Άσκηση

$$f(x) = \begin{cases} \eta\mu x, & -\pi \le x \le 0 \\ \ln(x+1), & x > 0. \end{cases}$$

- $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \eta\mu x = 0$

- $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \ln(x+1) = 0$

Άρα $\lim\limits_{x \to 0} f(x) = 0$ και $f(0) = 0$

Συνεχής στο 0!

$$\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{\eta\mu x - 0}{x-0} = \lim_{x \to 0^-} \frac{\eta\mu x}{x} = 1$$

$$\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{\ln(x+1)}{x} = \lim_{x \to 0} \frac{1}{x+1} = 1$$

$f'(0) = 1$

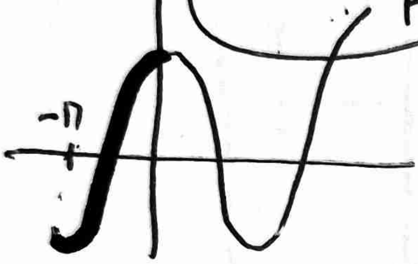
$-\pi \le x \le 0$

$f_1(x) = ημx$

$f_1'(x) = συνx$

$\rightarrow συνx = 0$

$x = -\frac{\pi}{2}$ Κ.Σ



$x > 0$

$f_2(x) = \ln(x+1)$

$f_2'(x) = \frac{1}{x+1}$

$\rightarrow \frac{1}{x+1} = 0$

$1 = 0$ Αδύν

Κρίσιμα Σημεία

$x = -\frac{\pi}{2}$.

Πιθανές θέσεις ακροτάτων

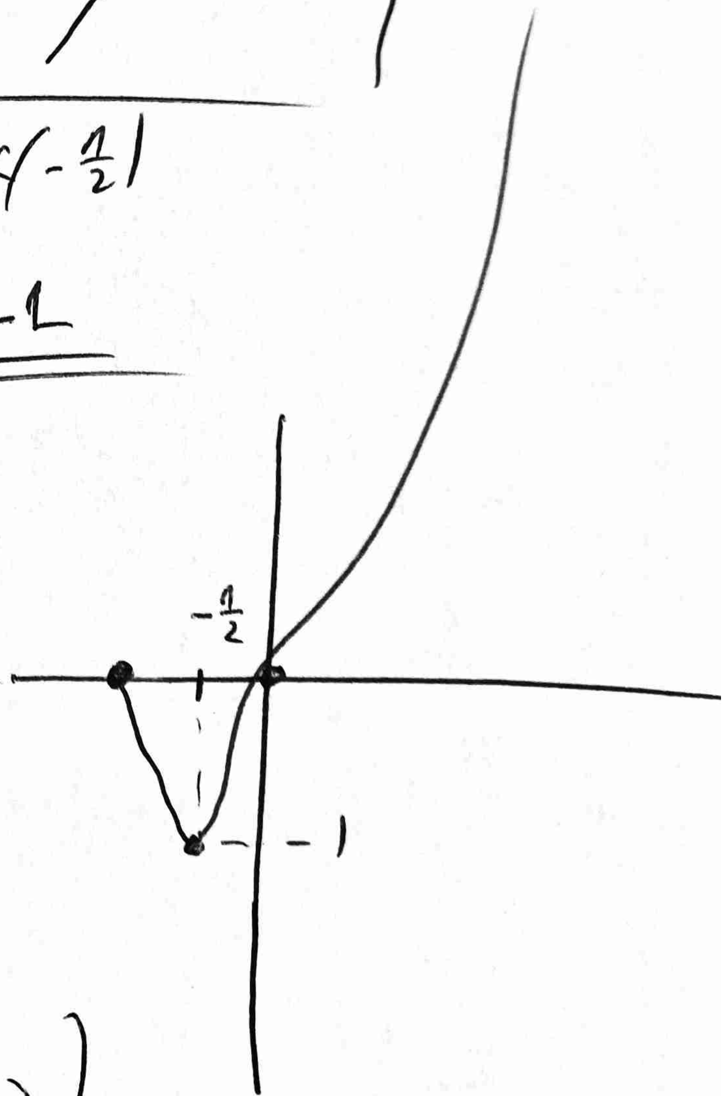
Κ.Σ	Άκρα
$-\frac{\pi}{2}$	$-\pi$

x	$-\pi$		$-\frac{1}{2}$		0	
f_1'		$-$	0	$+$		//////
f_2'		//////		//////		$+$
f'		$-$		$+$		$+$
f	0	$\searrow$	-1	$\nearrow$		$\nearrow$

$$f(x) \geq f\left(-\frac{1}{2}\right)$$

$$\underline{\underline{f(x) \geqslant -1}}$$

$$f(-\pi) = 0$$



$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\sum T_f = [-1, +\infty).$$

Ενότητα 24

24. $f(1)=f(2)=0$ $\qquad f''(x)<0$

(α) Νδο $\exists\ \xi\in(1,2)$ τ.ω $f'(\xi)=0$

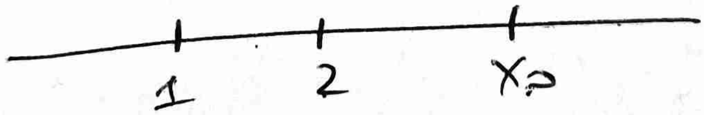
Αφού $f(1)=f(2)$

Rolle $f'(\xi)=0$

(β) Εξίσωση $f(x)=0$

Προφανείς ρίζες $x=1,\ x=2$.

Έστω ότι έχει 3 ρίζες.



$$f(1)=f(2)=f(x_0)=0$$

$f'(\xi)=0 \qquad f'(\kappa)=0$

Rolle

$f''(\rho)=0$ Άτοπο! Έχω το πολύ 2 ρίζες

33. $f: (0,+\infty) \to \mathbb{R}$

$f(1) = 0$

$x^2 f'(x) > 1 - x f(x)$

(i) N.δ.ο. $f(x) < \dfrac{\ln x}{x}$ $\quad x \in (0,1)$.

$\underbrace{x f(x) - \ln x}_{g(x)} < 0$

$g'(x) = f(x) + x f'(x) - \dfrac{1}{x} = \dfrac{x f(x) + x^2 f'(x) - 1}{x}$

$g'(x) = \dfrac{\overset{+}{x f(x) + x^2 f'(x) - 1}}{x}$

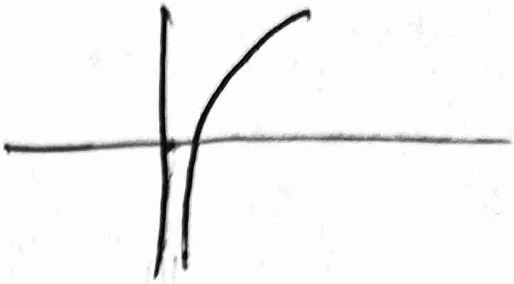
$g'(x) > 0$

$g \nearrow$

Αν $x < 1$ $\Rightarrow g(x) < g(1) \Rightarrow x f(x) - \ln x < 0$

$f(x) < \dfrac{\ln x}{x}$

(B) $\lim_{x\to 0} f(x)$



$$f(x) = \frac{\ln x}{x}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln x}{x}$$

$$\to \lim_{x\to 0^+} \ln x \quad \frac{1}{x} = (-\infty)(+\infty)$$

$$= -\infty$$

$$\lim_{x\to 0^+} f(x) = -\infty$$

$$\underline{\lim_{x\to 0^+} f(x) = -\infty}$$

36. $f(x) = (x^2+1)e^x - 1 \quad , x \in \mathbb{R}$

(α)

$f'(x) = 2x\,e^x + (x^2+1)\,e^x$

$f'(x) = e^x\,(x^2+2x+1)$

$f'(x) = e^x\,(x+1)^2 \geq 0$

$f \nearrow$

(β) $f(e^x - 1) + f(2x) = f(x)$

προφανής ρίζα $\underline{x=0}$

$$\boxed{f(e^x-1) + f(2x) - f(x) = 0}$$

$x>0$

- $x>0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1 \Rightarrow e^x - 1 > 0 \Rightarrow$

  $f(e^x-1) > f(0)$

  $\underline{f(e^x-1) > 0}$

- $x < 2x \Rightarrow f(x) < f(2x)$

  $f(2x) - f(x) > 0$

(+)

$f(e^x-1) + f(2x) - f(x) > 0$.

Συμπέρασμα

Αν $x > 0$ το $f(e^x-1) + f(2x) - f(x) > 0$

$x < 0$

- $x < 0 \Rightarrow e^x < e^0 \Rightarrow e^x < 1 \Rightarrow e^x - 1 < 0$

$f(e^x-1) < f(0)$

$f(e^x-1) < 0$

- $x > 2x \Rightarrow f(x) > f(2x) \Rightarrow f(2x) - f(x) < 0$

$f(e^x-1) + f(2x) - f(x) < 0$

Άρα προφανή ρίζα $x=0$

37. $f(x) = e^{-x} + x - 1$

(α1) $f(x) < f(x^2) + \ln x$

$f(x) - f(x^2) - \ln x < 0$

προφανώς
ρίζα
$x=1$

$f(x) - f(x^2) - \ln \frac{x^2}{x} < 0$

$f(x) - f(x^2) - \ln x^2 + \ln x < 0$

$f(x) + \ln x < f(x^2) + \ln x^2$

$\underline{g(x) = f(x) + \ln x}$ $\quad x > 0$

$g(x) < g(x^2)$

$g \nearrow$

$x < x^2$

$\underline{1 < x}$

Μονοτονία $f(x)$

$f'(x) = -e^{-x} + 1$

$\to -e^{-x} + 1 = 0$

$e^{-x} = 1$

$x = 0$

x		0	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

• $e^x \geq x + 1$

$e^{-x} \geq -x + 1$

$e^{-x} + x - 1 \geq 0$

$\underline{f(x) \geq 0}$

• $\forall x>0$ η $f \nearrow$

$$\left.\begin{array}{l} \bullet\ x_1<x_2 \Rightarrow f(x_1)<f(x_2) \\ \bullet\ x_1<x_2 \Rightarrow \ln x_1 < \ln x_2 \end{array}\right\} \oplus \quad g \nearrow$$

Συμπέρασμα.

Όταν $x>1$ $\quad f(x)<f(x^2)+\ln x$,

(B) $e^{x-1}-1+f(x^5)=f(x^3)$ στο $[1,+\infty)$.

Προφανής ρίζα το $x=1$

$$\underbrace{e^{x-1}-1}_{\oplus}+f(x^5)-f(x^3)=0$$

• $x>1 \Rightarrow x-1>0 \Rightarrow e^{x-1}>e^0 \Rightarrow e^{x-1}>1$

$e^{x-1}-1>0$

• $x>1 \Rightarrow x^3<x^5 \Rightarrow f(x^3)<f(x^5)$ $\quad \oplus$

$f(x^5)-f(x^3)>0$

$$e^{x-1}-1+f(x^5)-f(x^3)>0$$

|

40. $f(x) = e^x(x^2+3)$

(a) $f'(x) = e^x(x^2+3) + e^x \cdot (2x)$

$$\boxed{f'(x) = e^x(x^2+2x+3)}$$

$f''(x) = e^x(x^2+2x+3) + e^x(2x+2)$

$f''(x) = \overset{\oplus}{e^x(x^2+4x+5)} > 0$

$\Delta = -4 < 0$

$f' \nearrow$

(B) $f(2x^2) + 2f(x+2) > f(2x+4) + 2f(x^2)$

$(0, +\infty)$

$f(2x^2) - 2f(x^2) > f(2x+4) - 2f(x+2)$

$$\boxed{g(x) = f(2x) - 2f(x)}$$

$g(x^2) = g(x+2)$ $\quad$ $x=2$ $\quad$ $x=-1$

g 1-1

$x^2 = x+2$ $\qquad$ $x^2-x-2=0$

$$g'(x) = f'(2x)\,2 - 2f'(x)$$

$$g'(x) = 2f'(2x) - 2f'(x)$$

$$g'(x) = 2\left(f'(2x) - f'(x)\right) > 0$$

$g \nearrow$

$\rightarrow \quad x < 2x \overset{f' \nearrow}{\Rightarrow} f'(x) < f'(2x)$

$$f'(2x) - f'(x) > 0$$

(8) $f(2x^4+6) - f(2x^2+6) = 2f(x^4+3) - 2f(x^2+3)$

$$f(2x^4+6) - 2f(x^4+3) = f(2x^2+6) - 2f(x^2+3)$$

$$g(x^4+3) = g(x^2+3)$$

g 81–1

$$x^4+3 = x^2+3$$

$$x^4 = x^2$$

$$x^4 - x^2 = 0 \qquad x=0$$

$$\qquad x=1$$

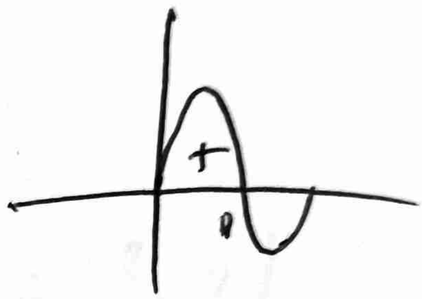
$$x(x^2-1) = 0 \qquad x=-1$$

45. $f(x) = \dfrac{\eta\mu x}{x} \quad x \in (0, 1]$

(α) $f'(x) = \dfrac{x\,\sigma\upsilon\nu x - \eta\mu x}{x^2}$

$\boxed{\varphi(x) = x\,\sigma\upsilon\nu x - \eta\mu x}$ $\qquad \varphi(0) = 0$

$\varphi'(x) = \cancel{\sigma\upsilon\nu x} - x\,\eta\mu x - \cancel{\sigma\upsilon\nu x}$



$\varphi'(x) = -x\,\eta\mu x < 0$

(+) (+)

x	0	1
φ'	$-$	
φ	$\searrow$ $-$	
x^2	$+$	
f'	$-$	
f	$\searrow$	

$x > 0 \Rightarrow \varphi(x) < \varphi(0)$

$\varphi(x) < 0$

B) i) Νδο $\frac{ημx}{ημy} > \frac{x}{y}$ $\forall x, y \in (0, π)$

$x < y$

$$\frac{ημx}{x} > \frac{ημy}{y}$$

$$f(x) > f(y)$$

$$f \downarrow$$

$$x < y$$

ii). Νδο $ημx < \frac{2x}{π}$ $\forall x \in \left(\frac{π}{2}, π\right)$.

$$\frac{ημx}{x} < \frac{2}{π}$$

$$f\left(\frac{π}{2}\right) = \frac{ημ\frac{π}{2}}{\frac{π}{2}} = \frac{2}{π}$$

$$f(x) < \frac{2}{π}.$$

$$f(x) < f\left(\frac{π}{2}\right)$$

$$f \downarrow$$

$$x > \frac{π}{2}$$

(γ) $\lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{\pi x}{\pi \eta\mu x - 2x} = \lim\limits_{x \to \frac{\pi}{2}^-} \pi x \dfrac{1}{\underset{(-)}{\pi \eta\mu x - 2x}}$

$= \pi \cdot \frac{\pi}{2} \cdot \dfrac{1}{0} = \frac{\pi^2}{2}(-\infty) = -\infty$

Από πριν $\quad \eta\mu x < \dfrac{2x}{\pi}$

$\cancel{\pi \eta\mu x < 2x}$

$\pi \eta\mu x - 2x < 0$

48. (α) Νδο $e^x > 1 + x + \frac{1}{2}x^2$ $\forall x > 0$.

$$\underbrace{e^x - 1 - x - \frac{1}{2}x^2}_{g(x)} > 0$$

$\boxed{\text{Αρκει νδο} \quad g(x) > 0 \quad \forall x > 0}$

$g'(x) = e^x - 1 - x \geq 0$ $\quad g \nearrow$

- $e^x \geq x + 1$

$e^x - x - 1 \geq 0$

Αν $x > 0 \Rightarrow g(x) > g(0)$

$e^x - 1 - x - \frac{1}{2}x^2 > 0$

(B) $h(x) = e^x - x$

$\varphi(x) = \frac{x^2}{2} + 1$

Αρκει νδο το συστημα.

$$\begin{cases} h'(x) = \varphi'(x) \\ h(x) = \varphi(x) \end{cases}$$

Βασικη

$$\begin{cases} e^x - 1 = \frac{1}{2} 2x \Rightarrow e^x = x + 1,\quad x = 0 \\ e^x - x = \frac{x^2}{2} + 1 \end{cases}$$

$e^0 - 0 = 0 + 1$

$1 = 1$ ✓

Για $x=0$ οι h και φ διχονει και εφαπτονται στο κοινο σημειο

Επομενο Μαθημα

Παλια

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