# Ενότητα 23

8. (α) $f(x) = \dfrac{x}{(x-1)^2}$ $\quad \underline{\underline{x \neq 1}}$

$$f'(x) = \frac{(x-1)^2 - x \cdot 2(x-1)}{(x-1)^4} = \frac{x-1-2x}{(x-1)^3} = \frac{-1-x}{(x-1)^3}$$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{-1-x}{(x-1)^3} = 0 \Rightarrow -1-x = 0$

$\underline{\underline{x = -1}}$

| $x$ | $-\infty$ | $-1$ | | $1$ | $+\infty$ |
|---|---|---|---|---|---|
| $-1-x$ | $+$ | $0$ | $-$ | ‖ | $-$ |
| $(x-1)^3$ | $-$ | | $-$ | ‖ | $+$ |
| $f'$ | $-$ | | $+$ | ‖ | $-$ |
| $f$ | ↘ | | ↗ | ‖ | ↘ |

Η $f$ γν. φθίνουσα $(-\infty, -1]$ και $(1, +\infty)$

Η $f$ γν. αύξουσα $[-1, 1)$,

(γ) $f(x) = \dfrac{2-x^2}{x^4}$ , $x \neq 0$

$$f'(x) = \frac{-2x \cdot x^4 - (2-x^2) \cdot 4x^3}{x^8} = \frac{-2x^2 - 4(2-x^2)}{x^5}$$

$$f'(x) = \frac{-2x^2 - 8 + 4x^2}{x^5} = \frac{2x^2 - 8}{x^5} = \frac{2(x^2-4)}{x^5}$$

$$\boxed{f'(x) = \frac{2(x^2-4)}{x^5}}$$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{2(x^2-4)}{x^5} = 0 \Rightarrow 2(x^2-4) = 0$

$x = 2$

$x = -2$

| $x$ | | $-2$ | | $0$ | | $2$ | |
|---|---|---|---|---|---|---|---|
| $x^2-4$ | $+$ | $0$ | $-$ | ‖ | $-$ | $0$ | $+$ |
| $x^5$ | $-$ | | $-$ | ‖ | $+$ | | $+$ |
| $f'$ | $-$ | | $+$ | ‖ | $-$ | | $+$ |
| $f$ | ↘ | | ↗ | ‖ | ↘ | | ↗ |

9. ⓡ $f(x) = \sqrt{x^2-9}$ $D_f = (-\infty, -3] \cup [3, +\infty)$

Прили $x^2 - 9 \geq 0$

| $x$ | | $-3$ | $3$ |
|---|---|---|---|
| $x^2-9$ | $+$ | $-$ | $+$ |

$$f'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-9}} = \frac{x}{\sqrt{x^2-9}}$$

$$f'(x) = 0 \Rightarrow \frac{x}{\sqrt{x^2-9}} = 0 \Rightarrow \underline{\underline{x = 0}}$$

| $x$ | | $-3$ | $0$ | $3$ | |
|---|---|---|---|---|---|
| $x$ | $-$ | $-$ | $0$ $+$ | | $+$ |
| $\sqrt{x^2-9}$ | $+$ | $+$ | $+$ | | $+$ |
| $f'$ | $-$ | | | | $+$ |
| $f$ | $\searrow$ | | | | $\nearrow$ |

$x \leq -3$

$f(x) \geq f(-3)$

$f(x) \geq 0$

$x \geq 3$

$f(x) \geq f(3)$

$f(x) \geq 0$

9. ⓐ $f(x)=\sqrt{x^2+4}\quad , x\in\mathbb{R}$

$$f'(x)=\frac{1}{2\sqrt{x^2+4}}\,(x^2+4)'$$

$$f'(x)=\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}}$$

$$\boxed{f'(x)=\frac{x}{\sqrt{x^2+4}}}$$

$\longrightarrow f'(x)=0 \Rightarrow \dfrac{x}{\sqrt{x^2+4}}=0 \Rightarrow \underline{\underline{x=0}}$

| $x$ | | 0 | |
|---|---|---|---|
| $x$ | $-$ | | $+$ |
| $\sqrt{x^2+4}$ | $+$ | | $+$ |
| $f'$ | $-$ | | $+$ |
| $f$ | $\searrow$ | | $\nearrow$ |

$f(x)\geq f(0)$

$$\underline{\underline{f(x)\geq 2}}$$

ε) $f(x) = \sqrt{x} + \sqrt{2-x}$

Πρέπει $x \geq 0$ και $2 - x \geq 0$

$2 \geq x$

$D_f = [0,2]$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{2-x}} = \frac{\sqrt{2-x} - \sqrt{x}}{2\sqrt{x}\sqrt{2-x}}$$

$$f'(x) = 0 \Rightarrow \frac{\sqrt{2-x} - \sqrt{x}}{2\sqrt{x}\sqrt{2-x}} = 0 \Rightarrow \sqrt{2-x} - \sqrt{x} = 0$$

$\sqrt{2-x} = \sqrt{x}$

$2 - x = x$

$2 = 2x$

$x = 1$

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\sqrt{2-x} - \sqrt{x}$ | + | 0 | − |
| $2\sqrt{x}\sqrt{2-x}$ | + | | + |
| $f'$ | + | | − |
| $f$ | ↗ | | ↘ |

$f(x) \leq f(1)$

$f(x) \leq 2$

**10.** (a) $f(x) = e^x - x + 3$ , $x \in R$.

$$f'(x) = e^x - 1$$

$\to f'(x) = 0 \Rightarrow e^x - 1 = 0$

$e^x = 1$

$x = 0$

| $x$ | | $0$ |
|---|---|---|
| $f'$ | $-$ | $+$ |
| $f$ | $\searrow$ | $\nearrow$ |

$f(x) \geq f(0)$

$f(x) \geq 4$

(γ) $f(x) = (x-2)e^x$

$f'(x) = e^x + (x-2)e^x = e^x(1 + x - 2) \Rightarrow$

$f'(x) = (x-1)e^x$

$\to f'(x) = 0 \Rightarrow (x-1)e^x = 0 \Rightarrow x - 1 = 0$

$x = 1$

| $x$ | | $1$ |
|---|---|---|
| $x-1$ | $-$ | $+$ |
| $e^x$ | $+$ | $+$ |
| $f'$ | $-$ | $+$ |
| $f$ | $\searrow$ | $\nearrow$ |

$f(x) \geq f(1)$

$f(x) \geq -e$

ε $f(x) = \dfrac{x^2}{e^x}$ $\qquad$ $\underline{\underline{x \in \mathbb{R}}}$

$$f'(x) = \frac{2x\,e^x - x^2 e^x}{(e^x)^2} = \frac{2x - x^2}{e^x}$$

$$f'(x) = 0 \Rightarrow \frac{2x - x^2}{e^x} = 0 \Rightarrow 2x - x^2 = 0$$

$$x(2 - x) = 0$$

$x = 0$ $\qquad$ $x = 2$

| $x$ | | $0$ | | $2$ | |
|---|---|---|---|---|---|
| $2x - x^2$ | $-$ | | $+$ | | $-$ |
| $e^x$ | $+$ | | $+$ | | $+$ |
| $f'$ | $-$ | | $+$ | | $-$ |
| $f$ | $\searrow$ | | $\nearrow$ | | $\searrow$ |

11. (a) $f(x) = e^{-2x} + x - 1 \quad , x \in R.$

$$f'(x) = -2e^{-2x} + 1$$

$$\rightarrow f'(x) = 0 \quad \Rightarrow -2e^{-2x} + 1 = 0$$

$$1 = 2e^{-2x}$$

$$\frac{1}{2} = e^{-2x}$$

$$\ln\frac{1}{2} = -2x$$

$$\cancel{\ln}1 - \ln 2 = -2x$$

$$x = \frac{\ln 2}{2}$$

| $x$ | | $\frac{\ln 2}{2}$ | |
|---|---|---|---|
| $f'$ | $-$ | | $+$ |
| $f$ | $\searrow$ | | $\nearrow$ |

$$f(x) \geq f\left(\frac{\ln 2}{2}\right)$$

(γ) $f(x) = e^{2x} - 3e^x + x - 1$

$$f'(x) = 2e^{2x} - 3e^x + 1 = 2\left(e^x - 1\right)\left(e^x - \frac{1}{2}\right) =$$
$$= \left(e^x - 1\right)\left(2e^x - 1\right)$$

$\rightarrow f'(x) = 0 \Rightarrow 2e^{2x} - 3e^x + 1 = 0$

$$\underline{\underline{e^x = t}}$$

$$2t^2 - 3t + 1 = 0$$

$$\Delta = 1$$

$$t = \frac{3 \pm 1}{4} \quad \begin{matrix} 1 \\ \frac{1}{2} \end{matrix}$$

$t = 1$ $\qquad\qquad$ $t = \frac{1}{2}$

$e^x = 1$ $\qquad\qquad$ $e^x = \frac{1}{2}$

$\underline{\underline{x = 0}}$ $\qquad\qquad$ $x = \ln\frac{1}{2} = \ln 1 - \ln 2$

$$\underline{\underline{x = -\ln 2}}$$

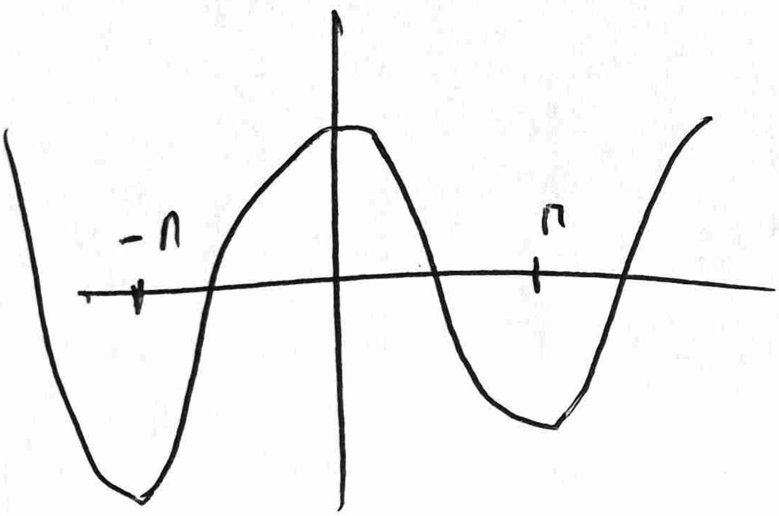
| $x$ | | $-\ln 2$ | | $0$ | |
|---|---|---|---|---|---|
| $e^x - 1$ | $-$ | | $-$ | $0$ | $+$ |
| $2e^x - 1$ | $-$ | $0$ | $+$ | | $+$ |
| $f'$ | $+$ | | $-$ | | $+$ |
| $f$ | $\nearrow$ | | $\searrow$ | | $\nearrow$ |

**15.** (a) $f(x) = \dfrac{\eta\mu x}{1+\sigma\upsilon\nu x}, \quad x \in (-\pi, \pi)$

$$f'(x) = \frac{\sigma\upsilon\nu x \cdot (1+\sigma\upsilon\nu x) - \eta\mu x(-\eta\mu x)}{(1+\sigma\upsilon\nu x)^2}$$

$$f'(x) = \frac{\sigma\upsilon\nu x + \sigma\upsilon\nu^2 x + \eta\mu^2 x}{(1+\sigma\upsilon\nu x)^2}$$

$$f'(x) = \frac{\cancel{\sigma\upsilon\nu x + 1}}{(1+\sigma\upsilon\nu x)^{\cancel{2}}} \Rightarrow f'(x) = \frac{1}{\underset{(+)}{1+\sigma\upsilon\nu x}}$$




$\sigma\upsilon\nu x$

$f'(x) > 0$

$f \nearrow$

$-1 \leq \sigma\upsilon\nu x \leq 1$

$0 \leq 1+\sigma\upsilon\nu x \leq 2$

16. ⓐ $f(x) = \begin{cases} x^2-2x+3, & x<1 \\ \frac{1}{x}+1, & x \geq 1 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2-2x+3) = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x}+1 = 2$$

$$\lim_{x\to 1} f(x) = 2 \qquad f(1)=2$$

Συνεχής.

$x<1$

$f_1(x) = x^2-2x+3$

$f_1'(x) = 2x-2$

$\to 2x-2=0$

$2x=2$

$x=1$

$x>1$

$f_2(x) = \frac{1}{x}+1$

$f_2'(x) = -\frac{1}{x^2}<0$

| $x$ | | $1$ | |
|---|---|---|---|
| $f_1'$ | $-$ | $0$ | |
| $f_2'$ | | | $-$ |
| $f'$ | $-$ | | $-$ |
| $f$ | $\searrow$ | | $\searrow$ |

**20.** α) $f(x) = e^{x-1} - \ln x \quad , x > 0 .$

$$f'(x) = e^{x-1} - \frac{1}{x} \qquad f'(1) = 0$$

$$f''(x) = e^{x-1} + \frac{1}{x^2} > 0$$

| $x$ | $0$ | | $1$ | |
|---|---|---|---|---|
| $f''$ | | $+$ | | $+$ |
| $f'$ | | $\nearrow$ $-$ | $0$ | $+$ $\nearrow$ |
| $f$ | | $\searrow$ | | $\nearrow$ |

$x < 1 \Rightarrow f'(x) < f'(1)$

$f'(x) < 0$

$x > 1 \quad f'(x) > f'(1)$

$f'(x) > 0$

$$f(x) \geq f(1)$$

$$\Rightarrow f(x) \geq 1$$

(B) $f(x) = (x-1)\ln x \qquad , x > 0$

$$f'(x) = \ln x + (x-1)\frac{1}{x}$$

$$f'(x) = \ln x + 1 - \frac{1}{x} \qquad f'(1) = 0$$

$$f''(x) = \frac{1}{x} + \frac{1}{x^2} > 0.$$

| $x$ | | 1 | |
|---|---|---|---|
| $f''$ | + | | + |
| $f'$ | $\nearrow$ − | 0 | + $\nearrow$ |
| $f$ | $\searrow$ | | $\nearrow$ |

$$f(x) \geq f(1)$$

$$\underline{f(x) \geq 0}$$

(δ) $f(x) = e^x - \frac{x^3}{6} - \frac{x^2}{2} - x - 1$

$f'(x) = e^x - \frac{1}{6} 3x^2 - \frac{1}{2} 2x - 1$

$f'(x) = e^x - \frac{1}{2} x^2 - x - 1$ $\quad f'(0) = 0$

$f''(x) = e^x - x - 1 \geq 0$

• $e^x \geq x + 1$

$e^x - x - 1 \geq 0$

| | | |
|---|---|---|
| $x$ | | $0$ |
| $f''$ | $+$ | $+$ |
| $f'$ | $\nearrow$ $-$ | $0$ $+$ $\nearrow$ |
| $f$ | $\searrow$ | $\nearrow$ |

$f(x) \geq f(0)$

## Β' τρόπος

$f''(x) = e^x - x - 1$ $\qquad f''(0) = 0$

$f'''(x) = e^x - 1$ $\qquad f'''(0) = 0$

$f''''(x) = e^x > 0$

| $x$ | | $0$ | |
|---|---|---|---|
| $f''''$ | $+$ | | $+$ |
| $f'''$ | $\nearrow$ $-$ | $0$ | $+$ $\nearrow$ |
| $f''$ | $+$ $\searrow$ | $0$ | $+$ $\nearrow$ |
| $f'$ | $\nearrow$ $-$ | $0$ | $+$ $\nearrow$ |
| $f$ | $\searrow$ | | $\nearrow$ |

$f''(x) \geq f''(0)$

$f''(x) \geq 0$

**21.** ⓐ $f(x) = x - \frac{\ln x}{x}$ $\qquad \underline{x > 0}$

$$f'(x) = 1 - \frac{\frac{1}{x}x - \ln x}{x^2}$$

$$f'(x) = 1 - \frac{1 - \ln x}{x^2}$$

$$\boxed{f'(x) = \frac{x^2 - 1 + \ln x}{x^2}}$$

$\varphi(1) = 0 \qquad \varphi(x) = x^2 - 1 + \ln x$

$\varphi'(x) = 2x + \frac{1}{x} > 0$

| $x$ | $0$ | | $1$ | |
|---|---|---|---|---|
| $\varphi'$ | | $+$ | | $+$ |
| $\varphi$ | | ↗ $-$ | $0$ | $+$ ↗ |
| $x^2$ | | $+$ | | $+$ |
| $f'$ | | $-$ | | $+$ |
| $f$ | | ↘ | | ↗ |

Συνέχεια 24

3. ⓐ $f(x) = x - \ln(x^2+1)$

$D_f = \mathbb{R}$

$$f'(x) = 1 - \frac{2x}{x^2+1} = \frac{x^2+1-2x}{x^2+1}$$

$$f'(x) = \frac{(x-1)^2}{x^2+1} \geq 0 \qquad f \nearrow$$

ⓑ i) Εξίσωση $x = \ln(x^2+1)$

$$x - \ln(x^2+1) = 0$$

$$f(x) = 0$$

$$f(x) = f(0)$$

$$f \text{ 1-1}$$

$$\underline{x=0}$$

ii). ανίσωση $f(x^2+1) > f(e^x)$

$$f \nearrow$$

$$x^2+1 > e^x$$

$\ln(x^2+1) > \ln e^x$

$\ln(x^2+1) > x \Rightarrow 0 > x - \ln(x^2+1) \Rightarrow$ $0 > f(x)$, $f(0) > f(x)$, $0 > x$

iii).

$$e^{f(x)} - 1 = f^2(x)$$

$$e^{f(x)} = f^2(x) + 1$$

$$\ln e^{f(x)} = \ln\left(f^2(x)+1\right)$$

$$f(x) - \ln\left(f^2(x)+1\right) = 0$$

$$f\left(f(x)\right) = f(0)$$

$f$ 3 1-1

$$f(x) = 0$$

$$f(x) = f(0)$$

$f$ 3 1-1

$$x = 0 .$$

(γ) Νδο $\ln(n^2+1) + f(\ln n) < n$

$$f(\ln n) < n - \ln(n^2+1)$$

$$f(\ln n) < f(n)$$

$$\ln n < n$$

- $\ln x \le x - 1$

$\ln n \le n - 1 \Rightarrow \ln n < n$ ✓

(δ) $\lim_{x\to 0} \dfrac{\ln x}{f(3x)-f(2x)} = \lim_{x\to 0^+} \dfrac{\ln x}{f(3x)-f(2x)}$

$= \lim_{x\to 0^+} \ln x \cdot \dfrac{1}{\underset{(+)}{f(3x)-f(2x)}}$

$= -\infty \cdot (+\infty) = -\infty.$

• $2x<3x \Rightarrow f(2x)<f(3x) \Rightarrow$

$f(3x)-f(2x)>0$

(ε) Αν $f(a)<0$ νδο $e^a<1$

$f(a)<0$

$f(a)<f(0)$

$f \nearrow$

$a<0$

$e^a<e^0 \Rightarrow \underline{e^a<1}$

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6. (α)

$$f(x) = x - \ln(1+e^x) \qquad x \in \mathbb{R}.$$

$$f'(x) = 1 - \frac{e^x}{1+e^x} = \frac{1+e^x - e^x}{1+e^x}$$

$$f'(x) = \frac{1}{1+e^x} > 0 \qquad f \nearrow$$

$$f''(x) = \frac{-e^x}{(1+e^x)^2} < 0 \qquad f' \searrow$$

(β)

$$2f'(f(x)+\ln 2) < 1$$

$$f'(f(x)+\ln 2) < \frac{1}{2}$$

$$f'(f(x)+\ln 2) < f'(0)$$

$$f' \searrow$$

$$f(x)+\ln 2 > 0$$

$$f(x) > -\ln 2$$

$$f(x) > f(0)$$

$$f \nearrow \quad x > 0.$$

11).
$$f^{1}\left(\ln\frac{1+e^{x}}{1+e^{x^2}}\right)=f^{1}(x-x^{2})$$

$$f^{1} \text{ g } 1-1$$

$$\ln\frac{1+e^{x}}{1+e^{x^2}}=x-x^{2}$$

$$\ln(1+e^{x})-\ln(1+e^{x^2})=x-x^{2}$$

$$x^{2}-\ln(1+e^{x^2})=x-\ln(1+e^{x})$$

$$f(x^{2})=f(x)$$

$$f \text{ g } 1-1$$

$$x^{2}=x$$

$$x^{2}-x=0 \Rightarrow x(x-1)=0$$

$$x=0 \quad x=1$$

III) $f(x) + 2f'(1-e^{x}) = \ln e - \ln 2$

$f(x) + 2f'(1-e^{x}) = 1 - \ln 2$

$$\underbrace{f(x) + 2f'(1-e^{x}) - 1 + \ln 2}_{\varphi(x)} = 0$$

$\varphi(x) = 0$

$\varphi(x) = \varphi(0)$

$\varphi$ 1-1

$x = 0$

• $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

• $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

$-e^{x_1} > -e^{x_2}$

$1-e^{x_1} > 1-e^{x_2}$

$f'(1-e^{x_1}) < f'(1-e^{x_2})$

$\varphi \nearrow$

$\varphi(0) = f(0) + 2f'(0) - 1 + \ln 2$

$\varphi(0) = -\cancel{\ln 2} + \cancel{2}\frac{1}{\cancel{2}} - 1 + \cancel{\ln 2} = 0$

(i) $\lim\limits_{x \to 0} \dfrac{\ln x}{f(x) + \ln 2} =$

$= \lim\limits_{x \to 0^+} \ln x \dfrac{1}{f(x) + \ln 2} = -\infty \cdot +\infty$

$= -\infty.$

$\forall x > 0 \Rightarrow f(x) + \ln 2 > 0$

οσο αρι ✓

# Επομενο Μαθημα.

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13 αγε

14 αγ.

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23 αγε

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