x^2 \ln^2 x}$$

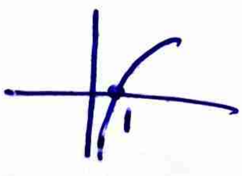
$$f'(x) = \frac{\ln x - x + 1}{x^2 \ln^2 x} \leq 0 \qquad f\downarrow$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-1}{x\ln x} = \frac{-1}{0} = +\infty.$$

⊖ τείνει στο 0^+

$$\to \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0^+} -\frac{x^{\cancel{2}}}{\cancel{x}} = 0$$



$$\lim_{x\to +\infty} \frac{x-1}{x\ln x} = \lim_{x\to +\infty} \frac{1}{\ln x + 1} = 0.$$