Θεωρημα σταθερης συναρτησης

Εστω f ορισμενη στο Δ

1. Αν η f συνεχης στο Δ.
2. Αν $f'(x) = 0$ $\forall x$ εσωτερικο του Δ

Τοτε η f σταθερη σε ολο το Δ.

Πορισμα

Εστω f, g ορισμενες στο Δ.

1. Αν f, g συνεχεις στο Δ.
2. Αν $f'(x) = g'(x)$ $\forall x$ εσωτερικο του Δ.

Τοτε $f(x) = g(x) + c$ $\forall x \in \Delta$.

SOS

Τα παραπανω δυο θεωρηματα ισχυουν ΜΟΝΟ για ενωσαιο διαστημα και οχι για ενωση διαστηματων.

Προσοχή

Έστω ότι $f'(x) = f(x) \quad \forall x \in \mathbb{R}$

Τότε $f(x) = c e^{x}$

Απόδειξη

$f'(x) = f(x)$

$e^{x} f'(x) = e^{x} f(x)$

$e^{x} f'(x) - e^{x} f(x) = 0$

$$\frac{e^{x} f'(x) - e^{x} f(x)}{e^{2x}} = 0$$

$$\left(\frac{f(x)}{e^{x}}\right)' = 0$$

$g'(x) = 0 \Rightarrow g(x) = c$

$\frac{f(x)}{e^{x}} = c \Rightarrow \underline{\underline{f(x) = ce^{x}}}$

4. Εστω $f: \mathbb{R} \to \mathbb{R}$ παρ/μη

$$f'(x) = f(x) + 2xe^{x} \quad \forall x \in \mathbb{R}$$

ⓐ Νδο $g(x) = \dfrac{f(x)}{e^{x}} - x^{2}$ σταθερη

$$g'(x) = \frac{f'(x)e^{x} - f(x)e^{x}}{e^{2x}} - 2x = \frac{f'(x) - f(x)}{e^{x}} - 2x$$

$$g'(x) = \frac{f'(x) - f(x) - 2xe^{x}}{e^{x}} = \frac{\cancel{f(x)} + 2xe^{x} - \cancel{f(x)} - 2xe^{x}}{e^{x}}$$

$$g'(x) = \frac{2x\cancel{e^{x}} - 2x\cancel{e^{x}}}{e^{x}} = 0 \Rightarrow g'(x) = 0$$

Αφου $g'(x) = 0$ $\forall x$ εσωτερικο του $\mathbb{R}$

Η $g(x)$ συνεχη στο $\mathbb{R}$ ω π.δ.σ

αφου f παρ/μη αρα και συνεχη

Αρα $g(x) = C$.

(B) Av $f(0)=1$

$$g(x)=C \Rightarrow \frac{f(x)}{e^x}-x^2=C$$

$x=0$

$$\frac{f(0)}{e^0}-0^2=c$$

$$1=c$$

$$\frac{f(x)}{e^x}-x^2=1$$

$$\frac{f(x)}{e^x}=x^2+1$$

$$\underline{\underline{f(x)=(x^2+1)e^x}}$$

Πανελλαδικά 2025 Δ

- $f:(0,+\infty)\to \mathbb{R}$ παρ/μη
- F παράγουσα της f $\Rightarrow$ $F'(x)=f(x)$
- $x f(x) = 2F(x)\ln x \quad \forall x>0$
- Η εφαπτομένη της C_f στο $M(1,f(1))$

είναι παράλληλη στην $\varepsilon: y=2x$.

Δ1 Νδο $g(x)=\dfrac{F(x)}{x^{\ln x}}$, $x>0$ σταθερή

$$g'(x)=\frac{F'(x)\,x^{\ln x}-F(x)\,\left(x^{\ln x}\right)'}{\left(x^{\ln x}\right)^2}$$

$$\longrightarrow\ x^{\ln x}=e^{\ln x^{\ln x}}=e^{\ln x\ln x}=e^{\ln^2 x}$$

$$\longrightarrow\ \left(x^{\ln x}\right)'=\left(e^{\ln^2 x}\right)'=e^{\ln^2 x}\left(\ln^2 x\right)'=e^{\ln^2 x}\,2\ln x\,\frac{1}{x}$$

$$\Rightarrow \left(x^{\ln x}\right)'=x^{\ln x}\,\frac{2\ln x}{x}$$

$$g'(x) = \frac{f(x)\, x^{\ln x} - F(x)\, x^{\ln x}\, \frac{2\ln x}{x}}{\left(x^{\ln x}\right)^2}$$

$$g'(x) = \frac{f(x) - F(x)\, \frac{2\ln x}{x}}{x^{\ln x}} = \frac{x f(x) - F(x)\, 2\ln x}{x \cdot x^{\ln x}}$$

$$g'(x) = \frac{\cancel{2\ln x\, F(x)} - \cancel{F(x)\, 2\ln x}}{x \cdot x^{\ln x}} = 0$$

$\Rightarrow g(x)$ σταθερή.

Δ2 i) $\lim\limits_{x \to 1} \frac{f(x)}{\ln x}$

$x f(x) = 2F(x) \ln x$

$\underline{x = 1}$

$\underline{\underline{f(1) = 0}}$

α' τρόπος

$$\lim_{x\to 1}\frac{f(x)}{\ln x}\ \overset{\left(\frac{0}{0}\right)}{\underset{DLH}{=}}\ \lim_{x\to 1}\frac{f'(x)}{\frac{1}{x}} = \frac{f'(1)}{1} = \frac{2}{1} = 2$$

Αφού η εφαπτομένη της C_f στο $x_0=1$

είναι παράλληλη $y=2x$.

$$y-f(1)=f'(1)(x-1) \quad // \quad y=2x$$

$$f'(1)=2$$

Για να το κάνω αυτό πρέπει νδο η $f'(x)$ συνεχ.

$$x f(x) = 2F(x)\cdot \ln x$$

$$f(x)=\frac{2F(x)\ln x}{x} \qquad f \text{ παρ/μη}$$

$$f'(x)=2\,\frac{\left(F(x)\ln x\right)' x - 2F(x)\ln x\cdot 1}{x^2}$$

$$f'(x)=2\,\frac{\left(f(x)\ln x + F(x)\frac{1}{x}\right)x - 2F(x)\ln x}{x^2}$$

$$f'(x) = 2\,\frac{x f(x)\ln x + F(x) - 2F(x)\ln x}{x^2}$$

Η $f'(x)$ είναι παρ/μη $(0,+\infty)$

ως π.π.δ.

Άρα $f'(x)$ συνεχής.

B' τρόπος

$f'(1)=2$

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=2$$

$$\lim_{x\to 1}\frac{f(x)}{\ln x}=\lim_{x\to 1}\frac{\frac{f(x)}{x-1}}{\frac{\ln x}{x-1}}=\frac{2}{1}=2,$$

$$\lim_{x\to 1}\frac{f(x)-0}{x-1}=2$$

$$\to \lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1$$

$$\lim_{x\to 1}\frac{f(x)}{x-1}=2$$

ii) Νδο $F(1)=1$ και οτι $F(x)=x^{\ln x}$

$$x f(x) = 2F(x)\ln x$$

$$F(x) = \frac{x f(x)}{2\ln x}$$

$$\lim_{x\to 1} F(x) = \lim_{x\to 1} \frac{x f(x)}{2\ln x} \overset{\left(\frac{0}{0}\right)}{\underset{DLH}{=}} \lim_{x\to 1} \frac{f(x)+x f'(x)}{2\frac{1}{x}}$$

$$= \frac{f(1)+f'(1)}{2} = \frac{0+2}{2} = 1$$

$F(x)$ συνεχ. $\Rightarrow F(1) = \lim_{x\to 1} F(x) = 1$

$$\underline{\underline{F(1)=1}}$$

Αφου $g(x)$ σταθερη

$\Rightarrow g(x) = c$

$$\frac{F(x)}{x^{\ln x}} = c \longrightarrow \underline{\underline{F(x) = x^{\ln x}}}$$

$\underline{x=1}$

$$\frac{F(1)}{1} = c \Rightarrow c = 1$$

Δ3

$$F(x) = x^{\ln x}$$

$$F'(x) = \left(x^{\ln x}\right)' = x^{\ln x} \frac{2\ln x}{x}$$

$$\rightarrow F'(x) = 0 \Rightarrow x^{\ln x} \cdot \frac{2\ln x}{\cancel{x}} = 0$$

$$2\ln x = 0$$

$$x = 1$$

x		1
F'	$-$	$+$
F	↘	↗

$$F(x) \geq F(1)$$

$$\boxed{F(x) \geq 1}$$

Εξίσωση $F(x^2) = F(x) - (x-1)^2$

$$F(x^2) - F(x) + (x-1)^2 = 0.$$

Προφανή ρίζα $\underline{\underline{x=1}}$

Αν $x>1$

- $x<x^2 \Rightarrow F(x)<F(x^2) \Rightarrow F(x^2)-F(x)<0$
- $(x-1)^2>0$

Άρα $F(x^2)-F(x)+(x-1)^2>0$

Αν $0<x<1$

- $x>x^2 \Rightarrow F(x)<F(x^2) \Rightarrow F(x^2)-F(x)>0$
- $(x-1)^2>0$

$F(x^2)-F(x)+(x-1)^2>0$

Μοναδική ρίζα $x=1$

7. $f: (0,+\infty) \to \mathbb{R}$ παρ/μη.

$f(1) = 1$

$x^2 f'(x) + x f(x) = 1 \quad \forall x > 0.$

ΝΔΟ $f(x) = \dfrac{1+\ln x}{x}, \; x > 0$

$x f(x) = 1 + \ln x$

$$\underbrace{x f(x) - 1 - \ln x}_{g(x)} = 0$$

$g'(x) = f(x) + x f'(x) - \dfrac{1}{x}$

$g'(x) = \dfrac{x f(x) + x^2 f'(x) - 1}{x} = \dfrac{0}{x} = 0$

$\Rightarrow g(x) = c \Rightarrow x f(x) - 1 - \ln x = c$

$\underline{x=1}$

$f(1) - 1 = c$

$c = 0$

$x f(x) - 1 - \ln x = 0$

$x f(x) = 1 + \ln x$

$f(x) = \dfrac{1+\ln x}{x}$

9. $f: \mathbb{R} \to \mathbb{R}$ παρ/μη $\qquad f(0) = 0$

$$x\left(f(x) + x f'(x)\right) = \sqrt{x^2+1} - f'(x)$$

Νδο $\quad f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$\underbrace{f(x)\sqrt{x^2+1} - x}_{g(x)} = 0$$

$$g'(x) = f'(x)\sqrt{x^2+1} + f(x)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} - 1$$

$$g'(x) = \frac{f'(x)\sqrt{x^2+1}^{\,2} + f(x)\,x - \sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$g'(x) = \frac{(x^2+1)f'(x) + x f(x) - \sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$\Rightarrow g'(x) = 0$$

$$x f(x) + x^2 f'(x) = \sqrt{x^2+1} - f'(x)$$

$$x f(x) + x^2 f'(x) + f'(x) - \sqrt{x^2+1} = 0$$

$$x f(x) + f'(x)(x^2+1) - \sqrt{x^2+1} = 0$$

$g(x) = c \rightarrow f(x)\sqrt{x^2+1} - x = c$

$x = 0$

$f(0) - 0 = c$

$0 = c.$

$f(x)\sqrt{x^2+1} - x = 0$

$f(x) = \frac{x}{\sqrt{x^2+1}}$

11. $f: (0,+\infty) \to \mathbb{R}$ $\quad f(1)=0 \quad$ παρ/μη

$$f'(x) = f(x) + \frac{e^x}{x} \qquad \forall x>0$$

Νδο $f(x) = e^x \ln x$

$$\underbrace{f(x) - e^x \ln x}_{g(x)} = 0$$

$$g'(x) = f'(x) - e^x \ln x - e^x \frac{1}{x}$$

$$g'(x) = f(x) + \cancel{\frac{e^x}{x}} - e^x \ln x - \cancel{\frac{e^x}{x}}$$

$$g'(x) = f(x) - e^x \ln x$$

$$g'(x) = g(x)$$

$$g(x) = ce^x$$ (Θέλει απόδειξη)

$$f(x) - e^x \ln x = ce^x \longrightarrow f(x) = e^x \ln x$$

$\underline{x=1}$

$$f(1) = ce \qquad 0 = ce \qquad c = 0$$

Ενότητα 24

45. $f(x) = \dfrac{ημx}{x}, \quad x \in (0, π)$

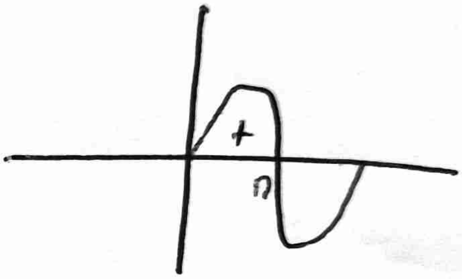
(α) $f'(x) = \dfrac{συνx \cdot x - ημx}{x^2}$

$$\boxed{g(x) = x\,συνx - ημx}$$

$g'(x) = συνx - x\,ημx - συνx$

$g'(x) = -x\,ημx < 0$

(+)



x	0		π
g'		−	
g	0	↘ −	
x^2		+	
f'		−	
f		↘	

$x > 0 \Rightarrow g(x) < g(0) \Rightarrow g(x) < 0$

③ i) $\frac{\ln x}{\ln y} > \frac{x}{y} \quad \forall x,y \in (0,e) \quad x<y$

$\frac{\ln x}{x} > \frac{\ln y}{y} \Rightarrow f(x) > f(y)$

$f \downarrow$

$x<y$

ii). $\ln x < \frac{2x}{e} \quad \forall x \in \left(\frac{e}{2}, e\right]$

$\frac{\ln x}{x} < \frac{2}{e}$

$f(x) < f\left(\frac{e}{2}\right)$

$f \downarrow$

$x > \frac{e}{2}$,

⑥ $\lim\limits_{x \to \frac{e}{2}^-} \frac{ex}{e\ln x - 2x} = \lim\limits_{x \to \frac{e}{2}^-} ex \, \frac{1}{\underset{\ominus}{e\ln x - 2x}}$

$= \frac{e^2}{2} \cdot (-\infty) = -\infty$.

$\ln x < \frac{2x}{e}$

$e\ln x - 2x < 0$

46. $f(x) = e^{-x} + \ln(x+1) - 1.$

ⓐ $\dfrac{P,7J}{f(x) = 0}$

$f'(x) = -e^{-x} + \dfrac{1}{x+1}$ $\quad f'(0) = 0$

$f(x) = f(0)$

f 1-1

$\boxed{x = 0}$

$f'(x) = -\dfrac{1}{e^x} + \dfrac{1}{x+1}$

$f'(x) = \dfrac{e^x - x - 1}{e^x(x+1)} \geq 0$

x	-1	0	$+\infty$
$f(x)$	$\nearrow$ $-$	0	$+$ $\nearrow$

$f \nearrow$

$x < 0 \Rightarrow f(x) < f(0) \Rightarrow f(x) < 0$

$x > 0 \Rightarrow f(x) > f(0) \Rightarrow f(x) > 0.$

Ⓑ i) $\dfrac{1}{e^{x^2-1}} + 2\ln x = 1.$

$e^{1-x^2} + \ln x^2 - 1 = 0$

$f(x^2 - 1) = f(0)$

$x^2 - 1 = 0$

$x = \pm 1$

11). $\ln \frac{x^2+1}{x+1} > \frac{e^{x^2}-e^x}{e^{x^2+x}}$

$$\ln(x^2+1) - \ln(x+1) > \frac{e^{x^2}}{e^{x^2+x}} - \frac{e^x}{e^{x^2+x}}$$

$$\ln(x^2+1) - \ln(x+1) > \frac{\cancel{e^{x^2}}}{\cancel{e^{x^2}} \cdot e^x} - \frac{\cancel{e^x}}{e^{x^2} \cdot \cancel{e^x}}$$

$$\ln(x^2+1) - \ln(x+1) > e^{-x} - e^{-x^2}$$

$$e^{-x^2} + \ln(x^2+1) - 1 > e^{-x} + \ln(x+1) - 1$$

$$f(x^2) > f(x)$$

$$f \nearrow$$

$$x^2 > x$$

$$x^2 - x > 0$$

x		0		1	
x^2-x	$+$		$-$		$+$

$x \in (-\infty, 0) \cup (1, +\infty)$

$\downarrow$

$x \in (-1, 0) \cup (1, \infty)$

(γ) $\lim\limits_{x \to 0} \dfrac{1}{f(e^x) - f(x+1)} = +\infty$.

$+$

- $e^x \geq x+1 \Rightarrow f(e^x) \geq f(x+1)$

$f(e^x) - f(x+1) \geq 0$

47. $f(x) = x - \ln(x^2+1)$

ⓐ $f'(x) = 1 - \dfrac{2x}{x^2+1} = \dfrac{x^2+1-2x}{x^2+1} = \dfrac{(x-1)^2}{x^2+1}$

$f'(x) \geq 0 \qquad f \nearrow$

ⓑ. $x - 2 < \ln\left((2x-1)^2+1\right) - \ln(x^2+2x+2)$.

$x - 2 < \ln\left((2x-1)^2+1\right) - \ln(x^2+2x+1+1)$

$x - 2 < \ln\left((2x-1)^2+1\right) - \ln\left((x+1)^2+1\right)$

$2x-1 \quad - \ln\left((2x-1)^2+1\right) < x+1 \quad - \ln\left((x+1)^2+1\right)$

$f(2x-1) < f(x+1)$

$f \nearrow$

$2x-1 < x+1$

$\underline{x < 2}$

(γ) $f\left(\ln\left(x^4+1+\frac{1}{x^4+1}\right)\right) < f\left((x^4+1)-x^2\right)$

$\ln\left(x^4+1+\frac{1}{x^4+1}\right) < (x^4+1)-x^2$

$\ln\frac{(x^4+1)^2+1}{x^4+1} < x^4+1-x^2$

$\ln\left((x^4+1)^2+1\right) - \ln(x^4+1) < x^4+1-x^2$

$x^2 - \ln(x^4+1) < x^4+1 - \ln\left((x^4+1)^2+1\right)$

$f(x^2) < f(x^4+1)$

$f \nearrow$

$x^2 < x^4+1$

$0 < x^4-x^2+1$ ✓

(5) $$(4x^2+1)e^{-2x} = 1$$

$$\frac{4x^2+1}{e^{2x}} = 1$$

$$4x^2+1 = e^{2x}$$

$$\ln\left((2x)^2+1\right) = 2x$$

$$2x - \ln\left((2x)^2+1\right) = 0$$

$$f(2x) = f(0)$$

$$x = 0$$

Ενότητα 24

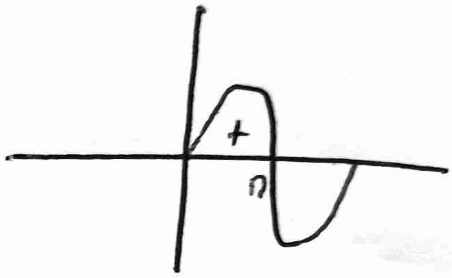
45. $f(x) = \dfrac{ημx}{x} \ , \ x \in (0, π)$

(α) $f'(x) = \dfrac{συνx \cdot x - ημx}{x^2}$

$$\boxed{g(x) = x\,συνx - ημx}$$

$g'(x) = συνx - x\,ημx - συνx$

$g'(x) = -x\,\underset{+}{ημx} < 0$



x	0		π
g'		$-$	
g	0	↘ $-$	
x^2		$+$	
f'		$-$	
f		↘	

$x > 0 \Rightarrow g(x) < g(0) \Rightarrow g(x) < 0$

(3) i) $\frac{\eta\mu x}{\eta\mu y} > \frac{x}{y} \quad \forall x,y \in (0,\pi) \qquad x<y$

$$\frac{\eta\mu x}{x} > \frac{\eta\mu y}{y} \Rightarrow f(x) > f(y)$$

$f \downarrow$

$x<y$

ii). $\eta\mu x < \frac{2x}{\pi} \quad \forall x \in \left(\frac{\pi}{2}, \pi\right)$

$$\frac{\eta\mu x}{x} < \frac{2}{\pi}$$

$$f(x) < f\left(\frac{\pi}{2}\right)$$

$f \downarrow$

$x > \frac{\pi}{2}$.

(4) $\lim_{x \to \frac{\pi}{2}^-} \frac{\pi x}{\pi\eta\mu x - 2x} = \lim_{x \to \frac{\pi}{2}^-} \pi x \, \frac{1}{\underbrace{\pi\eta\mu x - 2x}_{-}}$

$$= \frac{\pi^2}{2} \cdot (-\infty) = -\infty.$$

$\eta\mu x < \frac{2x}{\pi}$

$\pi\eta\mu x - 2x < 0$

Επομενο Μαθημα

22	23
2	2 α γ
3	3
5	4
6	5 α β γ
8	6 α β
10	8
	9
	10