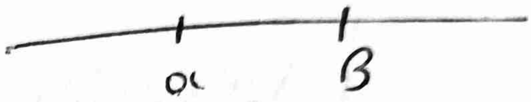
Ενότητα 21

24. $|f'(x)| \le 2$

(α) Ν.δ.ο $|f(\beta) - f(\alpha)| \le 2|\beta - \alpha|$

1η.

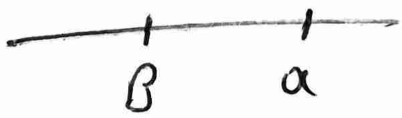


$$f'(\xi) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$$

όμως $|f'(\xi)| \le 2 \Rightarrow \left| \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \right| \le 2$

$$\frac{|f(\beta) - f(\alpha)|}{|\beta - \alpha|} \le 2 \Rightarrow |f(\beta) - f(\alpha)| \le 2|\beta - \alpha|.$$

2η



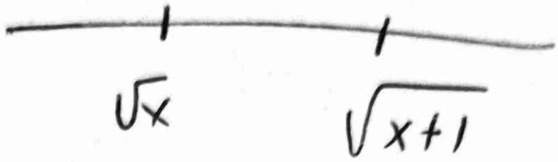
$$f'(\xi) = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

$$f'(\xi) = \frac{-(f(\beta) - f(\alpha))}{-(\beta - \alpha)} = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$$

όμοια.

(3) $\lim\limits_{x \to +\infty} f(\sqrt{x+1}) - f(\sqrt{x})$

$$f'(\xi) = \frac{f(\sqrt{x+1}) - f(\sqrt{x})}{\sqrt{x+1} - \sqrt{x}}$$



and $|f'(x)| \leq 2$

$$|f'(\xi)| \leq 2 \iff \left| \frac{f(\sqrt{x+1}) - f(\sqrt{x})}{\sqrt{x+1} - \sqrt{x}} \right| \leq 2$$

$$| f(\sqrt{x+1}) - f(\sqrt{x}) | \leq 2 |\sqrt{x+1} - \sqrt{x}|$$

$$-2|\sqrt{x+1} - \sqrt{x}| \leq f(\sqrt{x+1}) - f(\sqrt{x}) \leq 2(\sqrt{x+1} - \sqrt{x})$$

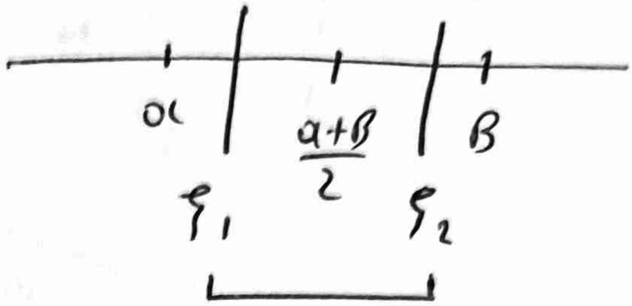
- $\lim\limits_{x \to +\infty} \sqrt{x+1} - \sqrt{x} = \lim\limits_{x \to +\infty} \dfrac{1}{\sqrt{x+1} + \sqrt{x}} = 0$
- $\lim\limits_{x \to +\infty} -2|\sqrt{x+1} - \sqrt{x}| = 0$
- $\lim\limits_{x \to +\infty} 2|\sqrt{x+1} - \sqrt{x}| = 0$

$$\left.\begin{matrix} \lim\limits_{x \to +\infty} -2|\sqrt{x+1}-\sqrt{x}| = 0 \\ \lim\limits_{x \to +\infty} 2|\sqrt{x+1}-\sqrt{x}| = 0 \end{matrix}\right\} \lim\limits_{x \to +\infty} f(\sqrt{x+1}) - f(\sqrt{x}) =$$

28. ⓐ $f' \uparrow$

$$f\left(\frac{\alpha+\beta}{2}\right) < \frac{f(\alpha)+f(\beta)}{2}$$

$$f'(\xi_1) = \frac{f\left(\frac{\alpha+\beta}{2}\right) - f(\alpha)}{\frac{\alpha+\beta}{2} - \alpha}$$



$$f'(\xi_1) = \frac{f\left(\frac{\alpha+\beta}{2}\right) - f(\alpha)}{\frac{\beta-\alpha}{2}}$$

$$\boxed{f'(\xi_1) = 2\,\frac{f\left(\frac{\alpha+\beta}{2}\right) - f(\alpha)}{\beta-\alpha}}$$

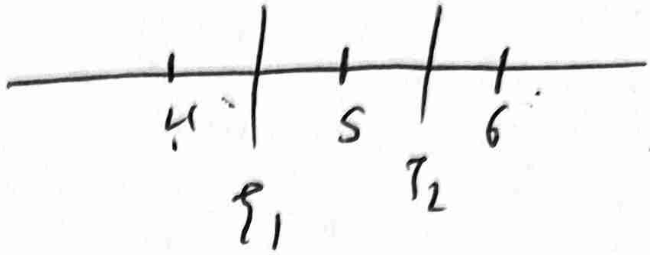
$$f'(\xi_2) = \frac{f(\beta) - f\left(\frac{\alpha+\beta}{2}\right)}{\beta - \frac{\alpha+\beta}{2}} = 2\,\frac{f(\beta) - f\left(\frac{\alpha+\beta}{2}\right)}{\beta-\alpha}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$\cancel{2}\,\frac{f\left(\frac{\alpha+\beta}{2}\right) - f(\alpha)}{\cancel{\beta-\alpha}} < \cancel{2}\,\frac{f(\beta) - f\left(\frac{\alpha+\beta}{2}\right)}{\cancel{\beta-\alpha}}$$

$$2f\left(\frac{\alpha+\beta}{2}\right) < f(\alpha) + f(\beta).$$

③ Ν.δ.ο. $\eta\mu 4 + \eta\mu 6 > 2\eta\mu 5$



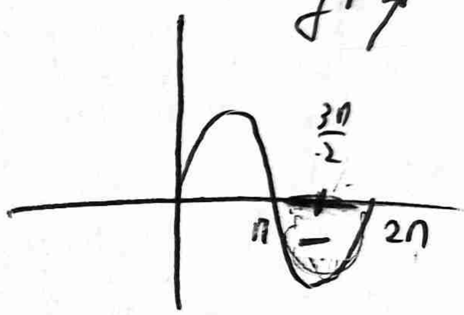
$$\boxed{\begin{aligned} f(x) &= \eta\mu x \\ f'(x) &= \sigma\upsilon\nu x \end{aligned}}$$

$$f'(\xi_1) = \frac{f(5)-f(4)}{5-4} = \eta\mu 5 - \eta\mu 4$$

$$f'(\xi_2) = \frac{f(6)-f(5)}{6-5} = \eta\mu 6 - \eta\mu 5 .$$

$f''(x) = -\eta\mu x > 0$

$f' \uparrow$



$\xi_1 < \xi_2$

$f' \uparrow$

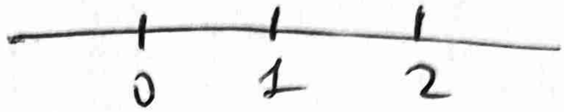
$f'(\xi_1) < f'(\xi_2)$

$\eta\mu 5 - \eta\mu 4 < \eta\mu 6 - \eta\mu 5$

$2\eta\mu 5 < \eta\mu 6 - \eta\mu 4$.

26. $f(0)=-1$ $\quad$ $f(2)=1$

$f'(x)\ge 1$ $\quad$ Νδο $f(1)=0$.



$$f'(\xi_1)=\frac{f(1)-f(0)}{1-0}=f(1)+1$$

$$f'(\xi_2)=\frac{f(2)-f(1)}{2-1}=1-f(1)$$

$$f'(\xi_1)\ge 1 \Rightarrow f(1)+1\ge 1 \Rightarrow f(1)\ge 0$$

$$f'(\xi_2)\ge 1 \Rightarrow \cancel{1}-f(1)\ge \cancel{1}$$

$$-f(1)\ge 0$$

$$f(1)\le 0$$

$$f(1)=0$$

31. $f(1)=2$ $\qquad f(2)=4$

(α) Ν.δ.ο $\exists x_0 \in (1,2)$ τ.ω $f(x_0)=2(3-x_0)$

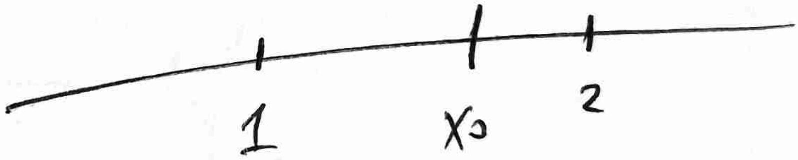
$$f(x)=2(3-x)$$

$$\underbrace{f(x)-2(3-x)}_{g(x)}=0$$

$$g(1)=f(1)-2\cdot 2 = 2-4=-2<0$$

$$g(2)=f(2)-2. = 4-2=2>0 \quad \checkmark$$

(β). Ν.δ.ο $\exists \xi_1, \xi_2 \in (1,2)$ $\quad f'(\xi_1)f'(\xi_2)=4$



$$f'(\xi_1)=\frac{f(x_0)-f(1)}{x_0-1}=\frac{2(3-x_0)-2}{x_0-1}=\frac{6-2x_0-2}{x_0-1}$$

$$=\frac{4-2x_0}{x_0-1}=\frac{2(2-x_0)}{x_0-1}$$

$$f'(\xi_2) = \frac{f(2) - f(x_0)}{2 - x_0} = \frac{4 - 2(3 - x_0)}{2 - x_0} =$$

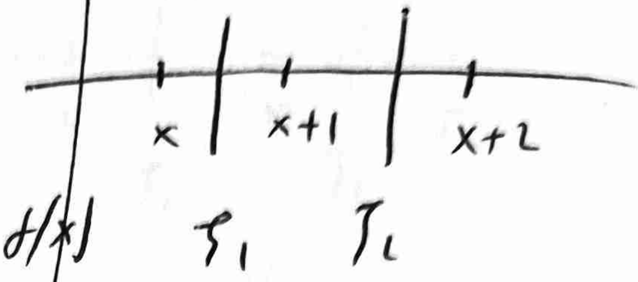
$$= \frac{4 - 6 + 2x_0}{2 - x_0} = \frac{2x_0 - 2}{2 - x_0}$$

$$= \frac{2(x_0 - 1)}{2 - x_0}.$$

$$f'(\xi_1)\, f'(\xi_2) = \frac{2\cancel{(2 - x_0)}}{\cancel{x_0 - 1}} \cdot \frac{2\cancel{(x_0 - 1)}}{\cancel{2 - x_0}}$$

$$= 4.$$

29. (a) $f(x)+f(x+2)=2f(x+1)$



$$f'(\xi_1)=\frac{f(x+1)-f(x)}{x+1-x}=f(x+1)-f(x)$$

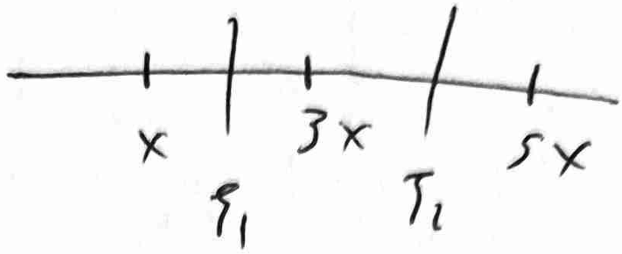
$$f'(\xi_2)=\frac{f(x+2)-f(x+1)}{x+2-x-1}=f(x+2)-f(x+1)$$

$$f(x+2)-f(x+1)=f(x+1)-f(x)$$

$$\underbrace{f'(\xi_1)=f'(\xi_2)}_{\text{Rolle}}$$

$$\underline{f''(\xi)=0}$$

(13) $f(x) + f(5x) = 2f(3x)$.



$$f'(\xi_1) = \frac{f(3x)-f(x)}{3x-x} = \frac{f(3x)-f(x)}{2x}$$

$$f'(\xi_2) = \frac{f(5x)-f(3x)}{5x-3x} = \frac{f(5x)-f(3x)}{2x}$$

$$\frac{f(5x)-f(3x)}{2x} = \frac{f(3x)-f(x)}{2x}$$

$$\underbrace{f'(\xi_1) = f'(\xi_2)}_{\text{Rolle}}$$

$$f'(\xi) = 0.$$

27. $f^2(x) = x - \ln x$ $\qquad$ $f(1) = 1$

$$f^2(x) = \sqrt{x - \ln x}^{\,2}$$

$$|f(x)| = \left|\sqrt{x - \ln x}\right|$$

$$|f(x)| = \sqrt{x - \ln x}$$

Ρ,7λ $f(x)$

$f(x) = 0$

$|f(x)| = 0$

$\sqrt{x - \ln x} = 0$

$x - \ln x = 0$

Άτοπο!

Άρα $f(x) \neq 0$ συνεχ

$\Rightarrow f(x) > 0$ ή $f(x) < 0$.

$f(1) = 1$

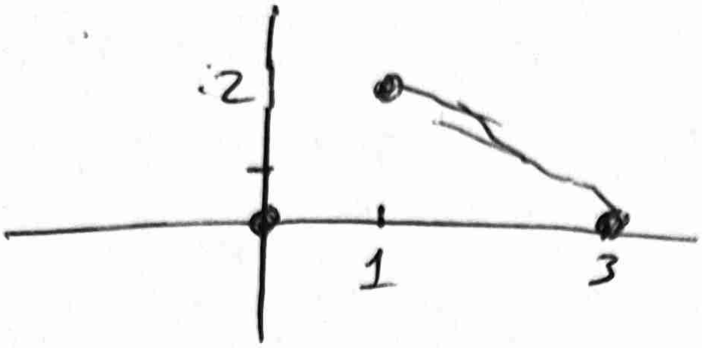
$$f(x) = \sqrt{x - \ln x}$$

. $\ln x \leq x - 1$

$1 \leq x - \ln x$

$0 < x - \ln x$

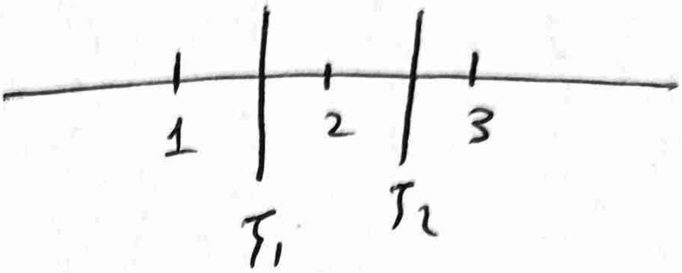
33.



$f(1) = 2$

$f(3) = 0$.

x	1	3
$f(x)$	2 $\searrow$	0



$$f'(\xi_1) = \frac{f(2) - f(1)}{2-1} = \frac{f(2)-2}{1}$$

$$f'(\xi_2) = \frac{f(3) - f(2)}{3-2} = \frac{0 - f(2)}{1}$$

$$f'(\xi_1) + f'(\xi_2) = \cancel{f(2)} - 2 - \cancel{f(2)}$$

$$= -2$$

22 Ενότητα

$f(0) = 1$

8. $x \cdot \left(f(x) + x f'(x) \right) = - f'(x)$

$x f(x) + x^2 f'(x) = - f'(x)$

$(x^2+1) f'(x) = - x f(x)$

Νδο $f(x) = \dfrac{1}{\sqrt{x^2+1}}$

$$\underbrace{f(x)\sqrt{x^2+1} - 1}_{g(x)} = 0$$

$$g'(x) = f'(x)\sqrt{x^2+1} + f(x)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}$$

$$g'(x) = \frac{f'(x)\sqrt{x^2+1}^{\,2} + x f(x)}{\sqrt{x^2+1}}$$

$$g'(x) = \frac{f'(x)(x^2+1) + x f(x)}{\sqrt{x^2+1}}$$

$$g'(x) = \frac{-x f(x) + x f(x)}{\sqrt{x^2+1}} = 0$$

$g(x) = c \Rightarrow$

$$f(x)\sqrt{x^2+1}-1 = C$$

$$x = 0$$

$$1-1 = 0 = C.$$

$$f(x) = \frac{1}{\sqrt{x^2+1}}$$

10. $f'(x) + f(x)\,εφx = 0$ $\qquad f(0)=1$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Νδο $f(x) = συν\,x$

$$\underbrace{f(x) - συνx}_{g(x)} = 0$$

$$g'(x) = f'(x) + ημx = -f(x)\,εφx + ημx$$

$$g'(x) = -f(x)\frac{ημx}{συνx} + ημx$$

Δεν βγαίνει!

$$f(x) = συνx$$

$$\frac{f(x)}{συνx} = 1 \Rightarrow \underbrace{\frac{f(x)}{συνx} - 1}_{g(x)} = 0$$

$$g'(x) = \frac{f'(x)\,συνx + f(x)\,ημx}{συν^2x} = \frac{-f(x)\,εφx\,συνx + f(x)\,ημx}{συν^2x}$$

$$g'(x) = \frac{-f(x)\,\dfrac{\eta\mu x}{\cancel{\sigma\upsilon\nu x}}\,\cancel{\sigma\upsilon\nu x} + f(x)\,\eta\mu x}{\sigma\upsilon\nu^2 x}$$

$$g'(x) = \frac{\cancel{-f(x)\,\eta\mu x} + \cancel{f(x)\,\eta\mu x}}{\sigma\upsilon\nu^2 x}$$

$$g'(x) = 0$$

$$g(x) = c$$

$$\frac{f(x)}{\sigma\upsilon\nu x} - 1 = c$$

$\underline{x = 0}$

$$f(0) - 1 = c.$$

$$c = 0.$$

$$\underline{f(x) = \sigma\upsilon\nu x}$$

16. $f'(x) = \dfrac{(x+1)^2}{x^2+1} f(x).$

(α) Νδο $g(x) = \dfrac{f(x)}{x^2+1}$ νδο $g(x) = ce^x$

Θεωρία

$g'(x) = g(x)$

$\Rightarrow g(x) = ce^x$

$$g'(x) = \frac{f'(x)(x^2+1) - f(x)2x}{(x^2+1)^2}$$

$$g'(x) = \frac{f(x)\cdot\dfrac{(x+1)^2}{\cancel{x^2+1}}\cancel{(x^2+1)} - 2xf(x)}{(x^2+1)^2}$$

$$g'(x) = \frac{f(x)(x^2+\cancel{2x}+1\cancel{-2x})}{(x^2+1)^2}$$

$$g'(x) = \frac{f(x)\cancel{(x^2+1)}}{(x^2+1)^{\cancel{2}}}$$

$$g'(x) = \frac{f(x)}{x^2+1}$$

$\underline{\underline{g'(x) = g(x)}}$

$$g'(x)=g(x)$$

$$g'(x)-g(x)=0$$

$$e^{-x}g'(x)-e^{-x}g(x)=0$$

$$\left(e^{-x}g(x)\right)'=0$$

$$e^{-x}g(x)=c$$

$$\frac{1}{e^{x}}g(x)=c$$

$$g(x)=ce^{x}$$

(B) Αφου $g(x)=ce^{x}$ $\qquad f(0)=1$

$$\frac{f(x)}{x^2+1}=ce^{x}$$

$$f(0)=c=1$$

$$f(x)=(x^2+1)e^{x}$$

18. $2f(x)+4xf'(x)+(x^2+1)f''(x)=0$

$f(0)=0 \qquad f'(0)=1.$

$$f(x)=\frac{x}{x^2+1}$$

$$\underbrace{f(x)(x^2+1)-x}_{g(x)}=0$$

$$g'(x)=f'(x)(x^2+1)+f(x)2x-1$$

$$g''(x)=f''(x)(x^2+1)+f'(x)2x+f'(x)2x+f(x)2$$

$$g''(x)=f''(x)(x^2+1)+4xf'(x)+2f(x)$$

$$g''(x)=0$$

$$g'(x)=c$$

$$f'(x)(x^2+1)+2xf(x)-1=c$$

$\underline{x=0}$

$$f'(0)-1=c \Rightarrow c=0.$$

$$\underbrace{f'(x)(x^2+1)+2x\,f(x)}-1=0$$

$$\left(f(x)(x^2+1)-x\right)'=0$$

$$f(x)(x^2+1)-x=c$$

$$x=0$$

$$f(0)-0=c$$

$$c=0$$

$$f(x)=\frac{x}{x^2+1}$$

Θεωρημα Fermat

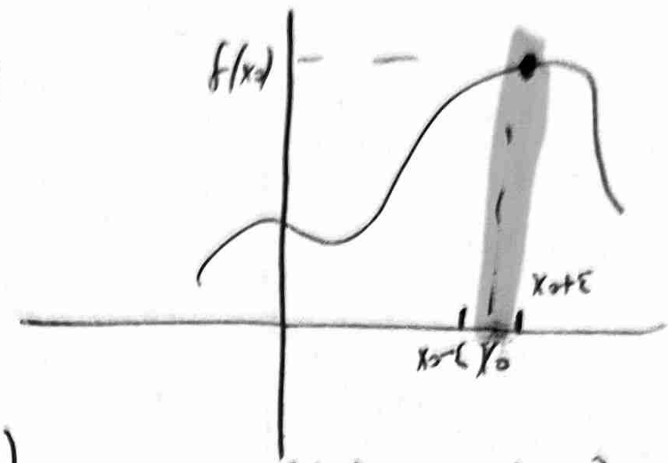
Εστω f ορισμενη στο Δ

1. Αν η f εχει ακροτατο στο x_0
2. Αν η f παρ/μη στο x_0
3. Αν το x_0 εσωτερικο του Δ

Τοτε το Θεωρημα του Fermat μας λεει οτι

$$f'(x_0) = 0.$$

Εστω οτι η f εχει Μεγιστο το $A(x_0, f(x_0))$



Στο $(x_0-\varepsilon, x_0+\varepsilon)$ το $f(x) \le f(x_0)$

$$\lim_{x\to x_0^-} \frac{\overset{\ominus}{f(x)-f(x_0)}}{\underset{\ominus}{x-x_0}} \ge 0$$

$$\lim_{x\to x_0^+} \frac{\overset{\ominus}{f(x)-f(x_0)}}{\underset{\oplus}{x-x_0}} \le 0.$$

Η f παρ/μη στο x_0

Το οριο αυτο $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ πρεπει να υπαρχει.

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$$

$f'(x_0)=0$

Επομενο Μαθημα

Εξεταζω σε

- Bolzano 1.22 + 1.23
- ΘΕΤ 1.24 + 1.25
- ΘΜΕΤ. 1.26
- Rolle 1.34 + 1.35
- ΘΜΤ 1.36 + 1.37
- Fermat. 1.40.

+ γεωμετρικα ερμηνεια.