Ενότητα 33

2. $f(x) = \text{συν}x$

(α) $F(x) = \text{ημ}x$

(β) $F(x) = \text{ημ}x + c$

(γ) $F(0) = 3$

$$F(0) = \text{ημ}0 + c = 3$$

$$c = 3$$

$$\boxed{F(x) = \text{ημ}x + 3}$$

4. (β) $f(x) = x$

$$F(x) = \frac{x^2}{2}$$

(δ) $f(x) = \frac{1}{x^3} = x^{-3}$

$$F(x) = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = -\frac{1}{2}\frac{1}{x^2}$$

$$F(x) = -\frac{1}{2x^2}$$

στ) $f(x) = \sqrt{x}$

$f(x) = x^{1/2}$

$$F(x) = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\sqrt{x^3}$$

$$F(x) = \frac{2}{3}\sqrt{x^2}\sqrt{x} \Leftrightarrow \underline{\underline{F(x) = \frac{2}{3}x\sqrt{x}}}$$

η) $f(x) = x\sqrt{x} \Rightarrow f(x) = x\,x^{\frac{1}{2}} = x^{\frac{3}{2}}$

$$F(x) = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}\sqrt{x^5}$$

$$F(x) = \frac{2}{5}\sqrt{x^2}\sqrt{x^2}\sqrt{x}$$

$$F(x) = \frac{2}{5}x^2\sqrt{x}$$

5. (β) $f(x) = \eta\mu x - \frac{1}{x^2} + \frac{1}{2\sqrt{x}}$

$F(x) = -\sigma\upsilon\nu x + \frac{1}{x} + \sqrt{x}$

(ϛ) $f(x) = \frac{1}{\sigma\upsilon\nu^2 x} - \frac{1}{\eta\mu^2 x}$

$F(x) = \varepsilon\varphi x + \sigma\varphi x$

9. (β) $f(x) = \frac{(3x-2)^2}{x^2}$

$f(x) = \frac{9x^2 - 12x + 4}{x^2}$

$f(x) = 9 - \frac{12}{x} + \frac{4}{x^2}$

$F(x) = 9x - 12\ln|x| - 4\frac{1}{x}$

10. (β) $f(x) = e^x\left(\ln x + \frac{1}{x}\right)$

$$f(x) = e^x \ln x + e^x \frac{1}{x}$$

$$\boxed{F(x) = e^x \ln x}$$

(δ) $f(x) = \frac{2x - x^2}{e^x} = \frac{2xe^x - x^2e^x}{e^{2x}}$

$$F(x) = \frac{x^2}{e^x}$$

11. (β) $f(x) = 3x^2 e^{x^3}$

$$F(x) = e^{x^3}$$

(δ) $f(x) = \frac{x}{\sqrt{x^2+4}}$

$$F(x) = \sqrt{x^2+4}$$

$$\Rightarrow F'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+4}}.$$

(ε) $f(x) = e^{x^2+\ln x}$

$$f(x) = e^{x^2} \cdot e^{\ln x}$$

$$f(x) = x e^{x^2}$$

$$F(x) = \frac{1}{2}e^{x^2}$$

12. (13) $f(x) = \dfrac{1}{x-2}$, $x > 2$

$F(x) = \ln|\overset{\oplus}{x-2}|$

$F(x) = \ln(x-2)$.

(19) $f(x) = \sigma\varphi\, x$

$f(x) = \dfrac{\sigma\upsilon\nu\, x}{\eta\mu\, x}$

$F(x) = \ln(\eta\mu\, x)$.

(20) $f(x) = \dfrac{1}{2+e^{-x}} = \dfrac{1}{2+\frac{1}{e^x}} = \dfrac{1}{\frac{2e^x+1}{e^x}}$

$f(x) = \dfrac{e^x}{2e^x+1}$

$F(x) = \dfrac{1}{2}\ln(2e^x+1)$

16. ①

$$f'(x) = 1 - \frac{3}{x^2} \qquad f(1) = 2$$

$$f'(x) = \left(x + \frac{3}{x}\right)'$$

$$f(x) = x + \frac{3}{x} + c$$

$$x = 1$$

$$f(1) = 1 + 3 + c$$

$$2 = 4 + c$$

$$c = -2$$

$$f(x) = x + \frac{3}{x} - 2$$

$$f'(x) = g'(x)$$

Τοτε

$$f(x) = g(x) + C$$

17. (B) $f'(x) = 3^x - 2x + \frac{1}{\sqrt{x}}$ $\quad f(1) = \frac{3}{\ln 3}$

$$f'(x) = \left(\frac{3^x}{\ln 3} - x^2 + 2\sqrt{x} \right)'$$

$$f(x) = \frac{3^x}{\ln 3} - x^2 + 2\sqrt{x} + C$$

$\underline{x = 1}$

$$f(1) = \frac{3}{\ln 3} - 1 + 2 + C$$

$$\cancel{\frac{3}{\ln 3}} = \cancel{\frac{3}{\ln 3}} + 1 + C$$

$$C = -1$$

$$\underline{f(x) = \frac{3^x}{\ln 3} - x^2 + 2\sqrt{x} - 1}$$

20. B $f'(x) = e^{-x} + \frac{1}{2x+3}$ $f(0) = \ln\sqrt{3}$

$f'(x) = \left(-e^{-x} + \frac{1}{2}\ln|2x+3|\right)'$

$f(x) = -e^{-x} + \frac{1}{2}\ln|2x+3| + C$

$f(0) = -1 + \frac{1}{2}\ln 3 + C$

$\ln\sqrt{3} = -1 + \frac{1}{2}\ln 3 + C$

$\cancel{\frac{1}{2}\ln 3} = -1 + \cancel{\frac{1}{2}\ln 3} + C$

$C = 1$

$f(x) = -e^{-x} + \frac{1}{2}\ln(2x+3) + 1$

(8) $f'(x) = \frac{e^x + 1}{e^x}$ $\quad\quad f(0) = -1$

$f'(x) = 1 + \frac{1}{e^x}$

$f'(x) = 1 + e^{-x}$

$f'(x) = \left(x - e^{-x}\right)'$

$f(x) = x - e^{-x} + C$

$x = 0$

$f(0) = -1 + C$

$-1 = -1 + C$

$C = 0.$

$f(x) = x - e^{-x}$

(ε) $f'(x) = e^{x^2 + \ln x}$

$f(1) = \frac{e}{2}$

$f'(x) = e^{x^2} \cdot e^{\ln x}$

$f'(x) = x e^{x^2}$

$f'(x) = \left(\frac{1}{2} e^{x^2}\right)'$

$f(x) = \frac{1}{2} e^{x^2} + c$

$f(1) = \frac{1}{2} e + c$

$\cancel{\frac{e}{2}} = \cancel{\frac{e}{2}} + c$

$c = 0$

21. (8) $f'(x) = \dfrac{(x-1)^2}{x^2+1}$ $\quad f(0)=0$

$$f'(x) = \frac{x^2-2x+1}{x^2+1}$$

$$f'(x) = \frac{x^2+1}{x^2+1} - \frac{2x}{x^2+1}$$

$$f'(x) = 1 - \frac{2x}{x^2+1}$$

$$f'(x) = \left(x - \ln(x^2+1)\right)'$$

$$f(x) = x - \ln(x^2+1) + C$$

$$\underline{x=0}$$

$$f(0) = C = 0$$

$$f(x) = x - \ln(x^2+1).$$

22. B $f'(x)=\dfrac{x\,\sigma\upsilon\nu x-\eta\mu x}{x^2}$ $\quad f\left(\frac{\pi}{2}\right)=\frac{1}{\pi}$

$$f'(x)=\left(\frac{\eta\mu x}{x}\right)'$$

$$f(x)=\frac{\eta\mu x}{x}+c$$

$$x=\frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right)=\frac{\eta\mu\frac{\pi}{2}}{\frac{\pi}{2}}+c$$

$$\frac{1}{\pi}=\frac{1}{\frac{\pi}{2}}+c$$

$$\frac{1}{\pi}=\frac{2}{\pi}+c$$

$$c=-\frac{1}{\pi}.$$

24. (B) $f''(x) = -\frac{1}{x^2}$ $\qquad f(1) = f'(1) = 1$

$$f''(x) = \left(\frac{1}{x}\right)'$$

$$f'(x) = \frac{1}{x} + C$$

$\underline{x=1}$

$$f'(1) = 1 + C$$

$$1 = 1 + C$$

$$C = 0$$

$$f'(x) = \frac{1}{x}$$

$$f'(x) = (\ln x)'$$

$$f(x) = \ln x + C.$$

$\underline{x=1}$

$$f(1) = \ln 1 + C$$

$$1 = C$$

$$f(x) = \ln x + 1$$

24. ⑧ $f''(x) = \cos x - e^{-x}$ $\qquad f(0)=f'(0)=1$

$f''(x) = \left(\sin x + e^{-x}\right)'$

$f'(x) = \sin x + e^{-x} + C$

$f'(0) = \sin 0 + e^{0} + c$

$1 = 1 + C$

$\underline{C = 0}$

$f'(x) = \sin x + e^{-x}$

$f'(x) = \left(-\cos x - e^{-x}\right)'$

$f(x) = -\cos x - e^{-x} + C$

$\underline{x = 0}$

$f(0) = -1 - 1 + C$

$1 = -2 + C$

$\underline{C = 3}$

29. $f'(x) = \begin{cases} 2x-1, & x<0 \\ 3x^2-1, & x>0 \end{cases}$ $\quad f(1)=2$, f συνεχ.

$\underline{x<0}$

$f'(x) = 2x-1$

$f'(x) = (x^2-x)'$

$\boxed{f(x) = x^2 - x + C_1}$

$\underline{x>0}$

$f'(x) = 3x^2-1$

$f'(x) = (x^3-x)'$

$f(x) = x^3 - x + C_2$

$f(1) = 1 - 1 + C_2$

$C_2 = 2$

$\boxed{f(x) = x^3 - x + 2}$

$$\boxed{f(x) = \begin{cases} x^2 - x + C_1, & x<0 \\ x^3 - x + 2, & x \geq 0 \end{cases}}$$

$$\lim_{x\to 0^-} x^2 - x + C_1 = \lim_{x \to 0^+} x^3 - x + 2$$

$$\underline{C_1 = 2}$$

$$f(x) = \begin{cases} x^3 - x + 2, & x<0 \\ x^3 - x + 2, & x \geq 0. \end{cases}$$

$f(0)=1$

30. $(x-1)\,f'(x) = 3x^2-2x-1 \quad , x\in\mathbb{R}$

$$f'(x) = \frac{3x^2-2x-1}{x-1} \quad , x\neq 1$$

$\Delta = 4+12 = 16$

$$x = \frac{2\pm 4}{6} \quad \begin{cases} 1 \\ -\frac{1}{3} \end{cases}$$

$$f'(x) = \frac{3\cancel{(x-1)}\left(x+\frac{1}{3}\right)}{\cancel{x-1}}$$

$f'(x) = 3x+1$

$x \neq 1.$

$$f'(x) = \left(\frac{3}{2}x^2+x\right)'$$

$$f(x) = \begin{cases} \frac{3}{2}x^2+x+C_1, & x<1 \\ \frac{3}{2}x^2+x+C_2, & x\geq 1. \end{cases}$$

$f(0) = C_1 = 1$

$$f(x) = \begin{cases} \frac{3}{2}x^2+x+1, & x<1 \\ \frac{3}{2}x^2+x+C_2, & x\geq 1. \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \frac{7}{2}$$

$$\lim_{x\to 1^+} f(x) = \frac{5}{2} + C_2 = \frac{7}{2}$$

$$C_2 = 1$$

$$f(x) = \begin{cases} \frac{3}{2}x^2 + x + 1, & x < 1 \\ \frac{3}{2}x^2 + x + 1, & x \geq 1 \end{cases}$$

$$f(x) = \frac{3}{2}x^2 + x + 1$$

32. (B) $x\left(f'(x)-2\right)+f(x)=0$ $\quad \underline{x>0}$ $\quad f(1)=0$

$\underbrace{xf'(x)+f(x)}\ -2x\ =0$

$\left(xf(x)-x^2\right)'=0$

$xf(x)-x^2=c$

$\underline{x=1}$

$f(1)-1=c$

$0-1=c \Rightarrow c=-1$

$xf(x)-x^2=-1$

$xf(x)=x^2-1$

$f(x)=\dfrac{x^2-1}{x}$

(8) $f(x) = x f'(x) \ln x \qquad , x > 1 \qquad f(e) = 1$

$f(x) - x f'(x) \ln x = 0$

$\frac{1}{x \ln x} f(x) - f'(x) = 0$

$$\boxed{f'(x) - \frac{1}{x \ln x} f(x) = 0}$$

$g(x) = -\frac{1}{x \ln x}$

$G(x) = -\ln(\ln x)$

$e^{G(x)} = e^{-\ln(\ln x)} = \frac{1}{e^{\ln(\ln x)}} = \frac{1}{\ln x}$

$\frac{1}{\ln x} f'(x) - \frac{1}{\ln x} \frac{1}{x \ln x} f(x) = 0$

$\left(\frac{1}{\ln x} f(x) \right)' = c$

$\frac{1}{\ln x} f(x) = c \qquad f(e) = c = 1 \qquad f(x) = \ln x$

Πολλαπλασιαστής Euler

$$f'(x) + g(x) f(x) = 0$$

Πολ/ζω παντου με $e^{G(x)}$

οπου $G(x)$ παραγουσα τη $g(x)$

$$e^{G(x)} f'(x) + g(x) e^{G(x)} f(x) = 0$$

$$\left(e^{G(x)} f(x)\right)' = 0 .$$

32. ε. $f'(x)\cdot \text{συν}\,x = e^x\,\text{συν}^2 x - f(x)\,\text{ημ}\,x$

$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$f'(x) = e^x\,\text{συν}x - f(x)\frac{\text{ημ}x}{\text{συν}x}$ $\qquad f(0)=1$

$$\boxed{f'(x)+\frac{\text{ημ}x}{\text{συν}x}f(x)=e^x\,\text{συν}x}$$

$g(x)=\frac{\text{ημ}x}{\text{συν}x}$

$G(x) = -\ln(\text{συν}x)$

$e^{G(x)} = e^{-\ln(\text{συν}x)} = \frac{1}{e^{\ln \text{συν}x}} = \frac{1}{\text{συν}x}$

$$\frac{1}{\text{συν}x}f'(x)+\frac{1}{\text{συν}x}\frac{\text{ημ}x}{\text{συν}x}f(x)=e^x$$

$$\left(\frac{1}{\text{συν}x}f(x)\right)' = \left(e^x\right)'$$

$\frac{f(x)}{\text{συν}x}=e^x+c \quad \xrightarrow{x=0} \quad f(0)=e^0+c$

$1=1+c \qquad c=0$

$f(x)=e^x\,\text{συν}x$

33. B $\quad x f'(x) = (x-1)\, e^{-f(x)} \qquad f(1) = 0$

$$x f'(x) = (x-1)\frac{1}{e^{f(x)}} \qquad x > 0$$

$$x f'(x)\, e^{f(x)} = x - 1$$

$$f'(x)\, e^{f(x)} = 1 - \frac{1}{x}$$

$$\left(e^{f(x)}\right)' = (x - \ln x)'$$

$$e^{f(x)} = x - \ln x + C$$

$$\underline{x = 1}$$

$$e^{f(1)} = 1 + C$$

$$e^{0} = 1 + C$$

$$C = 0$$

$$e^{f(x)} = x - \ln x$$

$$f(x) = \ln(x - \ln x).$$

8 $f'(x) + 2x f(x) = 0$ $\quad x > 0$

$f(1) = e$

$f(x) \neq 0$

$g(x) = 2x$

$G(x) = x^2$

$e^{G(x)} = e^{x^2}$

$e^{x^2} f'(x) + 2x e^{x^2} f(x) = 0$

$\left(e^{x^2} f(x)\right)' = 0$

$e^{x^2} f(x) = c$

$\underline{x = 1}$

$e f(1) = c$

$e^2 = c$

$e^{x^2} f(x) = e^2$

$f(x) = \dfrac{e^2}{e^{x^2}}$

$f(x) = e^{2-x^2}$