Ασκηση 5

1. Αν $2\cdot[5(\alpha-3)-2(\alpha-9)]<-3(1-\alpha)$

$2[5\alpha-15-2\alpha+18]<-3+3\alpha$

$2(3\alpha+3)<-3+3\alpha$

$6\alpha+6<3\alpha-3$

$6\alpha-3\alpha<-6-3$

$3\alpha<-9$

$\alpha<-3$ που ισχυει.

2. Av $\alpha < 3 < \beta$ vsa $\alpha\beta + 9 < 3(\alpha+\beta)$

$$\alpha\beta + 9 < 3\alpha + 3\beta$$

$$\alpha\beta + 9 - 3\alpha - 3\beta < 0$$

$$\alpha(\beta - 3) + 3(3 - \beta) < 0$$

$$\alpha(\beta-3) - 3(\beta-3) < 0$$

$$\overset{(+)}{(\beta-3)}\overset{(-)}{(\alpha-3)} < 0$$

- $3 < \beta \Rightarrow 0 < \beta - 3$
- $\alpha < 3 \Rightarrow \alpha - 3 < 0$ ✓

3. Αν $\alpha < -2 < \beta$ τότε $\alpha^2 - 2\beta > \alpha(\beta - 2)$

$$\alpha^2 - 2\beta > \alpha\beta - 2\alpha$$

$$\alpha^2 - 2\beta - \alpha\beta + 2\alpha > 0$$

$$\alpha(\alpha - \beta) + 2(-\beta + \alpha) > 0$$

$$(\alpha - \beta)(\alpha + 2) > 0$$

(−) (−)

- $\alpha < \beta \Rightarrow \alpha - \beta < 0$
- $\alpha < -2 \Rightarrow \alpha + 2 < 0$

4. Να δ.ο $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \geq 2$ $\quad \alpha, \beta$ θετικοί

$\alpha\beta \frac{\alpha}{\beta} + \alpha\beta \frac{\beta}{\alpha} \geq 2\alpha\beta$ $\quad \downarrow$

$\alpha^2 + \beta^2 \geq 2\alpha\beta$ $\quad \alpha\beta > 0$

$\alpha^2 - 2\alpha\beta + \beta^2 \geq 0$

$(\alpha - \beta)^2 \geq 0$ $\quad \checkmark$

5. Να δ.ο $\alpha^2 + \beta^2 - 2(\alpha - \beta) + 2 \geq 0$

$\alpha^2 + \beta^2 - 2\alpha + 2\beta + 2 \geq 0.$

$\alpha^2 - 2\alpha + 1 + \beta^2 + 2\beta + 1 \geq 0$

$(\alpha - 1)^2 + (\beta + 1)^2 \geq 0$

$\checkmark$

6. Nδο $2\alpha^2+\beta^2+9 \geq 2\alpha(3-\beta)$

$$2\alpha^2+\beta^2+9 \geq 6\alpha-2\alpha\beta$$

$$2\alpha^2+\beta^2+9-6\alpha+2\alpha\beta \geq 0.$$

$$\alpha^2+\alpha^2+\beta^2+9-6\alpha+2\alpha\beta \geq 0$$

$$\alpha^2+2\alpha\beta+\beta^2+\alpha^2-6\alpha+9 \geq 0$$

$$(\alpha+\beta)^2+(\alpha-3)^2 \geq 0$$

✓

5. Ενότητα 7

(A) $x^2+y^2-2x+1 \geq 0$

$x^2-2x+1+y^2 \geq 0$

$(x-1)^2+y^2 \geq 0$ ✓

$\alpha^2+\beta^2=0$

Τότε

$\alpha=\beta=0$

Ισότητα

$(x-1)^2+y^2=0$

$x-1=0$ και $\boxed{y=0}$

$\boxed{x=1}$.

(B) $x^2+y^2-4x+6y \geq -13$

$x^2+y^2-4x+6y+13 \geq 0$

$x^2-4x+4+y^2+6y+9 \geq 0$

$(x-4)^2+(y+3)^2 \geq 0$

Το "=" $\Leftrightarrow$ $x=4$ και $y=-3$!

6. (a) $x^2+y^2-6y+9=0$

$x^2+(y-3)^2=0$

$x=0$ και $y-3=0$

$y=3$

(B) $x^2+y^2-4x+2y+5=0$.

$x^2-4x+4+y^2+2y+1=0$

$(x-2)^2+(y+1)^2=0$

$x-2=0$ και $y+1=0$

$x=2$ $y=-1$

(8) $x^2+y^2-10x+4y = -29$

$x^2+y^2-10x+4y+29=0$

$x^2-10x+25+y^2+4y+4=0$

$(x-5)^2+(y+2)^2=0.$

$x-5=0$ και $y+2=0$

$x=5$ $y=-2$

19. ⓐ Av $3(a-\beta)+1 > 5a-(7\beta-1)$

vsa $a < 2\beta$.

$3a-3\beta+1 > 5a-7\beta+1$

$3a-5a > -7\beta+3\beta$

$-2a > -4\beta$

$a < 2\beta$

ⓑ Av $a(a-2) > a^2-2\beta$ vsa $a < \beta$

$a^2-2a > a^2-2\beta$

$-2a > -2\beta$

$a < \beta$

20.

(α) Αν $\alpha < \beta$ τότε $\alpha < \frac{\alpha+\beta}{2} < \beta$

$\alpha < \frac{\alpha+\beta}{2}$ και $\frac{\alpha+\beta}{2} < \beta$

$2\alpha < \alpha+\beta$ $\quad$ $\alpha+\beta < 2\beta$

$\alpha < \beta$ ✓ $\quad$ $\alpha < \beta$ ✓

(β) $\alpha < \frac{\alpha+2\beta}{3} < \beta$.

$\alpha < \frac{\alpha+2\beta}{3}$ και $\beta > \frac{\alpha+2\beta}{3}$

$3\alpha < \alpha+2\beta$ $\quad$ $3\beta > \alpha+2\beta$

$2\alpha < 2\beta$ $\quad$ $\beta > \alpha$ που ισχύει

$\alpha < \beta$ που ισχύει

19. ① Av $1-3(2\alpha-\beta) < 2\beta-(5\alpha-1)$

vb $\alpha > \beta$.

$1-6\alpha+3\beta < 2\beta-5\alpha+1$

$-6\alpha+5\alpha < 2\beta-3\beta$

$-\alpha < -\beta$

$\alpha > \beta$

19. (δ) Αν $\frac{1-\alpha}{2} - \frac{\beta-2}{4} > 1$

τότε $\beta < -2\alpha$.

$\frac{1-\alpha}{2} - \frac{\beta-2}{4} > 1 \Rightarrow 4\frac{1-\alpha}{2} - 4\frac{\beta-2}{4} > 4 \Rightarrow$

$\Rightarrow 2-2\alpha-\beta+2 > 4 \Rightarrow -\beta-2\alpha > 0 \Rightarrow -2\alpha > \beta$

28. (α) $x^2-4x+5 > 0$

$x^2-4x+4+1 > 0$

$(x-2)^2+1 > 0$ ✓

(β) $9x^2-6x+2 > 0$

$9x^2-6x+1+1 > 0$

$(3x-1)^2+1 > 0$ ✓

(γ) $2x^2+2x+1 > 0$

$x^2+x^2+2x+1 > 0$

$x^2+(x+1)^2 > 0$

(δ) $\alpha^2+4\beta+\beta^2 \geq 0$

$2\alpha^2+2\alpha\beta+2\beta^2 \geq 0$

$\alpha^2+\alpha^2+2\alpha\beta+\beta^2+\beta^2 \geq 0$

$\alpha^2+(\alpha+\beta)^2+\beta^2 \geq 0$.

21. (α) Αν $\alpha<0$ νδο $\alpha+\frac{1}{\alpha}\leq -2$

$$\alpha^2+\alpha\frac{1}{\alpha}\geq -2\alpha$$

$$\alpha^2+2\alpha+1\geq 0$$

$$(\alpha+1)^2\geq 0$$ ✓

(β) Αν $\alpha>0$, $\beta>0$

ii) νδο $\frac{1}{\alpha}+\frac{1}{\beta}\geq\frac{4}{\alpha+\beta}$. ΕΚΠ: $\alpha\beta(\alpha+\beta)$

$$\alpha\beta(\alpha+\beta)\frac{1}{\alpha}+\alpha\beta(\alpha+\beta)\frac{1}{\beta}\geq\alpha\beta(\alpha+\beta)\frac{4}{\alpha+\beta}$$

$$\beta(\alpha+\beta)+\alpha(\alpha+\beta)\geq 4\alpha\beta$$

$$\beta\alpha+\beta^2+\alpha^2+\alpha\beta\geq 4\alpha\beta$$

$$\beta^2-2\alpha\beta+\alpha^2\geq 0$$

$$(\beta-\alpha)^2\geq 0.$$

11). Vso $(a-\beta)\left(\frac{1}{a}-\frac{1}{\beta}\right) \leq 4$

$$\alpha \cdot \frac{1}{\alpha} - a\frac{1}{\beta} - \beta\frac{1}{a} + \beta\frac{1}{\beta} \leq 4$$

$$1 - \frac{\alpha}{\beta} - \frac{\beta}{a} + 1 \leq 4$$

$$2 - \frac{a}{\beta} - \frac{\beta}{a} \leq 4$$

$$-\frac{a}{\beta} - \frac{\beta}{a} \leq 2$$

$$-a\beta\frac{a}{\beta} - a\beta\frac{\beta}{a} \leq 2a\beta$$

$$-a^2 - \beta^2 \leq 2a\beta$$

$$0 \leq a^2 + 2a\beta + \beta^2$$

$$0 \leq (a+\beta)^2$$

α, β
θετικά.

iii) νδο $\frac{\alpha}{1+\alpha} < \frac{\alpha+\beta}{1+\beta}$.

$$\alpha(1+\beta) < (\alpha+\beta)(1+\alpha)$$

$$\cancel{\alpha} + \cancel{\alpha\beta} < \cancel{\alpha} + \alpha^2 + \beta + \cancel{\alpha\beta}$$

$$0 < \alpha^2 + \beta \quad \checkmark$$

22. ⓐ $\frac{1}{x^2+1} \leq 1$

$(x^2+1)\frac{1}{x^2+1} \leq 1 \cdot (x^2+1)$

$1 \leq x^2+1$

$0 \leq x^2$ ✓

ⓑ $-\frac{1}{2} \leq \frac{a}{a^2+1} \leq \frac{1}{2}$

$-\frac{1}{2} \leq \frac{a}{a^2+1}$ וגם $\frac{a}{a^2+1} \leq \frac{1}{2}$

$-(a^2+1) \leq 2a$

$-a^2-1 \leq 2a$

$0 \leq a^2+2a+1$

$0 \leq (a+1)^2$

$2a \leq a^2+1$

$0 \leq a^2-2a+1$

$0 \leq (a-1)^2$ ✓

23. (a) Αν $a<1$

Νδο $a(a-1)^2 < 1-a^2$

$a(a^2-2a+1) < 1-a^2$

$a^3-2a^2+a < 1-a^2$

$a^3-a^2+a-1 < 0$

$a^2(a-1)+a-1 < 0$

$(a-1)(a^2+1) < 0$

$\ominus \quad \oplus$

• $a<1 \Rightarrow a-1<0$ ✓

24. $f(x) = e^{2x-1}$

$(e^x)' = e^x$

$f'(x) = e^{2x-1} \cdot (2x-1)'$

$f'(x) = e^{2x-1} \cdot 2$

$(\ln x)' = \frac{1}{x}$.

25. $f(x) = \ln(x^2+1)$

$f'(x) = \frac{1}{x^2+1} \cdot (x^2+1)'$

$f'(x) = \frac{2x}{x^2+1}$

25. Αν $x>1$

να δ.ο $x^2-3x+3>\frac{1}{x}$

$$x^3-3x^2+3x>1$$

$$x^3-3x^2+3x-1>0$$

$$(x-1)^3>0$$

(+)

Αφού $x>1$

$x-1>0$

α, β θετικοί.

43. ⓐ Νδο $\frac{\alpha^3+\beta^3}{\alpha+\beta} \geq \alpha\beta$

$\alpha^3+\beta^3 \geq \alpha\beta(\alpha+\beta)$

$\alpha^3+\beta^3 \geq \alpha^2\beta+\alpha\beta^2$

$\alpha^3+\beta^3-\alpha^2\beta-\alpha\beta^2 \geq 0$.

$\alpha^2(\alpha-\beta)+\beta^2(\beta-\alpha) \geq 0$

$\alpha^2(\alpha-\beta)-\beta^2(\alpha-\beta) \geq 0$

$(\alpha-\beta)(\alpha^2-\beta^2) \geq 0$

$(\alpha-\beta)(\alpha-\beta)(\alpha+\beta) \geq 0$

$(\alpha-\beta)^2(\alpha+\beta) \geq 0$

(+) (+)

Βασική Άσκηση διατάξης

Έστω $1<\alpha<2$ και $2<\beta<3$

1. $\alpha+\beta$

- $1<\alpha<2$
- $2<\beta<3$

$\Big\}\oplus$ $\boxed{3<\alpha+\beta<5}$

2. $3\alpha+2\beta$

- $1<\alpha<2 \Rightarrow 3<3\alpha<6$
- $2<\beta<3 \Rightarrow 4<2\beta<6$

$\Big\}\oplus$

$\boxed{7<3\alpha+2\beta<11}$

3. $\frac{1}{\alpha} - 2\beta$

- $1 < \alpha < 2 \Rightarrow \frac{1}{1} > \frac{1}{\alpha} > \frac{1}{2} \Rightarrow 1 > \frac{1}{\alpha} > \frac{1}{2}$
- $2 < \beta < 3 \Rightarrow -4 > -2\beta > -6$ ⊕

$$-3 > \frac{1}{\alpha} - 2\beta > \frac{1}{2} - 6$$

$$\boxed{-3 > \frac{1}{\alpha} - 2\beta > -\frac{11}{2}}$$

4. $\alpha - \beta$

- $1 < \alpha < 2 \Rightarrow 2 > \alpha > 1$
- $2 < \beta < 3 \Rightarrow -2 > -\beta > -3$

} ⊕

$$\boxed{0 > \alpha - \beta > -2}$$

5. $\frac{\alpha}{\beta}$

- $1 < \alpha < 2 \Rightarrow 2 > \alpha > 1$
- $2 < \beta < 3 \Rightarrow \frac{1}{2} > \frac{1}{\beta} > \frac{1}{3}$

} ⊙

$$2 \cdot \frac{1}{2} > \alpha \cdot \frac{1}{\beta} > 1 \cdot \frac{1}{3}$$

$$\frac{2}{2} > \frac{\alpha}{\beta} > \frac{1}{3}$$

$$\boxed{1 > \frac{\alpha}{\beta} > \frac{1}{3}}$$

Βασική Άσκηση διάστημα

Έστω $1 < x < 2$

1. $x^2 - 1$

- $1 < x < 2 \Rightarrow 1^2 < x^2 < 4$

$1 < x^2 < 4$

$1 - 1 < x^2 - 1 < 4 - 1$

$\boxed{0 < x^2 - 1 < 3}$

2. $x^3 - \frac{1}{x}$

- $1 < x < 2 \Rightarrow 1^3 < x^3 < 2^3$

$1 < x^3 < 8$

- $1 < x < 2 \rightarrow \frac{1}{1} > \frac{1}{x} > \frac{1}{2} \Rightarrow 1 > \frac{1}{x} > \frac{1}{2}$

$-1 < -\frac{1}{x} < -\frac{1}{2}$

(+)

$0 < x^3 - \frac{1}{x} < 8 - \frac{1}{2}$

Συμβολισμοί.

- $\alpha < x < \beta$ ή $x \in (\alpha, \beta)$.
- $\alpha \leq x \leq \beta$ ή $x \in [\alpha, \beta]$
- $\alpha < x \leq \beta$ ή $x \in (\alpha, \beta]$
- $x < \alpha$ ή $x \in (-\infty, \alpha)$
- $x \geqslant \alpha$ ή $x \in [\alpha, +\infty)$