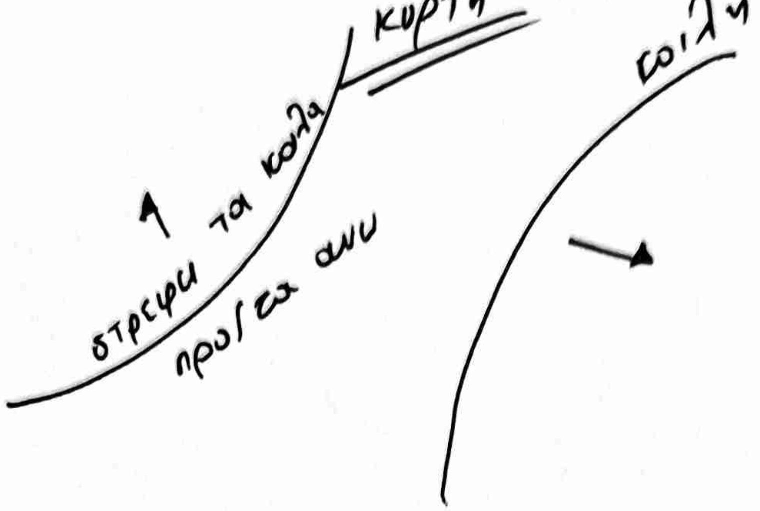
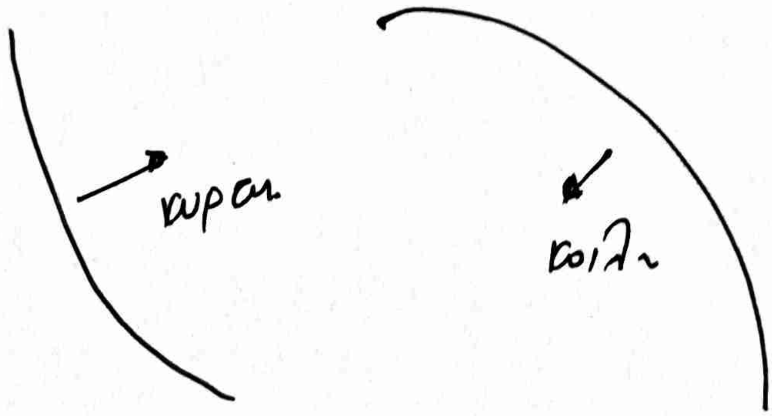
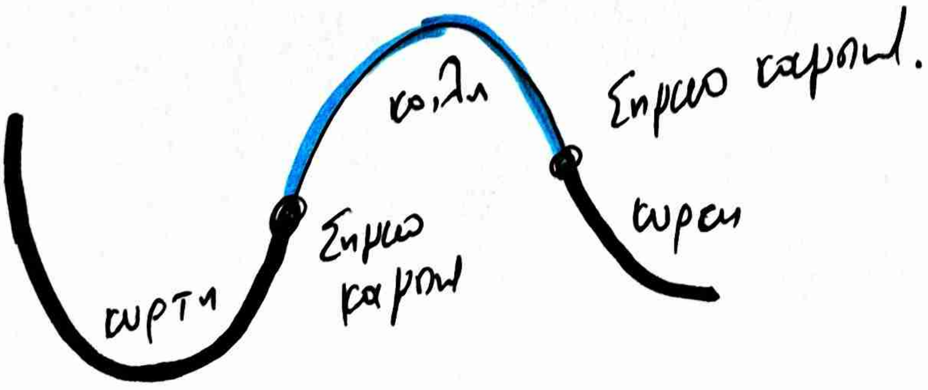
κυρτοτητα

$f \uparrow$



$f \downarrow$





$f''(x) \geq 0$ τοτε η f κυρτη

$f''(x) \leq 0$ τοτε η f κοιλη.

f κυρτη $\Longleftrightarrow f' \nearrow$

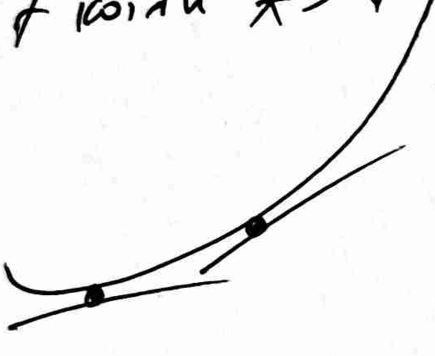
f κοιλη $\Longleftrightarrow f' \searrow$

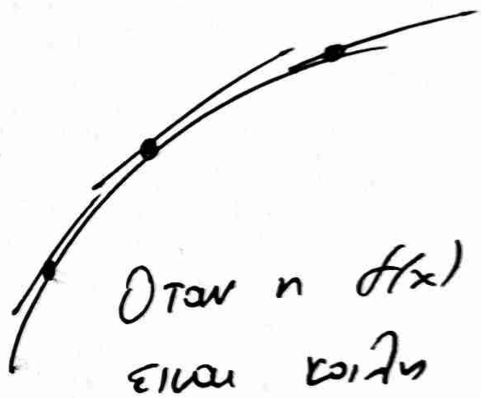
$f''(x) \geq 0 \Longrightarrow f$ κυρτη

$f''(x) \leq 0 \Longrightarrow f$ κοιλη

f κυρτη $\not\Longrightarrow f''(x) > 0$

f κοιλη $\not\Longrightarrow f''(x) < 0$





Οταν η $f(x)$ ειναι κοιλη τοτε η $f(x) \leq \alpha x + \beta$.

Οταν η $f(x)$ ειναι κυρτη τοτε $f(x) \geq \alpha x + \beta$ οπου $y = \alpha x + \beta$ εφαπτομενη της C_f

Ενότητα 29

2. (β) $f(x) = \ln x - x^3$, $x > 0$

$f'(x) = \frac{1}{x} - 3x^2$

$f''(x) = -\frac{1}{x^2} - 6x < 0$

f κοίλη.

(δ) $f(x) = x - x^4$

$f'(x) = 1 - 4x^3$ $\quad$ f κοίλη

$f''(x) = -12x^2 \leq 0$

(στ) $f(x) = 6e^x - x^3$

$f'(x) = 6e^x - 3x^2$

$f''(x) = 6e^x - 6x = 6(e^x - x) > 0$

f κυρτή.

- $e^x \geq x + 1$

$e^x - x \geq 1$ $\quad$ $e^x - x > 0$

3. ① $f(x) = 2\ln x - x^2 + 2 \quad , x > 0$

$$f'(x) = \frac{2}{x} - 2x = \frac{2-2x^2}{x} = 2\,\frac{1-x^2}{x}$$

$$\boxed{f'(x) = 2\,\frac{1-x^2}{x}}$$

$\rightarrow f'(x) = 0 \Rightarrow 2\,\dfrac{1-x^2}{x} = 0 \Rightarrow 1 - x^2 = 0$

$x = 1 \qquad x = -1$

x	0	1	
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$f(x) \leq f(1)$

$f(x) \leq 1$

$$f''(x) = -\frac{2}{x^2} - 2 < 0$$

f קעורה

5. (B) $f(x) = \frac{x^4}{6} - x^2$

$f'(x) = \frac{1}{6} 4x^3 - 2x$

$f''(x) = \frac{4}{6} 3x^2 - 2 \quad = \frac{12}{6} x^2 - 2 = 2x^2 - 2$

$$\boxed{f''(x) = 2(x^2 - 1)}$$

$\rightarrow f''(x) = 0 \quad \Rightarrow \quad x^2 - 1 = 0$

$x = 1 \quad x = -1$

x		-1		1	
f''	$+$	0	$-$	0	$+$
f	$\cup$		$\cap$		$\cup$
	κυρτή		κοίλη		κυρτή

Αν $x \in (-\infty, -1]$ κυρτή

Αν $x \in [-1, 1]$ κοίλη

Αν $x \in [1, +\infty)$ κυρτή

$A(-1, -\frac{5}{6})$ Σημείο καμπής

$B(1, -\frac{5}{6})$ Σημείο καμπής

(δ) $f(x) = x^5 - 10x^3$

$f'(x) = 5x^4 - 30x^2$

$f''(x) = 20x^3 - 60x = 20x(x^2-3)$

$\to f''(x) = 0 \Rightarrow 20x = 0$ ή $x^2-3=0$

$x=0$ $x=\sqrt{3}$ $x=-\sqrt{3}$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
$20x$	$-$		$-$	0	$+$		$+$
x^2-3	$+$	0	$-$		$-$	0	$+$
f''	$-$		$+$		$-$		$+$
f	$\frown$		$\smile$		$\frown$		$\smile$.

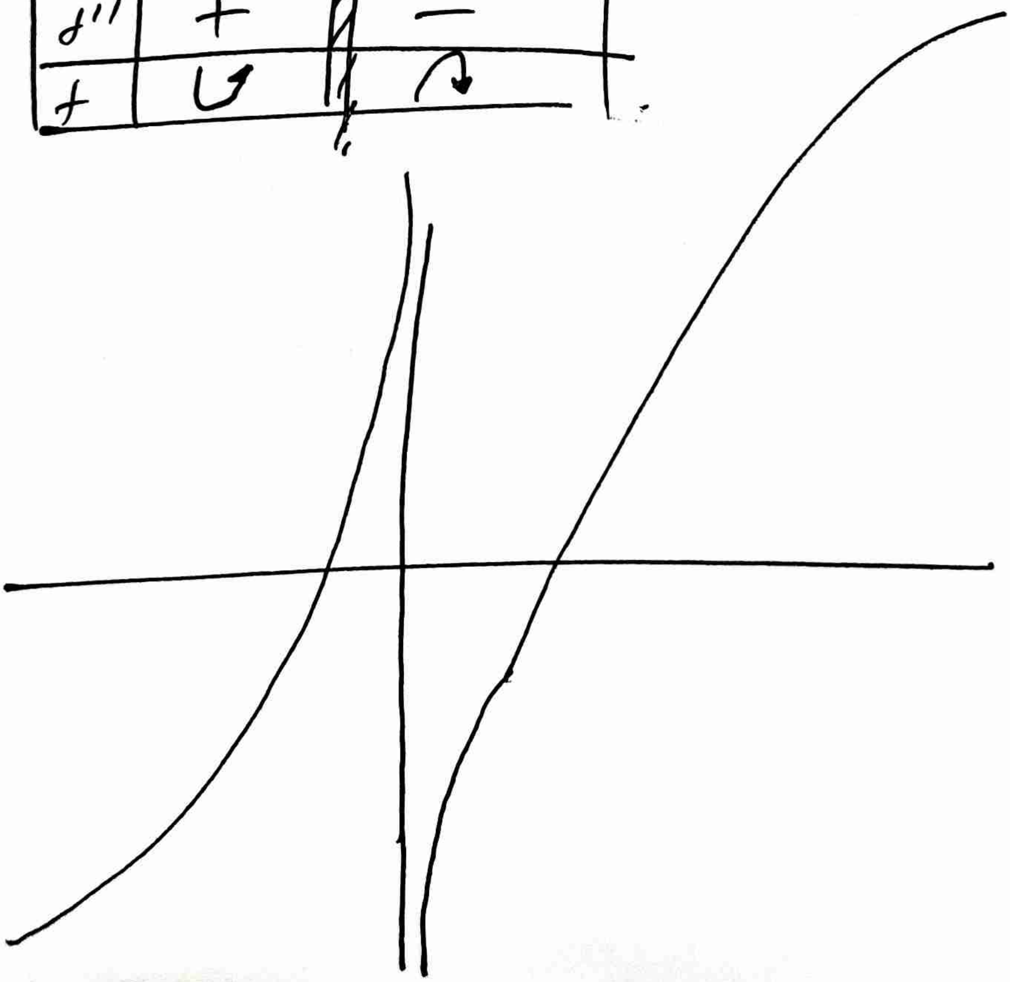
6. ⑧ $f(x) = x - \frac{1}{x}, x \neq 0$

$f'(x) = 1 + \frac{1}{x^2} > 0 \quad f \nearrow$

$f''(x) = \frac{-2x}{x^4} = \frac{-2}{x^3}$

$f''(x) = -\frac{2}{x^3}$

x		0	
-2	$-$		$-$
x^3	$-$		$+$
f''	$+$		$-$
f	∪		∩



$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to 0^-} f(x) = +\infty$$

$$\lim_{x \to 0^+} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

6. (δ) $f(x) = x^2 + \frac{1}{x}$, $x \neq 0$

$$f'(x) = 2x - \frac{1}{x^2}$$

$$f''(x) = 2 - \frac{-2x}{x^4} = 2 + \frac{2}{x^3} = \frac{2x^3+2}{x^3} = \frac{2(x^3+1)}{x^3}$$

$$\boxed{f''(x) = 2\,\frac{x^3+1}{x^3}}$$

$\longrightarrow f''(x) = 0 \Rightarrow x^3+1=0 \Rightarrow x^3 = -1$ $\quad x=-1$

x		-1		0	
x^3+1	$-$	0	$+$		$+$
x^3	$-$		$-$		$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

6. (α) $f(x) = \dfrac{3x^2 - 2}{x^3}, \ x \neq 0$

$$f'(x) = \frac{6x \cdot x^3 - (3x^2-2)3x^2}{x^6}$$

$$f'(x) = \frac{6x^4 - 3x^2(3x^2-2)}{x^6}$$

$$f'(x) = \frac{6x^2 - 3(3x^2-2)}{x^4}$$

$$f'(x) = \frac{6x^2 - 9x^2 + 6}{x^4} = \frac{6-3x^2}{x^4}$$

$$\boxed{f'(x) = 3\,\frac{2-x^2}{x^4}}$$

$$f''(x) = 3\,\frac{-2x\cdot x^4 - (2-x^2)4x^3}{x^8}$$

$$f''(x) = 3\,\frac{-2x^5 - 4x^3(2-x^2)}{x^8}$$

$$f''(x) = 3\,\frac{-2x^2 - 4(2-x^2)}{x^5}$$

$$f''(x) = 3\,\frac{-2x^2 - 8 + 4x^2}{x^5} = 3\,\frac{2x^2-8}{x^5}$$

$$\boxed{f''(x) = 6\,\frac{x^2-4}{x^5}}$$

x	-2		0		2
x^2-4	+	-	‖	-	+
x^5	-	-	‖	+	+
f''	-	+	‖	-	+
f	$\cap$	$\cup$	‖	$\cap$	$\cup$

7. (B) $f(x) = x\sqrt{1-x^2}$

$1-x^2 \geq 0$

x		-1		1	
$1-x^2$	$-$	0	$+$	0	$-$

$D_f = [-1,1]$

$$f'(x) = \sqrt{1-x^2} + x\,\frac{-2x}{2\sqrt{1-x^2}}$$

$$f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{\cancel{\sqrt{1-x^2}}^{\,2} - x^2}{\sqrt{1-x^2}}$$

$$\boxed{f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}}}$$

$$f''(x) = \frac{(1-2x^2)'\sqrt{1-x^2} - (1-2x^2)\sqrt{1-x^2}\,'}{\sqrt{1-x^2}^{\,2}}$$

$$f''(x) = \frac{-4x\sqrt{1-x^2} - (1-2x^2)\dfrac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$f''(x) = \frac{-4x\sqrt{1-x^2} + \dfrac{x(1-2x^2)}{\sqrt{1-x^2}}}{1-x^2}$$

$$f''(x) = \frac{-4x(1-x^2) + x(1-2x^2)}{(1-x^2)\sqrt{1-x^2}}$$

$$f''(x) = \frac{-4x + 4x^2 + x - 2x^3}{(1-x^2)\sqrt{1-x^2}}$$

$$f''(x) = \frac{-2x^3 + 4x^2 - 3x}{(1-x^2)\sqrt{1-x^2}}$$

$$f''(x) = \frac{x\,(\overset{\Delta<0}{-2x^2+4x-3})}{\underset{+}{(1-x^2)}\,\underset{+}{\sqrt{1-x^2}}}$$

x	-1		0		1
x		$-$		$+$	
$-2x^2+4x-3$		$-$		$-$	
f''		$+$		$-$	
f		$\cup$		$\cap$	

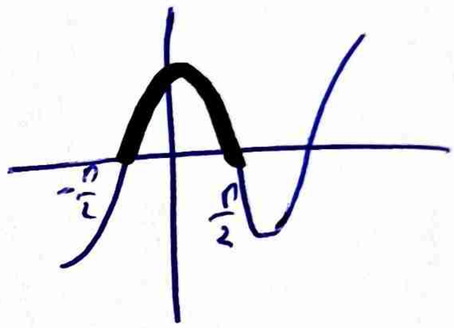
7. (δ) $f(x) = \varepsilon\varphi x - x - 2\ln(\sigma\upsilon\nu x)$ $\quad x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$f'(x) = \frac{1}{\sigma\upsilon\nu^2 x} - 1 - 2\,\frac{-\eta\mu x}{\sigma\upsilon\nu x}$$

$$f'(x) = \frac{1}{\sigma\upsilon\nu^2 x} - 1 + 2\,\varepsilon\varphi x$$

$$f''(x) = \frac{-2\sigma\upsilon\nu x(-\eta\mu x)}{\sigma\upsilon\nu^4 x} + 2\cdot\frac{1}{\sigma\upsilon\nu^2 x}$$

$$f''(x) = \frac{2\eta\mu x}{\sigma\upsilon\nu^3 x} + \frac{2\sigma\upsilon\nu x}{\sigma\upsilon\nu^3 x}$$



$$\boxed{f''(x) = 2\,\frac{\eta\mu x + \sigma\upsilon\nu x}{\underset{\oplus}{\sigma\upsilon\nu^3 x}}}$$

$\to f''(x) = 0 \Rightarrow \eta\mu x + \sigma\upsilon\nu x = 0$

$$\frac{\eta\mu x}{\sigma\upsilon\nu x} + 1 = 0$$

$$\varepsilon\varphi x = -1$$

$$x = \frac{3\pi}{4}$$

x	$-\frac{\pi}{2}$ $\quad$ $\frac{3\pi}{4}$	$\frac{\pi}{2}$
f''	$-$	$+$
f	$\frown$	$\smile$

8. (B) $f(x) = x e^{1-x}$

$f'(x) = e^{1-x} - x e^{1-x}$

$f'(x) = e^{1-x}(1-x)$

$f''(x) = -e^{1-x}(1-x) - e^{1-x}$

$f''(x) = e^{1-x}(-1+x-1)$

$f''(x) = e^{1-x}(x-2)$

$\longrightarrow f''(x) = 0 \Rightarrow e^{1-x}(x-2) = 0$

$x = 2$

x		2	
f''	$-$		$+$
f	$\cap$		$\cup$

8 . (8) $f(x) = e^{-x^2}$

$f'(x) = -2x\, e^{-x^2}$

$f''(x) = -2 \cdot e^{-x^2} + (-2x)(-2x) e^{-x^2}$

$f''(x) = -2 e^{-x^2} + 4x^2 e^{-x^2}$

$f''(x) = e^{-x^2} (4x^2 - 2)$

$\longrightarrow f''(x) = 0 \Rightarrow 4x^2 - 2 = 0$

$2x^2 - 1 = 0$

$x^2 = \frac{1}{2} \qquad x = \pm \frac{\sqrt{2}}{2}$

x		$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
f''	+	−	+
f	∪	∩	∪

9. (a) $f(x) = x^2(\ln x - 1) \quad x > 0$

$$f'(x) = 2x(\ln x - 1) + x^2 \cdot \frac{1}{x}$$

$$f'(x) = 2x\ln x - 2x + x$$

$$\boxed{f'(x) = 2x\ln x - x}$$

$$f''(x) = 2\ln x + 2x \cdot \frac{1}{x} - 1$$

$$f''(x) = 2\ln x + 2 - 1$$

$$\boxed{f''(x) = 2\ln x + 1}$$

$\rightarrow f''(x) = 0 \Rightarrow 2\ln x + 1 = 0$

$$\ln x = -\frac{1}{2}$$

$$x = e^{-\frac{1}{2}} = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}} = \frac{\sqrt{e}}{e}$$

x		$\sqrt{e}/e$	
f''	$-$		$+$
f	$\cap$		$\cup$

9. (8) $f(x) = e^x - \ln(x+1)$, $x > -1$

$$f'(x) = e^x - \frac{1}{x+1}$$

$$f''(x) = e^x - \frac{-1}{(x+1)^2} = e^x + \frac{1}{(x+1)^2} > 0$$

f κυρτή .

10. (B) $f(x) = x^2 - \ln^2 x$, $x > 0$

$$f'(x) = 2x - 2\ln x \cdot \frac{1}{x}$$

$$f'(x) = 2x - 2\frac{\ln x}{x}$$

$$f''(x) = 2 - 2\frac{\frac{1}{x}x - \ln x}{x^2} = 2 - 2\frac{1 - \ln x}{x^2}$$

$$\boxed{f''(x) = 2\frac{x^2 - 1 + \ln x}{x^2}}$$

$$\boxed{\varphi(x) = x^2 - 1 + \ln x}$$ $\varphi(1) = 0$

$$\varphi'(x) = 2x + \frac{1}{x}, \quad x > 0$$

x	0		1	
φ'		$+$		$+$
φ		$\nearrow$ $-$	0	$+$ $\nearrow$
f''		$-$		$+$
f		$\frown$		$\smile$

$x < 1 \Rightarrow \varphi(x) < \varphi(1)$

$\varphi(x) < 0$

$x > 1 \Rightarrow \varphi(x) > \varphi(1)$

$\varphi(x) > 0$

Ενότητα 26

9. $f(x) = \alpha x^3 - \beta x^2 + 3x - 1$

ⓐ Fermat $f'(1)=0$ $f'(3)=0$

$f'(x) = 3\alpha x^2 - 2\beta x + 3$

$f'(1) = 3\alpha - 2\beta + 3 = 0$ $\qquad f'(3) = \boxed{27\alpha - 9\beta + 8 = 0}$

$\boxed{3\alpha - 2\beta = -3}$

$$\begin{cases} 3\alpha - 2\beta = -3 & \Rightarrow -27\alpha + 18\beta = 27 \\ 27\alpha - 9\beta = -8 & \end{cases} \quad (+)$$

$9\beta = 18$

$\beta = 2$ $\qquad \alpha = \frac{1}{3}$

ⓑ $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 1$

$f'(x) = x^2 - 4x + 3$

$\rightarrow f'(x) = 0 \Rightarrow x^2 - 4x + 3 = 0$

$x = 1$ $\qquad x = 3$

x		1		3	
f'	+		−		+
f	↗		↘		↗

15. (a) $f(x) = x + 2\dfrac{\ln x}{x}$, $x > 0$

$$f'(x) = 1 + 2\,\frac{\frac{1}{x}x - \ln x}{x^2} = 1 + 2\,\frac{1-\ln x}{x^2}$$

$$f'(x) = \frac{x^2 + 2 - 2\ln x}{x^2}$$

$$\boxed{g(x) = x^2 + 2 - 2\ln x}$$

$$g'(x) = 2x - \frac{2}{x} = \frac{2x^2-2}{x} = 2\,\frac{x^2-1}{x}$$

x	0	1	
g'	$-$		$+$
g	$\searrow$ $+$		$+$ $\nearrow$
f	$\nearrow$		$\nearrow$

$g(x) \geq g(1)$

$g(x) \geq 3$

B) $f(x) = x^2 + 6\,\dfrac{\ln x}{x}$ $\quad x > 0$

$$f'(x) = 2x + 6\,\frac{\frac{1}{x}x - \ln x}{x^2}$$

$$f'(x) = \frac{2x^3 + 6 - 6\ln x}{x^2}$$

$$f'(x) = 2\,\frac{x^3 + 3 - 3\ln x}{x^2}$$

$$\boxed{g(x) = x^3 + 3 - 3\ln x}$$

$$g'(x) = 3x^2 - 3\,\frac{1}{x} = \frac{3x^3 - 3}{x} = 3\,\frac{x^3 - 1}{x}$$

x	0	1	
g'	$-$		$+$
g	$\searrow$		$\nearrow$
f'	$+$		$+$
f	$\nearrow$		$\nearrow$

$g(x) \geq g(1)$

$g(x) \geq 4$

13. α) $f(x) = e^x - x^2$

$f'(x) = e^x - 2x$

$f''(x) = e^x - 2$

$\longrightarrow f''(x) = 0 \Rightarrow e^x - 2 = 0$

$e^x = 2$

$x = \ln 2$

x		$\ln 2$	
f''	$-$		$+$
f'	$\searrow +$		$+ \nearrow$
f	$\nearrow$		$\nearrow$

$f'(x) \geq f'(\ln 2)$

$f'(x) \geq e^{\ln 2} - 2\ln 2$

$f'(x) \geq 2 - \ln 4$

$f'(x) \geq \ln e^2 - \ln 4$

$f'(x) \geq \ln \left(\frac{e^2}{4}\right)^{>1}$

$f'(x) \geq$ θετικό.

(B) $f(x) = e^x - (x+1)\ln(x+1)$, $x > -1$

$f'(x) = e^x - \ln(x+1) - 1$

<u>α' τροπος</u>

- $e^x \geq x+1$
- $\ln x \leq x-1$

$\ln(x+1) \leq x+1-1$

$\ln(x+1) \leq x$

$-\ln(x+1) \geq -x$

(+) $e^x - \ln(x+1) \geq x+1-x$

$e^x - \ln(x+1) - 1 \geq 0$

$f'(x) \geq 0$

$f \nearrow$

(Γ) $f(x) = \eta\mu x - \frac{x^3}{6} - x$

$f'(x) = \sigma\upsilon\nu x - \frac{1}{6} 3x^2 - 1$

$f'(x) = \sigma\upsilon\nu x - \frac{1}{2}x^2 - 1$ $\qquad f'(0) = 0$

$f''(x) = -\eta\mu x - x$ $\qquad f''(0) = 0$

$f'''(x) = -\sigma\upsilon\nu x - 1 \leq 0$

x		0	
f'''	$-$		$-$
f''	$\searrow$ $+$	0	$\searrow$ $-$
f'	$\nearrow$ $-$	0	$\searrow$ $-$
f	$\searrow$		$\searrow$

$-1 \leq \sigma\upsilon\nu x \leq 1$

$1 \geq -\sigma\upsilon\nu x \geq -1$ $\quad \Rightarrow 0 \geq -1 - \sigma\upsilon\nu x \geq -2$

12. $$f(x)=\frac{x}{e^{x-1}-1-\ln x}$$

πρεπει $e^{x-1}-1-\ln x \neq 0$

$\to g(x)=e^{x-1}-1-\ln x$ $\qquad$ $\underline{\underline{g(1)=0}}$

$g'(x)=e^{x-1}-\frac{1}{x}$ $\qquad g'(1)=0$

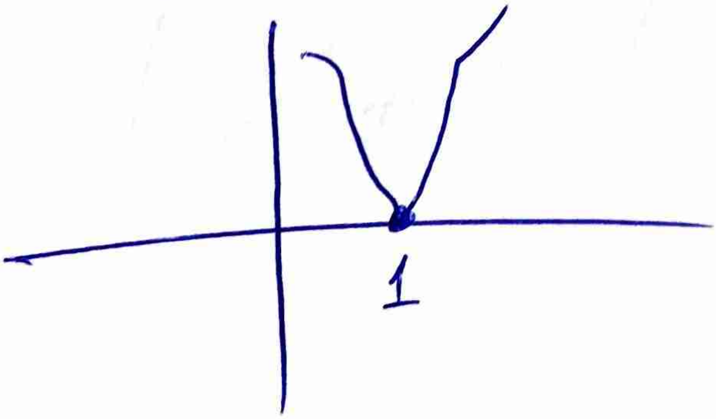
$g''(x)=e^{x-1}+\frac{1}{x^2}>0$

x		1	
g''	$+$		$+$
g'	$\nearrow$ $-$	0	$+$ $\nearrow$
g	$\searrow$		$\nearrow$

$g(x)\geq g(1)$

$g(x)\geq 0.$

Η εξισωση $g(x)=0$ εχει μοναδικη ριζα στο ακροτατο της $\underline{\underline{x=1}}$



B'

$g(x) = 0$

Προφανής ρίζα $x=1$

$x<1$

$g \downarrow$

$g(x) > g(1)$

$g(x) > 0$

$x>1$

$g \nearrow$

$g(x) > g(1)$

$g(x) > 0$

$D_g = (0,1) \cup (1,+\infty)$

Επομενο Μαθημα

29

2 α γ ε

3 α β

4

5 α γ

6 α γ ε

7 α γ

8 α γ

9 α γ ε

10 α.