Ενότητα 23

10. (1) $f(x) = (x-2)e^x$ $\quad D_f = \mathbb{R}$.

$$f'(x) = e^x + (x-2)e^x$$

$$f'(x) = e^x(1+x-2)$$

$$\boxed{f'(x) = e^x(x-1)}$$

$\longrightarrow f'(x) = 0 \Rightarrow e^x(x-1) = 0 \quad x=1$,

x		1	
e^x	+		+
$x-1$	−	0	+
f'	−		+
f	↘		↗

Η $f \downarrow (-\infty, 1]$

Η $f \uparrow [1, +\infty)$

$A(1, -e)$ Ο.Ε.

$$f(x) \geq f(1)$$

$$\underline{f(x) \geq -e}$$

10. σε $f(x) = \dfrac{e^x}{x^2+1}$, $D_f = \mathbb{R}$.

$$f'(x) = \frac{e^x(x^2+1) - e^x\, 2x}{(x^2+1)^2} = \frac{e^x(x^2+1-2x)}{(x^2+1)^2}$$

$$\boxed{f'(x) = \frac{e^x(x-1)^2}{(x^2+1)^2} > 0 \qquad f \nearrow}$$

11. Β $f(x) = x e^{\frac{1}{x}}$ $D_f = \mathbb{R}^*$

$$f'(x) = e^{\frac{1}{x}} - x\, e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)$$

$$f'(x) = e^{\frac{1}{x}}\left(1 + \frac{1}{x}\right) = e^{\frac{1}{x}}\left(\frac{x+1}{x}\right)$$

$$\boxed{f'(x) = \frac{x+1}{x}\, e^{\frac{1}{x}}}$$

x		-1		0	
$x+1$	$-$	0	$+$	||	$+$
e^x	$+$		$+$	||	$+$
x	$-$		$-$	0	$+$
f'	$+$		$-$	||	$+$
f	$\nearrow$		$\searrow$	||	$\nearrow$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{x+1}{x} e^{\frac{1}{x}} = 0 \Rightarrow \dfrac{x+1}{x} = 0$

$x + 1 = 0 \quad x = -1$,

11. (δ) $f(x) = e^x - e^{-x} - 2x$

$e^{-x} = \frac{1}{e^x}$

$f'(x) = e^x + e^{-x} - 2$

$f'(x) = e^x + \frac{1}{e^x} - 2$

$f'(x) = \frac{e^{2x} + 1 - 2e^x}{e^x}$

$f'(x) = \frac{(e^x - 1)^2}{e^x} \geq 0$ $\quad f \nearrow$

12. (B) $f(x) = x^2 \ln x \quad , \underline{x > 0}$

$f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x}$

$f'(x) = 2x \ln x + x$

$\boxed{f'(x) = x\,(2\ln x + 1)}$

x	0		$\sqrt{e}/e$	
x		+		+
$2\ln x+1$		−	0	+
f'		−		+
f		↘		↗

$f'(x) = 0 \Rightarrow 2\ln x + 1 = 0 \rightarrow \ln x = -\frac{1}{2}$

$e^{\ln x} = e^{-\frac{1}{2}}$

$x = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}} = \frac{\sqrt{e}}{e}$

12. (δ) $f(x) = \dfrac{\ln x}{\sqrt{x}}$, $x > 0$

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x}$$

$$f'(x) = \frac{2\cancel{x}\sqrt{x}\,\frac{1}{\cancel{x}}\sqrt{x} - \cancel{2}x\cancel{\sqrt{x}}\ln x\,\frac{1}{\cancel{2\sqrt{x}}}}{2x\sqrt{x}\;x}$$

$$f'(x) = \frac{2x - x\ln x}{2x^2\sqrt{x}} = \frac{\cancel{x}(2 - \ln x)}{2x^{\cancel{2}}\sqrt{x}} = \frac{2 - \ln x}{2x\sqrt{x}}$$

$$\rightarrow f'(x) = 0 \Rightarrow \frac{\cancel{x}(2 - \ln x)}{2x^{\cancel{2}}\sqrt{x}} = 0$$

$$\Rightarrow 2 - \ln x = 0 \Rightarrow 2 = \ln x$$

$$\underline{\underline{x = e^2}}$$

x	0	e^2	
$2 - \ln x$	$+$	0	$-$
$2x\sqrt{x}$	$+$		$+$
f'	$+$		$-$
f	$\nearrow$		$\searrow$

$f(x) \leq f(e^2)$

$f(x) \leq \dfrac{\ln e^2}{\sqrt{e^2}}$

$f(x) \leq \dfrac{2}{e}$

12. (στ) $f(x) = x + \ln(x^2+1)$, $x \in \mathbb{R}$.

$$f'(x) = 1 + \frac{2x}{x^2+1} = \frac{x^2+1+2x}{x^2+1} = \frac{(x+1)^2}{x^2+1} \geq 0$$

$f \nearrow$

(η) $f(x) = x^{\ln x}$

$$e^{\ln x} = x$$

$$\left[x^{\ln x} = e^{\ln x^{\ln x}} = e^{\ln x \ln x} = e^{\ln^2 x} \right]$$

$$f'(x) = \left(x^{\ln x}\right)' = \left(e^{\ln^2 x}\right)' = e^{\ln^2 x} \left(\ln^2(x)\right)'$$

$$f'(x) = e^{\ln^2 x} \, 2 \ln x \, \frac{1}{x}$$

$$\boxed{f'(x) = x^{\ln x} \frac{2 \ln x}{x}}$$

13. Ⓑ $f(x) = 2xe^{-x} + (x-1)^2, \; x \in \mathbb{R}.$

$$f'(x) = 2e^{-x} - 2xe^{-x} + 2(x-1)$$

$$f'(x) = 2e^{-x}(1-x) - 2(1-x)$$

$$f'(x) = (1-x)(2e^{-x} - 2)$$

$$\boxed{f'(x) = 2(1-x)(e^{-x} - 1)}$$

$\longrightarrow f'(x) = 0 \Rightarrow 1 - x = 0$ ή $e^{-x} - 1 = 0$

$x = 1$ 　　　 $e^{-x} = 1$

$x = 0$

x		0		1	
$1-x$	$+$		$+$	0	$-$
$e^{-x}-1$	$+$	0	$-$		$-$
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

13. (5) $f(x)=(x^2-x)\ln x+\dfrac{x^2}{2}$, $x>0$

$$f'(x)=(2x-1)\ln x+(x^2-x)\frac{1}{x}+\frac{1}{2}2x$$

$$f'(x)=(2x-1)\ln x+x-1+x$$

$$f'(x)=(2x-1)\ln x+2x-1$$

$$\boxed{f'(x)=(2x-1)(\ln x+1)}$$

$\rightarrow f'(x)=0 \Rightarrow 2x-1=0$ ή $\ln x+1=0$

$x=\frac{1}{2}$ $\qquad \ln x=-1$

$x=e^{-1}$

$x=\frac{1}{e}$

x	0	$\frac{1}{e}$	$\frac{1}{2}$	$+\infty$
$2x-1$	$-$	$-$	0 $+$	
$\ln x+1$	$-$	0 $+$	$+$	
f'	$+$	$-$	$+$	
f	$\nearrow$	$\searrow$	$\nearrow$	

13. στ) $f(x) = \dfrac{x}{\ln^2 x}$, $x \in (0,1) \cup (1,+\infty)$

$$f'(x) = \frac{\ln^2 x - x \cdot 2\ln x \, \frac{1}{x}}{\ln^4 x} = \frac{\ln^2 x - 2\ln x}{\ln^4 x}$$

$$f'(x) = \frac{\ln x - 2}{\ln^3 x}$$

$\rightarrow f'(x) = 0 \Rightarrow \dfrac{\ln x - 2}{\ln^3 x} = 0 \Rightarrow \ln x - 2 = 0$

$\ln x = 2$

$x = e^2$

x	0	1	e^2
$\ln x - 2$	$-$	$-$	0 $+$
$\ln^3 x$	$-$	$+$	$+$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

14. B $f(x) = \frac{x^2}{2} - x\cos x + \sin x$

$$f'(x) = \frac{1}{2}2x - (\cos x - x\sin x) + \cos x$$

$$f'(x) = x - \cancel{\cos x} + x\sin x + \cancel{\cos x}$$

$$f'(x) = x + x\sin x$$

$$\boxed{f'(x) = x(1+\sin x)}$$

$+$

$-1 \leq \sin x \leq 1$

$0 \leq 1 + \sin x \leq 2$.

$\rightarrow f'(x) = 0 \Rightarrow \underline{\underline{x = 0}}$

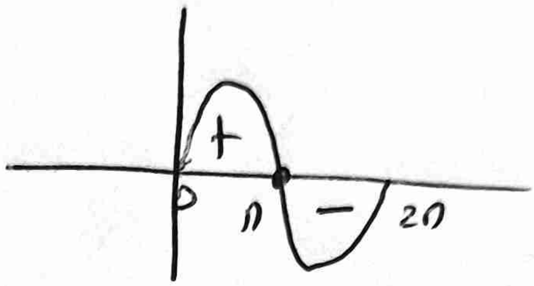
x		0	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

$f(x) \geq 0$

14. (δ) $f(x) = \dfrac{1}{1-\sigma\upsilon\nu x}$ $\quad x \in (0, 2\pi)$.

$$f'(x) = \frac{-(+\eta\mu x)}{(1-\sigma\upsilon\nu x)^2} = \frac{-\eta\mu x}{(1-\sigma\upsilon\nu x)^2}$$



x	0	π	2π
f'	−	+	
f	↘	↗	

$f(x) \geq f(\pi)$

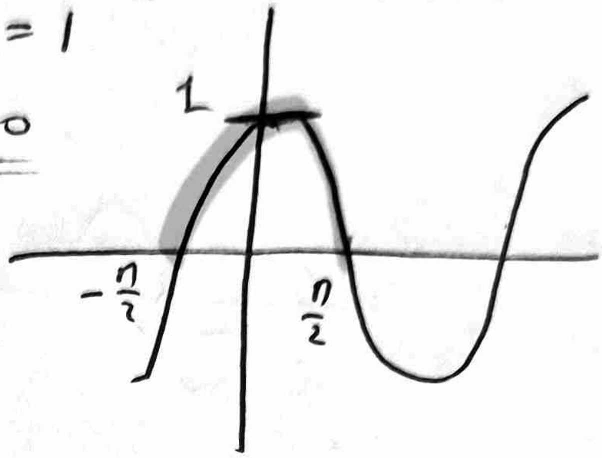
$f(x) \geq \frac{1}{2}$

15. (B) $f(x) = ημx - εφx \; , \; x \in \left(-\frac{π}{2}, \frac{π}{2}\right)$

$$f'(x) = συνx - \frac{1}{συν^2x} = \frac{συν^3x - 1}{συν^2x}$$

$\to f'(x) = 0 \Rightarrow συν^3x = 1$

$x = 0$



x	$-\frac{π}{2}$		0		$\frac{π}{2}$
f'		$-$	0	$-$	
f		$\searrow$		$\searrow$	

16. $f(x) = \begin{cases} x^3 - 3x \, , & x < 0 \\ x^2 - 4x \, , & x \geq 0 \end{cases}$ Συνεχής.

$\lim\limits_{x \to 0^-} f(x) = 0$
$\lim\limits_{x \to 0^+} f(x) = 0$

$\Big\} \lim\limits_{x \to 0} f(x) = 0$ $\quad f(0) = 0$

$x < 0$

$f_1(x) = x^3 - 3x$

$f_1'(x) = 3x^2 - 3$

$f_1'(x) = 3(x^2 - 1)$

$x = 1$ $\quad x = -1$.

$x > 0$

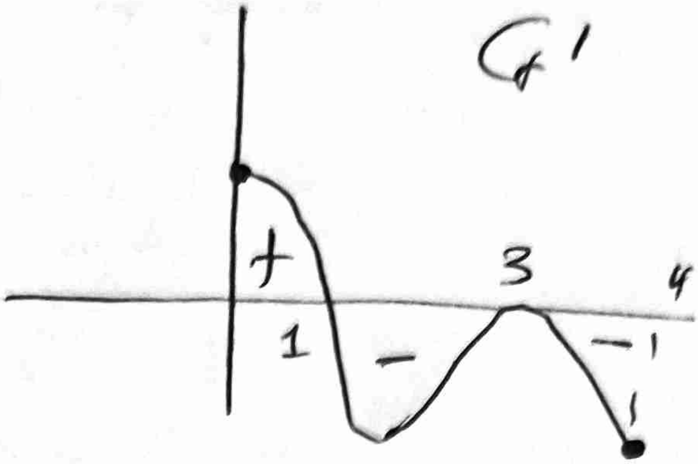
$f_2(x) = x^2 - 4x$

$f_2'(x) = 2x - 4$

$x = 2$

x		-1		0		1		2	
f_1'	$+$	0	$-$						
f_2'					$-$		$-$	0	$+$
f'	$+$		$-$		$-$		$-$		$+$
f	$\nearrow$		$\searrow$		$\searrow$		$\searrow$		$\nearrow$

17.



x		1		3	
f'	+	0	−		−
f	↗		↘		↘

20. (γ) $f(x) = e^{x-1} + x\ln x - 2x \quad , x>0$

$$f'(x) = e^{x-1} + \ln x + x\frac{1}{x} - 2$$

$$\boxed{f'(x) = e^{x-1} + \ln x - 1} \qquad \underline{f'(1)=0}$$

$$f''(x) = e^{x-1} + \frac{1}{x} > 0$$

x	0		1	
f''		+		+
f'		↗ −	0	+ ↗
f		↘		↗

$x<1 \Rightarrow f'(x) < f'(1) \Rightarrow f'(x)<0$

$x>1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x)>0$

$f(x) \geq f(1) \Rightarrow \underline{f(x) \geq 0}$

20. E $f(x) = e^x + \frac{x^2}{2} - 2x + \sin x$

$$f'(x) = e^x + \frac{1}{2}2x - 2 + \cos x$$

$$\boxed{f'(x) = e^x + x - 2 + \cos x} \qquad f'(0) = 0$$

$$f''(x) = e^x + \underbrace{1 - \sin x}_{+} > 0$$

$$-1 \le \sin x \le 1$$

$$1 \ge -\sin x \ge -1$$

$$2 \ge 1 - \sin x \ge 0$$

x		0	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

$x < 0 \Rightarrow f'(x) < f'(0) \Rightarrow f'(x) < 0$

$x > 0 \Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 0$

$$f(x) \ge f(0)$$

$$\underline{f(x) \ge 1}$$

21. (B) $f(x) = \dfrac{x^3 - 2\ln x}{2x}, \quad x > 0.$

$$f'(x) = \frac{\left(3x^2 - \frac{2}{x}\right)2x - (x^3 - 2\ln x)\,2}{4x^2}$$

$$f'(x) = \frac{6x^3 - 4 - 2x^3 + 4\ln x}{4x^2}$$

$$f'(x) = \frac{4x^3 - 4 + 4\ln x}{4x^2} = \frac{x^3 - 1 + \ln x}{x^2}$$

$$\boxed{f'(x) = \frac{x^3 - 1 + \ln x}{x^2}}$$

$\varphi(x) = x^3 - 1 + \ln x \qquad \varphi(1) = 0$

$\varphi'(x) = 3x^2 + \dfrac{1}{x} > 0$

x	0		1	
φ'		$+$		$+$
φ		$\nearrow$ $-$	0	$+$ $\nearrow$
x^2		$+$		$+$
f'		$-$		$+$
f		$\searrow$		$\nearrow$

21. (δ) $f(x)=\begin{cases} \dfrac{1-\sigma\upsilon\nu x}{x} & , x\in(0,\frac{\pi}{2}) \\ 0 & , x=0 . \end{cases}$

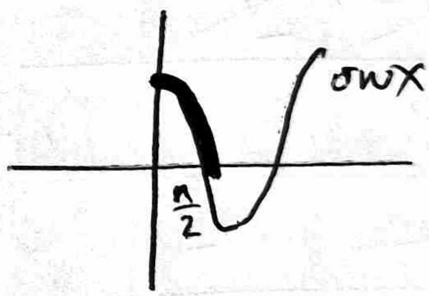
$$f'(x)=\frac{\eta\mu x\cdot x-(1-\sigma\upsilon\nu x)}{x^2}$$

$$\boxed{f'(x)=\frac{x\eta\mu x-1+\sigma\upsilon\nu x}{x^2}}$$

$\varphi(x)=x\eta\mu x-1+\sigma\upsilon\nu x$ $\qquad\varphi(0)=0.$

$\varphi'(x)=\cancel{\eta\mu x}+x\sigma\upsilon\nu x-\cancel{\eta\mu x}$

$\varphi'(x)=x\sigma\upsilon\nu x>0$



x	0	$\frac{\pi}{2}$
φ'		$+$
φ	0	$+$ $\nearrow$
x^2		$+$
f'		$+$
f		$\nearrow$

22. (B) $f(x) = x \ln x - \frac{x^2}{2}$, $x>0$

$f'(x) = \ln x + x\,\frac{1}{x} - \frac{1}{2}\,2x$

$$\boxed{f'(x) = \ln x + 1 - x}$$

- $\ln x \leq x-1$

$\ln x - x + 1 \leq 0$

$f'(x) \leq 0$

$f \downarrow$

1. $e^x \geq x+1 \quad \forall x \in \mathbb{R}$.

2. $\ln x \leq x-1 \quad \forall x > 0$

3. $|\sin x| \leq |x| \quad \forall x \in \mathbb{R}$.

22. δ) $f(x) = x^2 \ln x + \frac{1}{2} x^2 - \frac{2}{3} x^3$, $x>0$

$$f'(x) = 2x \ln x + x^2 \frac{1}{x} + \frac{1}{2} 2x - \frac{2}{3} 3x^2$$

$$f'(x) = 2x \ln x + x + x - 2x^2$$

$$f'(x) = 2x \ln x + 2x - 2x^2$$

$$f'(x) = \underset{\oplus}{2x} \underset{\ominus}{(\ln x + 1 - x)} \le 0$$

- $\ln x \le x - 1$

$\ln x - x + 1 \le 0$

$f \downarrow$

23. (B) $f(x) = \frac{1}{2}x^2 + x - x\ln x$

$f'(x) = \frac{1}{2}2x + 1 - (\ln x + x\frac{1}{x})$

$f'(x) = x + \cancel{1} - \ln x \cancel{- 1}$

$f'(x) = x - \ln x > 0 \qquad f \nearrow$

- $\ln x \leq x - 1$

$1 \leq x - \ln x$

$0 < x - \ln x$

(5) $f(x) = (x+1)\ln x + e^x \quad , x > 0$

$f'(x) = \ln x + (x+1)\frac{1}{x} + e^x \qquad f \nearrow$

$f'(x) = \underbrace{\ln x + 1 + \frac{1}{x}}_{(+)} + e^x > 0$

- $\ln x \leq x - 1 \Rightarrow \ln\frac{1}{x} \leq \frac{1}{x} - 1 \Rightarrow \ln 1 - \ln x \leq \frac{1}{x} - 1$

$0 \leq \ln x + \frac{1}{x} - 1$

$2 \leq \ln x + \frac{1}{x} + 1$

23. ΣΤ $f(x) = e^{1-x} + \ln x$, $x > 0$

$f'(x) = -e^{1-x} + \frac{1}{x}$ $\qquad f'(1) = 0$

Σεναριο 1

$f''(x) = e^{1-x} - \frac{1}{x^2}$

Εχω προβλημα δεν μπορω να βγαλω ακρη με το προσημο.

Σεναριο 2

$f'(x) = \frac{1 - x e^{1-x}}{x}$

$\varphi(x) = 1 - x e^{1-x}$

$\varphi(1) = 0$

$\varphi'(x) = -e^{1-x} + x e^{1-x}$

$\varphi'(x) = e^{1-x}(x-1)$

$\to \varphi'(x) = 0$

$x = 1$

x	0	1
φ'	$-$	$+$
φ	$\searrow$	$\nearrow$
f'	$+$	$+$
f	$\nearrow$	$\nearrow$

$\varphi(x) \geq \varphi(1)$

$\varphi(x) \geq 0$

Σεναριο 3

$f'(x) = \frac{-1}{e^{x-1}} + \frac{1}{x}$

$f'(x) = \frac{e^{x-1} - x}{x e^{x-1}}$ $\oplus$

- $e^x \geq x + 1$

$e^{x-1} \geq x - 1 + 1$

$e^{x-1} \geq x$

$e^{x-1} - x \geq 0$

$f'(x) \geq 0$

$f \nearrow$.

24. (Β) $f(x) = \dfrac{x}{e^x - x}$

$e^x \geq x+1$

$e^x - x \geq 1$

$e^x - x > 0$

$D_f = \mathbb{R}$

$$f'(x) = \frac{e^x - x - (e^x - 1)}{(e^x - x)^2} = \frac{\cancel{e^x} - x - \cancel{e^x} + 1}{(e^x - x)^2}$$

$$\boxed{f'(x) = \frac{1 - x}{(e^x - x)^2}}$$

x		1	
f'	+	0	−
f	↗		↘

(δ) $f(x) = \dfrac{1}{x} - \dfrac{1}{εφx}$ $\quad x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = -\frac{1}{x^2} - \frac{-\frac{1}{συν^2x}}{εφ^2x} = -\frac{1}{x^2} + \frac{1}{\cancel{συν^2x}\,\frac{ημ^2x}{\cancel{συν^2x}}}$$

$$f'(x) = -\frac{1}{x^2} + \frac{1}{ημ^2x} = \frac{\overset{(+)}{x^2 - ημ^2x}}{x^2\,ημ^2x} \geq 0$$

f ↗

• $|ημx| \leq |x| \Rightarrow ημ^2x \leq x^2$

$x^2 - ημ^2x \geq 0$

25. Β $f(x) = \frac{1}{2}x^2 + συνx \quad , x \in R.$

$$f'(x) = \frac{1}{2}2x - ημx$$

$$f'(x) = x - ημx \qquad f'(0) = 0$$

$$f''(x) = 1 - συνx \geqslant 0$$

$$-1 \leq συνx \leq 1$$

$$1 \geqslant -συνx \geqslant -1$$

$$2 \geqslant 1 - συνx \geqslant 0$$

x		0	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

25. (δ) $f(x) = x\,\varepsilon\varphi x$ $\qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f'(x) = \varepsilon\varphi x + x \cdot \left(\frac{1}{\sigma\upsilon\nu^2 x}\right)$$

$$f'(x) = \varepsilon\varphi x - \frac{x}{\sigma\upsilon\nu^2 x}$$

$$f'(x) = \frac{\eta\mu x}{\sigma\upsilon\nu x} - \frac{x}{\sigma\upsilon\nu^2 x} = \frac{\eta\mu x\,\sigma\upsilon\nu x - x}{\sigma\upsilon\nu^2 x}$$

$$\boxed{\varphi(x) = \eta\mu x\,\sigma\upsilon\nu x - x} \qquad \varphi(0) = 0$$

$$\varphi'(x) = \sigma\upsilon\nu^2 x - \eta\mu^2 x - 1$$

$$\varphi'(x) = -\eta\mu^2 x - \eta\mu^2 x < 0$$

x	$-\pi/2$	0	$\pi/2$
φ'	$-$		$-$
φ	$\searrow$	$+\ 0\ -$	$\searrow$
f'	$+$		$-$
f	$\nearrow$		$\searrow$

Επομαω Μαθημα

17

2 16
3 17
4
5
6
7
8
11
12
13
14
15

23

2 αγε
3 αγε
4 αγε
5 αγ
6 αγε
7 αβδε
8 αγ
9 αβδε
10 αβδε.