Ενότητα 23

12. (3) $f(x) = x\ln^2 x - 3x + 2 \quad , x>0$

$$f'(x) = \ln^2 + x\,\frac{2\ln x}{x} - 3$$

$$\boxed{f'(x) = \ln^2 x + 2\ln x - 3}$$

$f'(x) = 0 \Rightarrow \ln x = -3 \qquad \ln x = 1$

$x = e^{-3} \qquad x = e$

x	0	e^{-3}		e	$+\infty$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

13. ⓐ $f(x) = (x-2)e^x - \frac{1}{2}x^2 + x - 1$

$$f'(x) = e^x + (x-2)e^x - x + 1$$

$$f'(x) = e^x(1 + x - 2) - x + 1$$

$$f'(x) = e^x(x-1) - (x-1)$$

$$\boxed{f'(x) = (e^x - 1)(x-1)}$$

x		0		1	
$e^x - 1$	$-$	0	$+$		$+$
$x - 1$	$-$		$-$	0	$+$
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

13. E $f(x) = \dfrac{e^x}{x^2}, \quad x \neq 0$

$$f'(x) = \frac{e^x x^2 - e^x 2x}{x^4} = \frac{e^x(x^2-2x)}{x^4}$$

$$\boxed{f'(x) = \frac{e^x(x-2)}{x^3}}$$

x		0	2
$x-2$	$-$	$-$	$+$
x^3	$-$	$+$	$+$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

13. (γ) $f(x) = (x^2 - 4x)\ln x - \frac{x^2}{2} + 4x \quad , x>0$

$$f'(x) = (2x-4)\ln x + (x^2-4x)\frac{1}{x} - \frac{1}{2}2x + 4$$

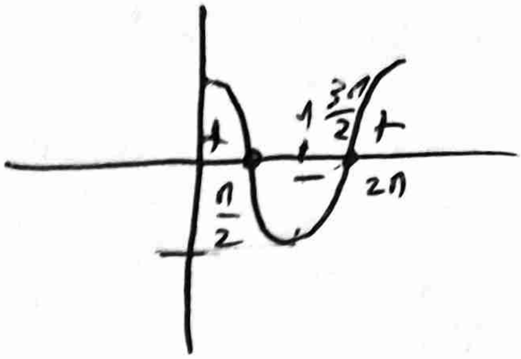
$$f'(x) = (2x-4)\ln x + \cancel{x} - \cancel{4} - \cancel{x} + \cancel{4}$$

$$\boxed{f'(x) = (2x-4)\ln x}$$

x	0 1	2	$+\infty$
$2x-4$	$-$	$-$	$+$
$\ln x$	$-$	$+$	$+$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

14. (α) $f(x) = e^{\eta\mu x}$, $x \in [0, 2\pi]$

$$\boxed{f'(x) = \underset{+}{e^{\eta\mu x}}\ \sigma\upsilon\nu x}$$

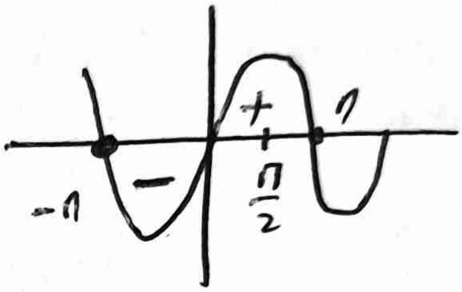


x	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π
f'	$+$	$-$	$+$	
f	↗	↘	↗	

(γ) $f(x) = \eta\mu^2 x + 4\sigma\upsilon\nu x$ $\qquad x \in [-\pi, \pi]$

$f'(x) = 2\eta\mu x\ \sigma\upsilon\nu x - 4\ \eta\mu x$

$f'(x) = 2\eta\mu x\,(\underset{-}{\sigma\upsilon\nu x - 2})$



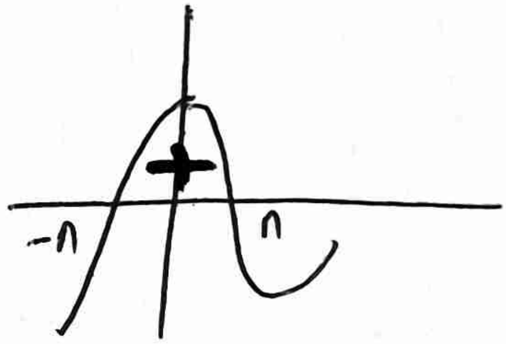
x	$-\pi$	0	π
$2\eta\mu x$	$-$	$+$	
$\sigma\upsilon\nu x - 2$	$-$	$-$	
f'	$+$	$-$	
f	↗	↘	

15. (a) $f(x) = \dfrac{\eta\mu x}{1+\sigma\upsilon\nu x}$ $\quad x \in (-\pi, \pi)$

$$f'(x) = \frac{\sigma\upsilon\nu x(1+\sigma\upsilon\nu x) - \eta\mu x\,(-\eta\mu x)}{(1+\sigma\upsilon\nu x)^2}$$

$$f'(x) = \frac{\sigma\upsilon\nu x + \sigma\upsilon\nu^2 x + \eta\mu^2 x}{(1+\sigma\upsilon\nu x)^2} = \cancel{\frac{\sigma\upsilon\nu x + 1}{(1+\sigma\upsilon\nu x)^2}}$$

$$f'(x) = \frac{1}{1+\sigma\upsilon\nu x} > 0.$$



$f \nearrow$

16. (a) $f(x) = \begin{cases} x^2-2x+3, & x<1 \\ \frac{1}{x}+1, & x \geq 1 \end{cases}$

- $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2-2x+3 = 2$
- $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{1}{x}+1 = 2$

$\left.\right\} \underline{\underline{\lim\limits_{x\to 1} f(x) = 2}}$

- $\underline{\underline{f(1) = 2}}$

Συνεχής στο 1!

$x<1$

$f_1(x) = x^2-2x+3$

$f_1'(x) = 2x-2 = 2(x-1)$

$\to f_1'(x) = 0 \Rightarrow \underline{\underline{x=1}}$

$x>1$

$f_2(x) = \frac{1}{x}+1$

$f_2'(x) = -\frac{1}{x^2} < 0$

x		1	
f_1'	$-$		
f_2'			$-$
f'	$-$		$-$
f	$\searrow$		$\searrow$

20. (a) $f(x) = e^{x-1} - \ln x \quad , x > 0$

$f'(x) = e^{x-1} - \frac{1}{x} \quad , \quad f'(1) = 0$

$f''(x) = e^{x-1} + \frac{1}{x^2} > 0$

x		1	
f''	$+$		$+$
f'	$\nearrow\ -$	0	$+\ \nearrow$
f	$\searrow$		$\nearrow$

$x < 1 \Rightarrow f'(x) < f'(1) \Rightarrow f'(x) < 0$

$x > 1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x) > 0$

(B) $f(x) = (x-1)\ln x \quad , x > 0$

$f'(x) = \ln x + \frac{x-1}{x}$

$$\boxed{f'(x) = \ln x + 1 - \frac{1}{x}}$$

$\underline{f'(1) = 0}$

$f''(x) = \frac{1}{x} + \frac{1}{x^2} > 0$

x	0		1	$+\infty$
f''		$+$		$+$
f'		$\nearrow\ -$	0	$+\ \nearrow$
f		$\searrow$		$\nearrow$

20. (δ) $f(x) = e^x - \frac{x^3}{6} - \frac{x^2}{2} - x - 1$

$$f'(x) = e^x - \frac{1}{6} 3x^2 - \frac{1}{2} 2x - 1$$

$$\boxed{f'(x) = e^x - \frac{1}{2}x^2 - x - 1} \qquad f'(0) = 0$$

$$f''(x) = e^x - \frac{1}{2} 2x - 1$$

$$f''(x) = e^x - x - 1 \geq 0$$

- $e^x \geq x + 1$

$e^x - x - 1 \geq 0$

x		0	
f''	$+$		$+$
f'	$\nearrow$ $-$		$+$ $\nearrow$
f	$\searrow$		$\nearrow$

21. (a) $f(x) = x - \dfrac{\ln x}{x}, \quad x > 0$

$$f'(x) = 1 - \frac{\frac{1}{x}x - \ln x}{x^2} = 1 - \frac{1 - \ln x}{x^2}$$

$$\boxed{f'(x) = \frac{x^2 - 1 + \ln x}{x^2}}$$

$\varphi(x) = x^2 - 1 + \ln x \qquad \varphi(1) = 0$

$\varphi'(x) = 2x + \frac{1}{x} > 0$

x	0		1	
φ'		$+$		$+$
φ		$\nearrow$ $-$	0	$+$ $\nearrow$
x^2		$+$		$+$
f'		$-$		$+$
f		$\searrow$		$\nearrow$

$x < 1 \Rightarrow \varphi(x) < \varphi(1)$

$\underline{\underline{\varphi(x) < 0}}$

21. γ $f(x) = 2\sqrt{x} - \dfrac{\ln x}{\sqrt{x}}$, $x > 0$

$$f'(x) = 2\,\frac{1}{2\sqrt{x}} - \frac{\frac{1}{x}\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x}$$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{2x\sqrt{x}\,\frac{1}{x}\sqrt{x} - 2x\sqrt{x}\,\ln x\,\frac{1}{2\sqrt{x}}}{2x\sqrt{x}\;x}$$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{2x - x\ln x}{2x^2\sqrt{x}}$$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{2 - \ln x}{2x\sqrt{x}}$$

$$\boxed{f'(x) = \frac{2x - 2 + \ln x}{2x\sqrt{x}}}$$

$$\boxed{\varphi(x) = 2x - 2 + \ln x}$$

$\varphi(1) = 0$

$$\varphi'(x) = 2 + \frac{1}{x} > 0 .$$

x	0	1
φ'	$+$	$+$
φ	$\nearrow$ $-$	0 $\nearrow$ $+$
$2x\sqrt{x}$	$+$	$+$
f'	$-$	$+$
f	$\searrow$	$\nearrow$

$x < 1 \Rightarrow \varphi(x) < \varphi(1) \Rightarrow \varphi(x) < 0$

$x > 1 \Rightarrow \varphi(x) > \varphi(1) \Rightarrow \varphi(x) > 0$.

24. (γ) $f(x) = \ln(x - \ln x)$, $x > 0$

και $x - \ln x > 0$

- $\ln x \leq x - 1$

$1 \leq x - \ln x$

$0 < x - \ln x$ ✓

$D_f = (0, +\infty)$.

$$f'(x) = \frac{1}{x - \ln x}\left(1 - \frac{1}{x}\right) = \underbrace{\frac{1}{x - \ln x}}_{(+)} \cdot \frac{x - 1}{x}$$

x		1	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

25. ۲ $f(x) = e^{-x}\sin x \quad x \in [0, \pi]$

$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$$

$$\boxed{f'(x) = e^{-x}(\cos x - \sin x)}$$

$\rightarrow f'(x) = 0 \quad \rightarrow \cos x - \sin x = 0$

$\cos x = \sin x$

$x = \frac{\pi}{4}$

x	0	$\pi/4$	π
f'	$+$	0	$-$
f	↗		↘

26. (γ) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$f'(x) = \frac{\sqrt{x^2+1} - x\,\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\sqrt{x^2+1}^{\,2}} = \frac{\sqrt{x^2+1}^{\,2} - x^2}{x^2+1}$$

$$f'(x) = \frac{1}{x^2+1} > 0 \qquad f \nearrow$$

$$\boxed{x^{\frac{3}{2}} = \sqrt[2]{x^3} = \sqrt{x^3} = \sqrt{x^2}\sqrt{x} = x\sqrt{x}}$$

(ε). $f(x) = \frac{1}{2}x^2 - \frac{8}{3}x^{\frac{3}{2}} + 3x$

$$\boxed{f(x) = \frac{1}{2}x^2 - \frac{8}{3}x\sqrt{x} + 3x}$$

$$f'(x) = \frac{1}{2}2x - \frac{8}{3}\left(\sqrt{x} + x\frac{1}{2\sqrt{x}}\right) + 3$$

$$f'(x) = x - \frac{8}{3}\sqrt{x} - \frac{8x}{6\sqrt{x}} + 3$$

$$f'(x) = x - \frac{8}{3}\sqrt{x} - \frac{4x}{3\sqrt{x}} + 3 = \frac{3x^2\sqrt{x} - 8x - 4x + 9\sqrt{x}}{3\sqrt{x}}$$

26. (ε) $f(x) = \frac{1}{2}x^2 - \frac{8}{3}x^{\frac{3}{2}} + 3x \quad , x > 0$

$$f'(x) = \frac{1}{2}2x - \frac{8}{3}\cdot\frac{3}{2}x^{1/2} + 3$$

$$f'(x) = x - 4\sqrt{x} + 3$$

$$f'(x) = \sqrt{x}^{\,2} - 4\sqrt{x} + 3$$

$\rightarrow f'(x) = 0 \rightarrow \sqrt{x}^{\,2} - 4\sqrt{x} + 3 = 0.$

$\underline{\underline{\sqrt{x} = t}}$

$t^2 - 4t + 3 = 0$

$t = 3 \qquad t = 1$

$\sqrt{x} = 3 \qquad \sqrt{x} = 1$

$x = 9 \qquad x = 1$

x	0		1		9	
f'		+	0	−	0	+
f		↗		↘		↗

Ενοτητα 24

14. (B) $f(x) = x^2 + 7 + \ln x$

$g(x) = 8x - 7\ln x$

Εστω $f(x) < g(x)$

$x^2 + 7 + \ln x < 8x - 7\ln x$

$\underbrace{x^2 - 8x + 7 + 8\ln x}_{h(x)} < 0 \Rightarrow h(x) < 0$

$h(x) < h(1)$

$h \nearrow$

$x < 1$

$h'(x) = 2x - 8 + \frac{8}{x} = \frac{2x^2 - 8x + 8}{x} = 2\,\frac{x^2 - 4x + 4}{x}$

$= 2\,\frac{(x-2)^2}{x} \geq 0.$

$h \nearrow$

Συμπερασμα

Οταν $x < 1$ τοτε C_f κατω C_g

Οταν $x > 1$ τοτε C_f πανω C_g

Οταν $x = 1$ τεμνονται

δ) $f(x) = \dfrac{e^x - 1}{x}$, $x \neq 0$

$g(x) = \dfrac{x}{2}$

Έστω $f(x) < g'(x)$.

$$\frac{e^x - 1}{x} < \frac{x}{2}$$

$$\frac{e^x - 1}{x} - \frac{x}{2} < 0$$

$$\frac{2e^x - 2 - x^2}{2x} < 0 . \Rightarrow h(x) < 0 \quad \text{Άτοπο!}$$

x		0	
φ'	$+$		$+$
φ	$\nearrow$ $-$		$+$ $\nearrow$
$2x$	$-$		$+$
$h(x)$	$+$		$+$

$\boxed{\varphi(x) = 2e^x - 2 - x^2}$ $\quad \varphi(0) = 0$

$\varphi'(x) = 2e^x - 2x = 2(e^x - x) > 0 .$

• $e^x \geq x + 1$

$e^x - x \geq 1$

$e^x - x > 0$

Άρα συμπεραίνω

$f(x) > g(x)$

$\forall x \in \mathbb{R}^*$.

15. 1 $e^x = x^4$ $(-1,0)$

$\underbrace{e^x - x^4}_{f(x)} = 0$

$f(-1) = e^{-1} - 1 < 0$

$f(0) = e^0 = 1 > 0$

Bolzano! ✓

$f'(x) = e^x - 4x^3 > 0$ $\forall x \in (-1,0)$,

$f \nearrow$ στο $(-1,0)$

άρα ρίζα $\exists$ του Bolzano

μοναδική!..

Προσοχή

$\ln e^x = \ln x^4$

$x \ln e = 4 \ln|x|$

$x = 4 \ln(-x)$.

Επομανο Μαθημα

23

(28) αβγ (Παλια ασκηση).

(26) αβδ.

(27) ολη

(29).

24

(1)

(2)

Το μαθημα της Τεταρτης

ξεκινα απο την ενοτητα

24 στην οποια πρεπει να

βγαλουμε πολλες ασκησεις