Ενότητα 25

11. $f(x) = \begin{cases} x^3+1, & -1 \le x < 1 \\ (x-2)^2+1, & 1 \le x \le 3 \end{cases}$

• $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x^3+1) = 2$

• $\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x-2)^2+1 = 2$

$\Bigg\}\ \lim_{x\to1} f(x) = 2$

$f(1) = 2$

$f(1) = \lim_{x\to1} f(x)$

Συνεχής στο 1!

• $\lim_{x\to1^-} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \dfrac{x^3+1-2}{x-1} =$

$= \lim_{x\to1^-} \dfrac{x^3-1}{x-1} = \lim_{x\to1^-} \dfrac{3x^2}{1} = 3$

• $\lim_{x\to1^+} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \dfrac{(x-2)^2+1-2}{x-1} =$

$= \lim_{x\to1^+} \dfrac{(x-2)^2-1}{x-1} = \lim_{x\to1^+} \dfrac{2(x-2)}{1} = -2$

Η f όχι παραγωγίσιμη στο 1 άρα το

1 είναι κρίσιμο σημείο.

$-1 \le x < 1$	$1 < x \le 3$
$f_1(x) = x^3 + 1$	$f_2(x) = (x-2)^2 + 1$
$f_1'(x) = 3x^2$	$f_2'(x) = 2(x-2)$
$\to 3x^2 = 0$	$\to 2(x-2) = 0$
$\boxed{x = 0}$	$\boxed{x = 2}$
Κ.Σ	Κ.Σ

Κρίσιμα Σημεία

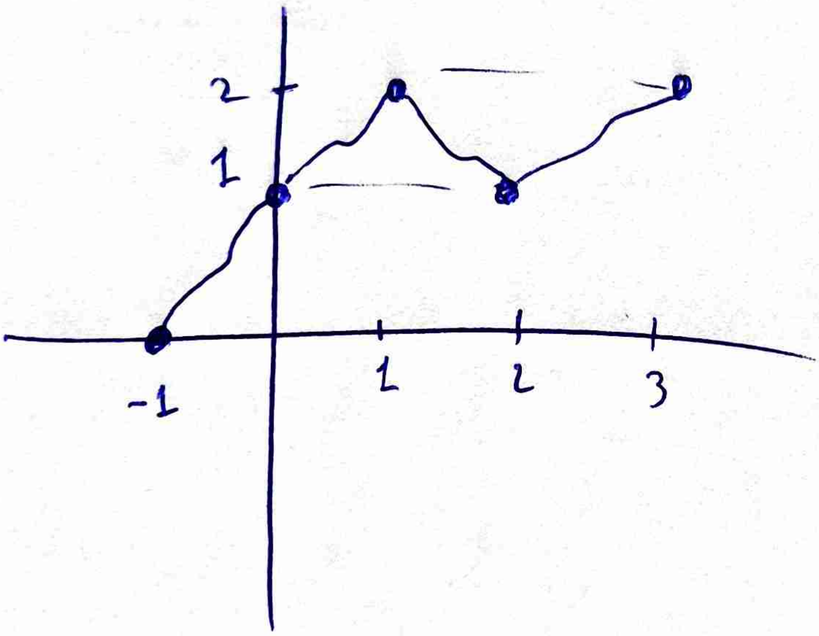
0, 1, 2

Πιθανά θέσεις ακροτάτων

0, 1, 2, −1, 3.

Σύνολο Τιμών

x	-1		0		1		2		3
f_1'		$+$	0	$+$					
f_2'						$-$	0	$+$	
f'		$+$		$+$		$-$		$+$	
f		↗		↗		↘		↗	



$T_f = [0,2]$

$f(-1) = 0$

$f(2) = 1$

$f(0) = 1$

$f(3) = 2$

$f(1) = 2$

14. (α) $f^3(x) + f(x) = x \quad , x \in R.$

Έστω ότι η f έχει ακρότατο

στο x_0

Fermat. $\underline{\underline{f'(x_0) = 0}}$

$$3f^2(x)\, f'(x) + f'(x) = 1$$

$$x = x_0$$

$$\cancel{3f^2(x_0)\, f'(x_0)} + \cancel{f'(x_0)} = 1$$

$$0 = 1 \; ,$$

Άτοπο!

Άρα δεν έχει ακρότατη

(B) $f^3(x)+2f(x)=x^3+x+1$

Έστω ότι η f έχει ακρότατο

στο x_0

Fermat $f'(x_0)=0$.

$3f^2(x)\,f'(x)+2f'(x)=3x^2+1$

$\xrightarrow{x=x_0}$

$3f^2(x_0)\,\cancel{f'(x_0)}\;\cancel{+2f'(x_0)}=3x_0^2+1$

$0=3x_0^2+1$

Άτοπο!

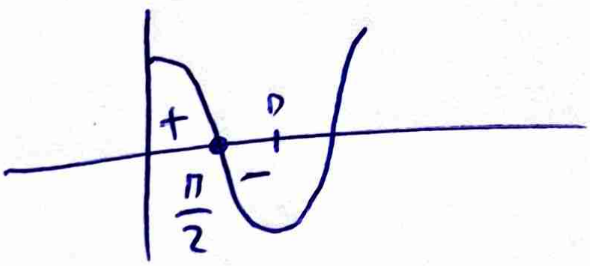
Ενότητα 26

5. (Β) $f(x) = \dfrac{ημx - συνx}{e^x}$ $\quad x \in [0, π]$

$$f'(x) = \frac{(συνx + ημx)e^x - (ημx - συνx)e^x}{(e^x)^2}$$

$$f'(x) = \frac{συνx + \cancel{ημx} - \cancel{ημx} + συνx}{e^x} = \frac{2συνx}{e^x}$$

$$f'(x) = \frac{2συνx}{e^x}, \quad x \in [0, π]$$



x	0		$\frac{π}{2}$		$π$
f'		$+$		$-$	
f		↗		↘	

(8) $f(x) = x\,\varepsilon\varphi x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f'(x) = \varepsilon\varphi x + x \cdot \frac{1}{\sigma\upsilon\nu^2 x} = \frac{\eta\mu x}{\sigma\upsilon\nu x} + \frac{x}{\sigma\upsilon\nu^2 x} = \frac{\eta\mu x\,\sigma\upsilon\nu x + x}{\sigma\upsilon\nu^2 x}$$

$$\boxed{\varphi(x) = \eta\mu x\,\sigma\upsilon\nu x + x} \qquad \varphi(0) = 0$$

$$\varphi'(x) = \sigma\upsilon\nu^2 x - \eta\mu^2 x + 1 = 2\sigma\upsilon\nu^2 x$$

x	$-\frac{\pi}{2}$		0		$\frac{\pi}{2}$
φ'		$+$		$+$	
φ		$\nearrow$ $-$	0	$+$ $\nearrow$	
f'		$-$		$+$	
f		$\searrow$		$\nearrow$	

$x < 0 \Rightarrow \varphi(x) < \varphi(0) \Rightarrow \varphi(x) < 0$

$x > 0 \Rightarrow \varphi(x) > \varphi(0) \Rightarrow \varphi(x) > 0$

10. $f(x) = x^2 + ax + B \ln x$

Fermat $f'(1) = 0$

$f'(3) = 0$

$f'(x) = 2x + a + \frac{B}{x}$

$f'(1) = 2 + a + B = 0 \Rightarrow \boxed{a + B = -2}$

$f'(3) = 6 + a + \frac{B}{3} = 0$

$18 + 3a + B = 0$

$3a + B = -18$

$\boxed{-3a - B = 18}$

(+)

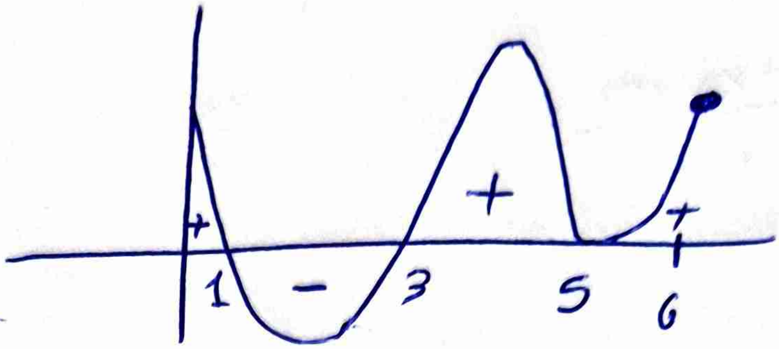
$-2a = 16$

$a = -8$

$B = 6$

7. $f:[0,6]\to\mathbb{R}$

f'



x	0	1	3	6
f'	+	−	+	
f	↗	↘	↗	

17. (a) $e^x > 2x$

$\underbrace{e^x - 2x}_{f(x)} > 0$

$f'(x) = e^x - 2$

$\to e^x - 2 = 0$

$e^x = 2$

$x = \ln 2$

x		$\ln 2$	
f'	−		+
f	↘		↗

$f(x) \geq f(\ln 2)$

$e^x - 2x \geq e^{\ln 2} - 2\ln 2$

$e^x - 2x \geq 2 - \ln 4$

$e^x - 2x \geq \ln e^2 - \ln 4$

$e^x - 2x \geq \ln \frac{e^2}{4} > 0$

$e^x - 2x > 0$

$\underline{\underline{e^x > 2x}}$

(β) $x^8 \geq 8x - 7 \quad \forall x \in \mathbb{R}.$

$$\underbrace{x^8 - 8x + 7}_{f(x)} \geq 0$$

$f'(x) = 8x^7 - 8 = 8(x^7 - 1)$

$\rightarrow x^7 - 1 = 0$

$x = 1$

x		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$x^8 - 8x + 7 \geq 0$

$x^8 - 8x \geq -7$

(γ) $x^2 \geq 2\ln x + 1 \quad , x > 0$

$$\underbrace{x^2 - 2\ln x - 1}_{f(x)} \geq 0$$

$f'(x) = 2x - \frac{2}{x} = 2\,\frac{x^2 - 1}{x}$

$\rightarrow f'(x) = 0 \quad x = 1 \quad x = -1$

x	0		1	
f'		$-$	0	$+$
f		$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$x^2 - 2\ln x - 1 \geq 0$

$x^2 \geq 2\ln x + 1$

(8) $x \leq \frac{1}{a} e^{x-1} + \ln a$

$$\underbrace{x - \frac{1}{a} e^{x-1} - \ln a}_{f(x)} \leq 0$$

$f'(x) = 1 - \frac{1}{a} e^{x-1}$

$\rightarrow f'(x) = 0$

$1 - \frac{1}{a} e^{x-1} = 0$

$1 = \frac{1}{a} e^{x-1}$

$a = e^{x-1}$

$\ln a = x - 1$

$x = \ln a + 1$

x		$\ln a + 1$	
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$f(x) \leq f(\ln a + 1)$

$f(x) \leq 0$

$f(\ln a + 1) = \cancel{\ln a} + 1 - \frac{1}{a} e^{\ln a + 1 - 1} \cancel{- \ln a} = 1 - \frac{1}{a} e^{\ln a} = 1 - \frac{1}{\cancel{a}} \cancel{a}$

$= 0$

18. $\sigma\upsilon\nu x < \dfrac{x^3}{6} - \dfrac{x^2}{2} + 1 \quad \forall x > 0$

$$\underbrace{\sigma\upsilon\nu x - \frac{x^3}{6} + \frac{x^2}{2} - 1}_{f(x)} < 0$$

$$f'(x) = -\eta\mu x - \frac{1}{6}3x^2 + \frac{1}{2}2x$$

$$f'(x) = -\eta\mu x - \frac{1}{2}x^2 + x \qquad f'(0) = 0$$

$$f''(x) = -\sigma\upsilon\nu x - \frac{1}{\cancel{2}}\cancel{2}x + 1$$

$$f''(x) = -\sigma\upsilon\nu x - x + 1 \qquad f''(0) = 0$$

$$f'''(x) = \eta\mu x - 1 \leq 0$$

x		0	
f'''			$-$
f''		0	$\searrow$ $-$
f'		0	$\searrow$ $-$
f			$\searrow$

$x > 0 \Rightarrow f''(x) < f''(0)$

$f''(x) < 0$

Αν $x > 0 \Rightarrow f(x) < f(0)$

$f(x) < 0$ ✓

Ενότητα 27

1. (B) $f(x) = x + \frac{2\ln x}{x}$

$$f'(x) = 1 + 2\,\frac{\frac{1}{x}x - \ln x \cdot 1}{x^2}$$

$$f'(x) = 1 + 2\,\frac{1-\ln x}{x^2} = \frac{x^2+2-2\ln x}{x^2}$$

$$\boxed{\varphi(x) = x^2+2-2\ln x}$$

$$\varphi'(x) = 2x - \frac{2}{x} = 2\,\frac{x^2-1}{x}$$

$\longrightarrow \varphi'(x)=0 \rightarrow x=\pm 1$

x	0	1	
φ'	$-$	0	$+$
φ	$\searrow +$		$+ \nearrow$
f'	$+$		$+$
f	$\nearrow$		$\nearrow$

$\varphi(x) \geq \varphi(1)$

$\varphi(x) \geq 3$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x + \frac{2\ln x}{x} = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x + \cancel{\frac{2\ln x}{x}} = +\infty$$

$$\to \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\Sigma\ T_f = \mathbb{R}\,.$$

10. ⓐ $x^3-3x+1=0$

$f(x)=x^3-3x+1 \quad , x\in\mathbb{R}.$

$f'(x)=3x^2-3 \quad =3(x^2-1)$

$\to f'(x)=0 \qquad \boxed{x=\pm 1}$

x	$-\infty$	-1	1	$+\infty$
f'	$+$	$-$	$+$	
f	$-\infty \nearrow$	$\searrow$	$\nearrow$	

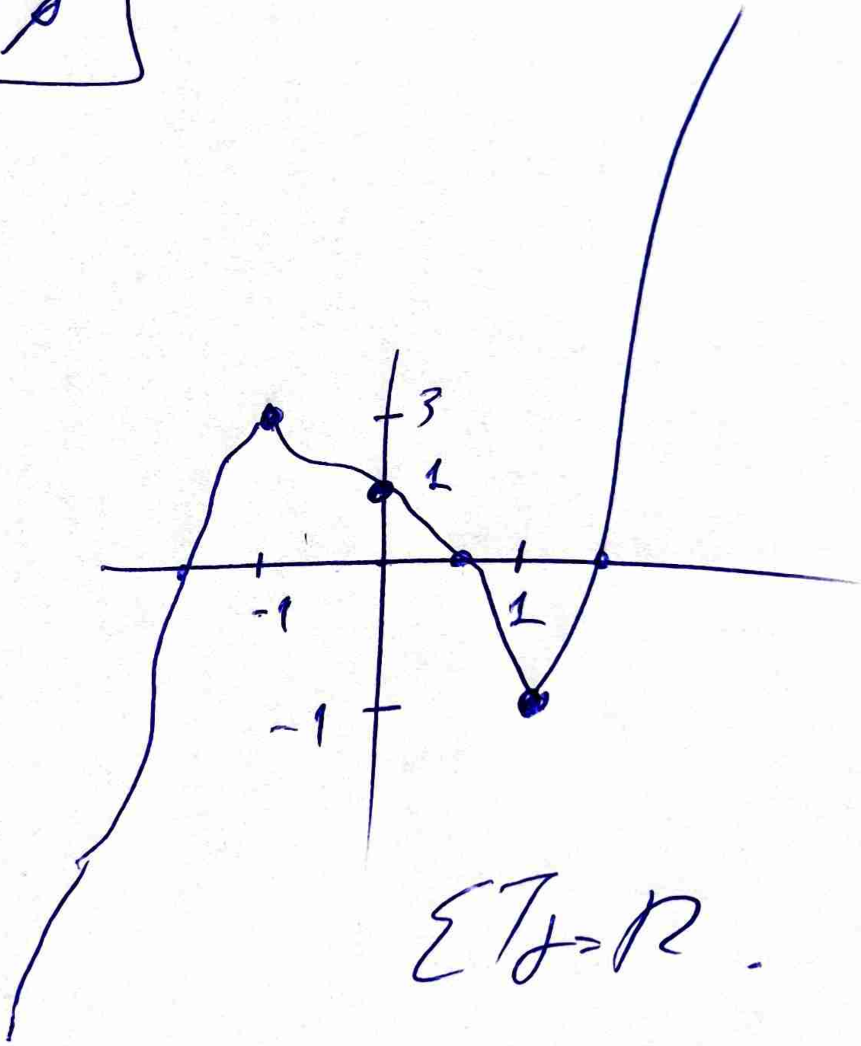
$\lim\limits_{x\to-\infty} f(x)=-\infty$

$f(-1)=3$

$f(1)=-1$

$f(0)=1$

$\lim\limits_{x\to+\infty} f(x)=+\infty$



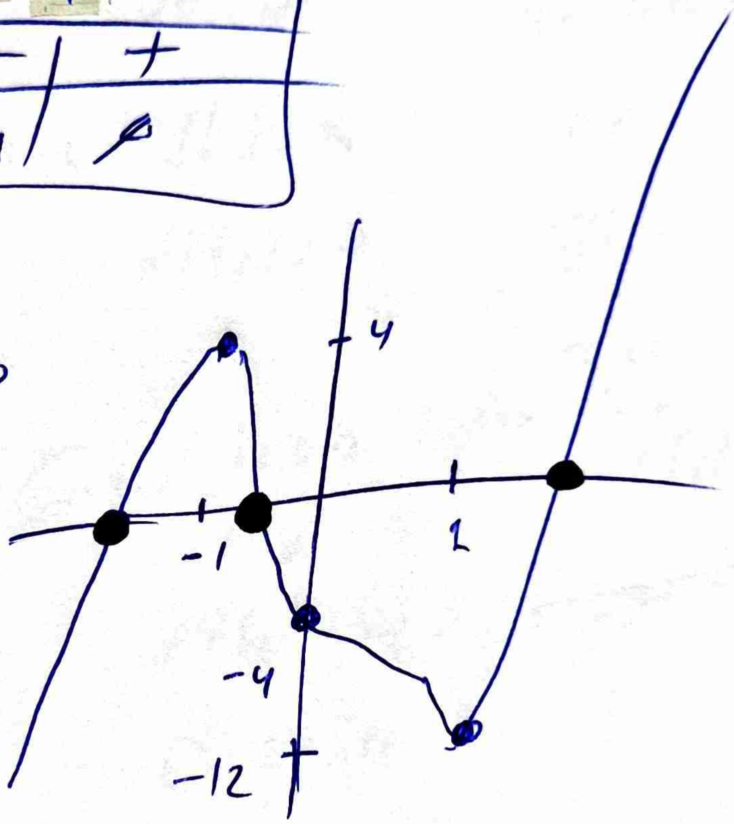
$\Sigma T_f=\mathbb{R}.$

21. $f(x) = x^4 - 6x^2 - 4x - 1$

$f'(x) = 4x^3 - 12x - 4$

$f''(x) = 12x^2 - 12 = 12(x^2-1)$

x	$-\infty$		-1		1	
f''		$+$		$-$		$+$
f'	$-\infty$	$\nearrow$		$\searrow$		$\nearrow$



$\lim\limits_{x \to -\infty} f'(x) = -\infty$

$f'(-1) = 4$

$f'(1) = -12$

$f'(0) = -4$

$\lim\limits_{x \to +\infty} f'(x) = +\infty$

$x < -1$

- f' συνεχής
- $f' \nearrow$
- $ΣΤ_{f'} = (-\infty, 4)$

Το $0 \in ΣΤ_{f'}$

άρα $\exists!\ \xi_1 < 0$

τ.ω $f'(\xi_1) = 0$

$-1 \le x \le 1$

- f' συνεχής
- $f' \searrow$
- $ΣΤ_{f'} = [-12, 4]$

Το $0 \in ΣΤ_{f'}$

άρα $\exists!\ \xi_2$ τ.ω

$f'(\xi_2) = 0$

Έστω $\xi_2 \ge 0$

$f' \searrow$

$f'(\xi_2) \le f'(0)$

$0 \le -4$

Άτοπο

$\xi_2 < 0$

$x > 1$

- f' συνεχής
- $f' \nearrow$
- $ΣΤ_{f'} = [-12, +\infty)$

Το $0 \in ΣΤ_{f'}$

άρα

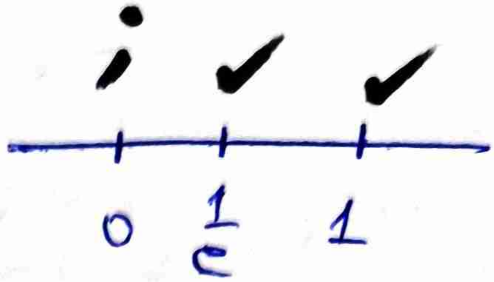
$\exists!\ \xi_3$

τ.ω

$f'(\xi_3) > 0$

13. (a) $2x+\ln x=1$ $\quad (0,1)$

$\underbrace{2x+\ln x-1}_{f(x)}=0$



$f'(x)=2+\frac{1}{x}>0$

$f \nearrow$

$f(1)=1$

$f(\frac{1}{e})=2\frac{1}{e}+\ln\frac{1}{e}-1=\frac{2}{e}+\ln 1-\ln e-1=$

$=\frac{2}{e}-2<0$

$f(1)f(\frac{1}{e})<0$

Bolzano $\exists\, \xi \in (\frac{1}{e},1)$ τ.ω

$f(\xi)=0$ και λόγω μονοτονίας

μοναδικό.

7. ① $f(x)=\begin{cases} -x^2+1, & x<0 \\ e^x - x, & x\geq 0 \end{cases}$

$D_{f^{-1}} = ΣΤ_f$.

$\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x) = 1 \\ \lim\limits_{x\to 0^+} f(x) = 1 \end{array}\right\} \lim\limits_{x\to 0} f(x) = 1 = f(0)$

Συνεχής στο 0!

$x<0$

$f_1(x) = -x^2+1$

$f_1'(x) = -2x > 0$

$x>0$

$f_2(x) = e^x - x$

$f_2'(x) = e^x - 1$

$\to e^x - 1 = 0$

$e^x = 0$

$\underline{\underline{x=0}}$

x	$-\infty$	0	$+\infty$
f_1'	$+$		//////
f_2'	//////	0	$+$
f'	$+$		$+$
f	$-\infty \nearrow$		$\nearrow +\infty$

$\underline{ΣΤ_f = \mathbb{R}.}$

$\lim\limits_{x\to -\infty} f(x) = -\infty$

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} e^x - x =$

$= \lim\limits_{x\to +\infty} e^x\left(1 - \frac{x}{e^x}\right) = +\infty$

$x < -1$

- f συνεχής
- $f \nearrow$
- $ΣΤf = (-\infty, 3)$

Το $0 \in ΣΤf$

άρα $\exists!\ \xi_1 < 0$

τ.ω $f(\xi_1) = 0$.

$-1 \leq x \leq 1$

- f συνεχής
- $f \searrow$
- $ΣΤf = [-1, 3]$

Το $0 \in ΣΤf$.

$\exists!\ \xi_2$ τ.ω

$f(\xi_2) = 0$

$x > 1$

- f συνεχής
- $f \nearrow$
- $ΣΤf = [-1, +\infty)$

Το $0 \in ΣΤf$

άρα $\exists!\ \xi_3 > 0$

τ.ω $f(\xi_3) = 0$

Έστω $\xi_2 < 0$

$f \searrow$

$f(\xi_2) > f(0)$

$0 > 1$

Άτοπο

$\xi_2 > 0$

Ενότητα 30

1. $f(x) = e^{x-1} - \ln x + x \quad , x>0$

ⓐ $f'(x) = e^{x-1} - \frac{1}{x} + 1$

$f''(x) = e^{x-1} + \frac{1}{x^2} > 0 \qquad f$ κυρτή

ⓑ $f'(e^x) > e + \frac{1}{2}$

$f'(e^x) > f'(2)$

$f' \nearrow$

$e^x > 2$

$x > \ln 2$

ⓒ $y - f(1) = f'(1)(x-1)$

$y - 2 = x - 1$

$\boxed{y = x + 1}$

(δ) i) Νδο $f(x)-1 \geq x \quad \forall x>0$

$$f(x) \geq x+1$$

$$\boxed{\begin{array}{l} \text{Αφού } f \text{ κυρτή} \quad f(x) \geq x+1 \\ \text{Το } \text{"="} \text{ για } x=1 \end{array}}$$

ii) $\frac{f(x)}{x} = 1 + \frac{1}{x}$

$$f(x) = x+1$$

$$\underline{\underline{x=1}}$$

(ε) i) $\frac{f(x^2+1)}{x^2+2} \geq 1$

$$f(x^2+1) \geq x^2+2$$

όμως $f(x) \geq x+1$

$$f(x^2+1) \geq x^2+1+1$$

$$f(x^2+1) \geq x^2+2$$

ii) $e^{-x}(f(e^x)-1) = 1$

$$f(e^x-1) = e^x$$

$f(x) \geq x+1$

$f(e^x-1) \geq e^x-1+1$

$f(e^x-1) \geq e^x$

To "=" για $e^x-1 = 1$

$e^x = 2$

$\underline{\underline{x = \ln 2}}$

2. $f(x) = \dfrac{x^2+3}{e^x}$

ⓐ $f'(x) = \dfrac{2xe^x - (x^2+3)e^x}{e^{2x}} = \dfrac{2x - x^2 - 3}{e^x}$

$\Delta < 0$

$f'(x) = \dfrac{-x^2+2x-3}{e^x} < 0$ $\quad f \downarrow$

$f''(x) = \dfrac{(-2x+2)e^x - (-x^2+2x-3)e^x}{e^{2x}}$

$f''(x) = \dfrac{-2x+2+x^2-2x+3}{e^x}$

$\Delta < 0$

$f''(x) = \dfrac{x^2-4x+5}{e^x} > 0$ $\quad f$ κυρτή.

ⓑ $y - f(0) = f'(0)(x-0)$

$y - 3 = -3x$

$\boxed{y = -3x+3}$

8) i) $f\left(f(x)+3x-3\right) = 3$

$$f\left(f(x)+3x-3\right) = f(0)$$

f: 1-1

$$f(x)+3x-3 = 0$$

$$f(x) = -3x+3 \Rightarrow \underline{\underline{x=0}}$$

Αφού f κυρτή τότε $f(x) \geq -3x+3$

Το "=" για $x=0$

ii) $f(ημ^2 x) = 3\,συν^2 x$

$$f(ημ^2 x) = 3(1-ημ^2 x)$$

$$f(ημ^2 x) = -3ημ^2 x + 3$$

$$f(x) \geq -3x+3$$

$$f(ημ^2 x) \geq -3ημ^2 x+3$$

Το "=" $ημ^2 x = 0 \Rightarrow ημx = 0$

$\underline{\underline{x = 2κπ}}$ $\underline{\underline{x = 2κπ+π}}$

① Νδο $\dfrac{x^2}{3} + 1 \geqslant (1-x)e^x$

$x^2 + 3 \geqslant 3(1-x)e^x$

$\dfrac{x^2+3}{e^x} \geqslant -3x+3$

$f(x) \geqslant -3x+3$ ✓

Επομενο Μαθημα

30

6
7
8
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12
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