1

Ενοτητα 32

$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

$\longrightarrow f'(x) = 0 \Rightarrow x^2 - 1 = 0$

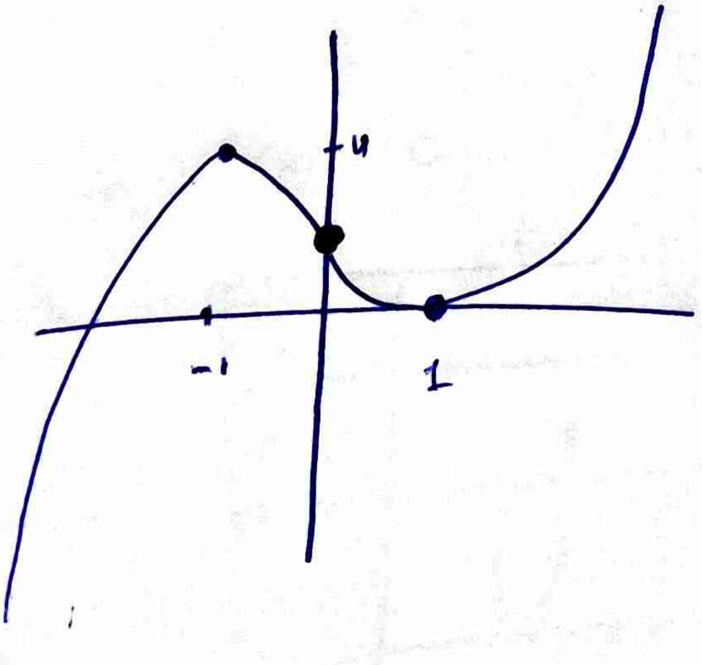
$x = \pm 1$

x		-1		1	
f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

$$f''(x) = 6x$$

x		0	
f''	$-$	0	$+$
f	$\cap$		$\cup$

Δεν εχει ασυμπτωτες. επειδη ειναι πολυωνυμο 3ου Βαθμου.



(B) $f(x) = a$

1. αν $a < 0$ 1 ριζα.
2. αν $a = 0$ 2 ριζες
3. αν $0 < a < 4$ 3 ριζες
4. αν $a = 4$ 2 ριζες
5. αν $a > 4$ 1 ριζα.

3. (a) $f(x) = \dfrac{e^x}{e^x+1}$, $x \in \mathbb{R}$.

$$f'(x) = \frac{e^x(e^x+1) - e^x e^x}{(e^x+1)^2} = \frac{\cancel{e^x e^x} + e^x \cancel{- e^x e^x}}{(e^x+1)^2}$$

$$f'(x) = \frac{e^x}{(e^x+1)^2} > 0 \quad f \nearrow$$

$$f''(x) = \frac{e^x (e^x+1)^2 - e^x\, 2(e^x+1) e^x}{(e^x+1)^4}$$

$$f''(x) = \frac{e^x(e^x+1) - 2e^x e^x}{(e^x+1)^3}$$

$$f''(x) = \frac{e^{2x} + e^x - 2e^{2x}}{(e^x+1)^3} = \frac{e^x - e^{2x}}{(e^x+1)^3} = \frac{e^x(1-e^x)}{(e^x+1)^3}$$

$$\to f''(x) = 0 \quad \Rightarrow \quad 1 - e^x = 0 \quad \Rightarrow \underline{\underline{x = 0}}$$

x		0	
f''	$+$		$-$
f	$\cup$		$\cap$

Αφου $D_f = \mathbb{R}$ δεν εχει κατακορυφη

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{e^x}{e^x+1} = \frac{0}{0+1} = 0$$

$E_1: y = 0$

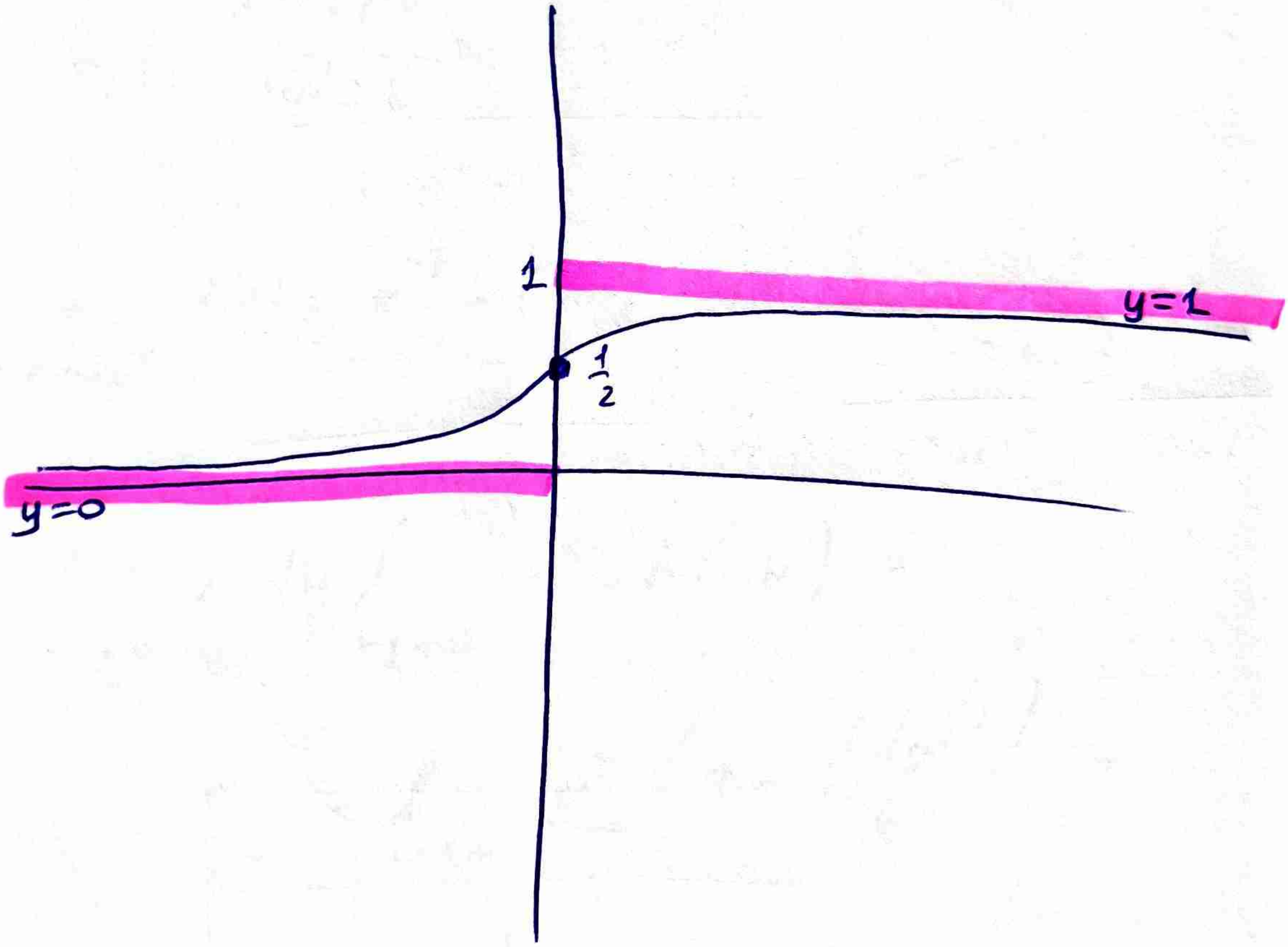
$-\infty$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^x}{e^x+1} = \lim_{x \to +\infty} \frac{e^x}{e^x} = 1$$

$E_2: y = 1$

$+\infty$

Δεν εχει πλαγια.



(δ) $f(x) = e^x - \ln(x+1)$, $x > -1$

$f'(x) = e^x - \frac{1}{x+1}$ $\quad f'(0) = 0$

$f''(x) = e^x + \frac{1}{(x+1)^2} > 0$ $\quad$ f κυρτή

x		0	
f''	$+$		$+$
f'	$\nearrow$ $-$	0	$+$ $\nearrow$
f	$\searrow$		$\nearrow$

$x < 0 \Rightarrow f'(x) < f'(0) \Rightarrow f'(x) < 0$

$x > 0 \Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 0$

$f(x) \geq f(0)$

$f(x) \geq 1$

$\lim_{x \to -1^+} f(x) = e^{-1} - (-\infty) = +\infty$

ε: $x = -1$ κατακόρυφη

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^x - \ln(x+1) =$

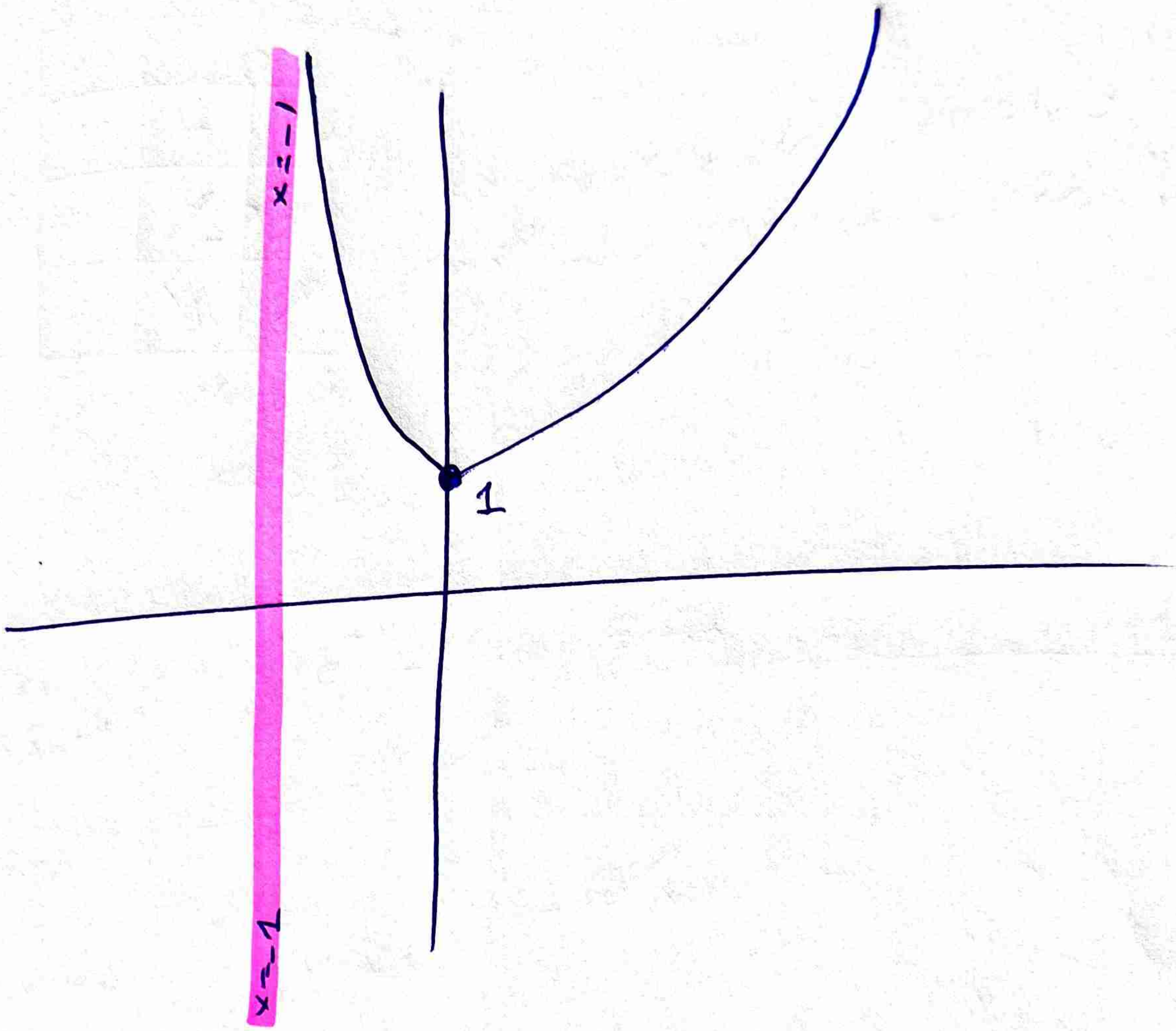
$= \lim_{x \to +\infty} e^x \left(1 - \cancel{\frac{\ln(x+1)}{e^x}}^{\,0}\right) = +\infty$

δεν έχει πλάγια

$\to \lim_{x \to +\infty} \frac{\ln(x+1)}{e^x} = \lim_{x \to +\infty} \frac{1}{(x+1)e^x} = 0$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{e^x - \ln(x+1)}{x} =$$

$$= \lim_{x \to +\infty} e^x - \frac{1}{x+1} = +\infty$$ δεν έχω πλάγια



11. (γ) $\int_0^1 e^{3x}\,dx = \left(\frac{e^{3x}}{3}\right)_0^1 = \frac{1}{3}\left(e^{3x}\right)_0^1 =$

$= \frac{1}{3}\left(e^{3}-1\right)$ **Ενότητα 34**

(ε) $\int_0^1 \frac{1}{e^{2x}}\,dx = \int_0^1 e^{-2x}\,dx =$

$= \left(\frac{e^{-2x}}{-2}\right)_0^1 = -\frac{1}{2}\left(e^{-2x}\right)_0^1$

$= -\frac{1}{2}\left(e^{-2}-1\right)$

9. (β) $\int_{-1}^0 \left(x^3-3x+5\right)dx =$

$\int_{-1}^0 x^3\,dx - \int_{-1}^0 3x\,dx + \int_{-1}^0 5\,dx$

$= \frac{1}{4}\left(x^4\right)_{-1}^0 - \frac{3}{2}\left(x^2\right)_{-1}^0 + 5\left(x\right)_{-1}^0$

$= \frac{1}{4}(0-1) - \frac{3}{2}(0-1) + 5(0+1)$

$= -\frac{1}{4} + \frac{3}{2} + 5 = \frac{5}{4} + \frac{20}{4} = \frac{25}{4}.$

20

Ενότητα 35

(a) $\int_0^1 \frac{2x+1}{x^2+x+1}\,dx = \left(\ln|x^2+x+1|\right)_0^1 = \ln 3$

21. (α) $\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx \overset{(*)}{=}$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-1)$$

$$1 = Ax + A + Bx - B$$

$$\boxed{1 = (A+B)x + A - B}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \overset{+}{\Leftrightarrow} 2A=1 \quad A=\frac{1}{2} \quad B=-\frac{1}{2}$$

$(*)$ $\frac{1}{2}\int_2^3 \frac{1}{x-1}\,dx - \frac{1}{2}\int_2^3 \frac{1}{x+1}\,dx =$

$= \frac{1}{2}\left(\ln|x-1|\right)_2^3 - \frac{1}{2}\left(\ln|x+1|\right)_2^3 = \frac{1}{2}\ln 2 - \frac{1}{2}(\ln 4 - \ln 3)$

$= \frac{1}{2}\left(\ln 2 - \ln\frac{4}{3}\right) = \frac{1}{2}\ln\frac{3}{2}.$

(B) $\displaystyle\int_0^{-1} \frac{2x-3}{x^2-4x+3}\,dx \quad \underline{\underline{\quad (*) \quad}}$

$$\frac{2x-3}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

$$2x-3 = A(x-1) + B(x-3)$$

$$2x-3 = Ax - A + Bx - 3B$$

$$2x-3 = (A+B)\,x - A - 3B$$

$$\begin{cases} 2 = A+B \\ -3 = -A-3B \end{cases} \quad (+) \qquad -1 = -2B \qquad \boxed{B = \frac{1}{2}}$$

$$2 = A + \frac{1}{2} \qquad A = 2 - \frac{1}{2} \qquad \boxed{A = \frac{3}{2}}$$

$$\underline{(*)} \int_0^{-1} \frac{\frac{3}{2}}{x-3} + \frac{\frac{1}{2}}{x-1}\,dx =$$

$$= \frac{3}{2}\int_0^{-1} \frac{1}{x-3}\,dx + \frac{1}{2}\int_0^{-1} \frac{1}{x-1}\,dx = \frac{3}{2}\Big(\ln|x-3|\Big)_0^{-1} + \frac{1}{2}\Big(\ln|x-1|\Big)_0^{-1}$$

23. ⓐ $\int_2^3 \frac{x^2+1}{x^2-x}\,dx = \int_2^3 \frac{1\cdot(x^2-x) \quad +x+1}{x^2-x}$

$$= \int_2^3 \cancel{\frac{x^2-x}{x^2-x}} + \frac{x+1}{x^2-x}\,dx$$

$$\begin{array}{r|l} x^2+1 & x^2-x \\ \cline{2-2} -(x^2-x) & 1 \\ \hline x+1 & \end{array}$$

$$= \int_2^3 1 + \frac{x+1}{x^2-x}\,dx =$$

$$= \int_2^3 1\,dx + \int_2^3 \frac{x+1}{x^2-x}\,dx$$

$$= (x)_2^3 + \int_2^3 \frac{-1}{x} + \frac{1}{x-1}\,dx = \text{ow}$$

$$\frac{x+1}{x^2-x} = \boxed{\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}}$$

$$x+1 = A(x-1) + Bx$$

$$x+1 = Ax - A + Bx$$

$$x+1 = (A+B)\,x - A$$

$$\begin{cases} 1 = A+B \\ 1 = -A \end{cases} \quad A=-1 \quad B=2$$

22. (a) $\int_0^1 \frac{x}{x+1}\,dx = \int_0^1 \frac{1\cdot(x+1)-1}{x+1}\,dx$

$$\begin{array}{r|l} x & x+1 \\ \hline -(x+1) & 1 \\ \hline -1 & \end{array}$$

$= \int_0^1 \frac{x+1}{x+1} - \frac{1}{x+1}\,dx$

$= \int_0^1 1\,dx - \int_0^1 \frac{1}{x+1}\,dx$

$= (x)\Big|_0^1 - \left(\ln|x+1|\right)\Big|_0^1$

$= 1 - (\ln 2 - 0) = 1 - \ln 2.$

(b) $\int_0^1 \frac{3x+2}{2x+1}\,dx = \int_0^1 \frac{\frac{3}{2}(2x+1)+\frac{1}{2}}{2x+1}\,dx$

$$\begin{array}{r|l} 3x+2 & 2x+1 \\ \hline -(3x+\frac{3}{2}) & \frac{3}{2} \\ \hline \frac{1}{2} & \end{array}$$

$= \int_0^1 \frac{\frac{3}{2}\cancel{(2x+1)}}{\cancel{2x+1}} + \frac{\frac{1}{2}}{2x+1}\,dx$

$= \int_0^1 \frac{3}{2}\,dx + \frac{1}{2}\int_0^1 \frac{1}{2x+1}\,dx$

$= \frac{3}{2}(x)\Big|_0^1 + \frac{1}{2}\cdot\frac{1}{2}\left(\ln|2x+1|\right)\Big|_0^1 = \frac{3}{2} + \frac{1}{4}\ln 3.$

Παραγοντική ολοκλήρωση

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$\int_{\alpha}^{\beta} (f(x)g(x))' \, dx = \int_{\alpha}^{\beta} f'(x)g(x) + f(x)g'(x) \, dx$$

$$[f(x)g(x)]_{\alpha}^{\beta} = \int_{\alpha}^{\beta} f'(x)g(x) \, dx + \int_{\alpha}^{\beta} f(x)g'(x) \, dx$$

$$(f(x)g(x))_{\alpha}^{\beta} - \int_{\alpha}^{\beta} f(x)g'(x) \, dx = \int_{\alpha}^{\beta} f'(x)g(x) \, dx$$

$$\boxed{\int_{\alpha}^{\beta} f'(x)g(x) \, dx = (f(x)g(x))_{\alpha}^{\beta} - \int_{\alpha}^{\beta} f(x)g'(x) \, dx}$$

1. $\int_0^{\pi} (2x-3)\,\text{ημ}x\,dx = \int_0^{\pi} (2x-3)(-\text{συν}x)'\,dx$

$= \left[-\text{συν}x\,(2x-3)\right]_0^{\pi} - \int_0^{\pi} -\text{συν}x \cdot 2\,dx$

$= -\left(\text{συν}x\,(2x-3)\right)_0^{\pi} + 2\int_0^{\pi} \text{συν}x\,dx$

$= -\left(\text{συν}\pi\,(2\pi-3) - \text{συν}0\,(-3)\right) + 2\left(\text{ημ}x\right)_0^{\pi}$

$= -\left(-(2\pi-3)+3\right) + 2\left(\cancel{\text{ημ}\pi - \text{ημ}0}\right)^{0}$

$= 2\pi - 3 - 3$

$= 2\pi - 6.$

Σχόλιο

$\int P(x)\,\text{ημ}x\,dx$

$\int P(x)\,\text{συν}x\,dx$ $\quad \int P(x)\ln x\,dx$

$\int P(x)\,e^x\,dx$

εξαίρεση

$\int e^x\,\text{ημ}x\,dx, \quad \int e^x\,\text{συν}x\,dx$

2. $\int_0^1 (x^2-2x+4)e^{-x}\,dx =$

$= \int_0^1 (x^2-2x+4)(-e^{-x})'\,dx$

$= \left(-(x^2-2x+4)e^{-x}\right)_0^1 - \int_0^1 -e^{-x}(2x-2)\,dx$

$= -\left((x^2-2x+4)e^{-x}\right)_0^1 + \int_0^1 e^{-x}(2x-2)\,dx$

$= -\left(3e^{-1}-4\right) + \int_0^1 (-e^{-x})'(2x-2)\,dx$

$= -\frac{3}{e}+4+\left(-e^{-x}(2x-2)\right)_0^1 - \int_0^1 -e^{-x}\cdot 2\,dx$

$= 4-\frac{3}{e} - \left(e^{-x}(2x-2)\right)_0^1 + 2\int_0^1 e^{-x}\,dx$

$= 4-\frac{3}{e} - \left(e^{-1}\cdot 0 - (-2)\right) + 2\left(-e^{-x}\right)_0^1$

$= 4-\frac{3}{e} - 2 - 2\left(e^{-x}\right)_0^1 = 2-\frac{3}{e} - 2\left(\frac{1}{e}-1\right) = 4-\frac{5}{e}.$

3. $\int_1^e (x^2+x)\ln x \, dx = \int_1^e \left(\frac{x^3}{3}+\frac{x^2}{2}\right)' \ln x \, dx$

$$= \left(\left(\frac{1}{3}x^3+\frac{1}{2}x^2\right)\ln x\right)_1^e - \int_1^e \left(\frac{x^3}{3}+\frac{x^2}{2}\right)\frac{1}{x} \, dx$$

$$\frac{1}{3}e^3+\frac{1}{2}e^2 - \int_1^e \frac{x^2}{3}+\frac{x}{2} \, dx$$

$$\frac{1}{3}e^3+\frac{1}{2}e^2 - \frac{1}{3}\int_1^e x^2 \, dx - \frac{1}{2}\int_1^e x \, dx$$

$$= \frac{1}{3}e^3+\frac{1}{2}e^2 - \frac{1}{3}\frac{1}{3}(x^3)_1^e + \frac{1}{6}\frac{1}{2}(x^2)_1^e$$

$$= \frac{1}{3}e^3+\frac{1}{2}e^2 - \frac{1}{9}(e^3-1) + \frac{1}{12}(e^2-1)$$

4. (κυκλικο)

$$I = \int_0^\pi e^x \,\sigma\upsilon\nu x \, dx$$

$$I = \int_0^\pi (e^x)' \,\sigma\upsilon\nu x \, dx$$

$$I = \left(e^x \,\sigma\upsilon\nu x\right)_0^\pi - \int_0^\pi e^x (-\eta\mu x) \, dx$$

$$I = -e^\pi - 1 + \int_0^\pi e^x \,\eta\mu x \, dx$$

$$I = -e^\pi - 1 + \int_0^\pi (e^x)' \,\eta\mu x \, dx$$

$$I = -e^\pi - 1 + \left(e^x \,\eta\mu x\right)_0^\pi - \int_0^\pi e^x \,\sigma\upsilon\nu x \, dx$$

$$I = -e^\pi - 1 + \cancel{\left(e^\pi \,\eta\mu \pi - 0\right)} - I$$

$$I = -e^\pi - 1 - I$$

$$2I = -e^\pi - 1$$

$$I = -\frac{e^\pi + 1}{2}$$

5. $\int_1^e \ln^2 x \, dx = \int_1^e 1 \cdot \ln^2 x \, dx$

$= \int_1^e (x)' \ln^2 x \, dx =$

$= \left(x \ln^2 x\right)_1^e - \int_1^e x \cdot 2 \ln x \, \frac{1}{x} \, dx$

$= e - 2 \int_1^e \ln x \, dx =$

$= e - 2 \int_1^e (x)' \ln x \, dx =$

$= e - 2 \left[\left(x \ln x\right)_1^e - \int_1^e x \cdot \frac{1}{x} \, dx \right]$

$= e - 2 \left[e - 0 - \int_1^e 1 \, dx \right]$

$= e - 2e + 2 (x)_1^e =$

$= -e + 2(e-1) = -e + 2e - 2$

$= e - 2.$

16. (δ) $\int_1^{\frac{1}{2}} \frac{e^{\frac{1}{x}}}{x^2}\, dx =$

$\frac{1}{x} = t$

$-\frac{1}{x^2}\, dx = dt \Rightarrow \frac{1}{x^2}\, dx = -dt$

$= -\int_1^2 e^t\, dt = -\left(e^t\right)_1^2 = -\left(e^2 - e\right)$

(ε) $\int_0^{\frac{\pi}{2}} \sigma\upsilon\nu x \cdot \sigma\upsilon\nu(\eta\mu x)\, dx =$

$\eta\mu x = t$

$\sigma\upsilon\nu x\, dx = dt$

$= \int_0^1 \sigma\upsilon\nu t\, dt = \left(\eta\mu t\right)_0^1$

$= \eta\mu 1.$

(62) $\int_e^{e^2} -\frac{1}{x\sqrt{\ln x}}\,dx = -\int_1^{\sqrt{2}} \frac{1}{\cancel{t}}\, 2\cancel{t}\,dt$

$\sqrt{\ln x} = t$

$\ln x = t^2$

$\frac{1}{x}\,dx = 2t\,dt$

$= -2\int_1^{\sqrt{2}} 1\,dt$

$= -2\,(t)_1^{\sqrt{2}}$

$= -2\,(\sqrt{2}-1)$.

18. ① $\int_0^1 x\sqrt{2-x^2}\,dx$ ⊛ **Ενότητα 35**

Θέτω $\sqrt{2-x^2} = t$

$2 - x^2 = t^2$

$-2x\,dx = 2t\,dt$

$-x\,dx = t\,dt$

$x\,dx = -t\,dt$

$$= \int_{\sqrt{2}}^{1} -t \cdot t\,dt = -\int_{\sqrt{2}}^{1} t^2\,dt$$

$$= -\frac{1}{3}\left(t^3\right)_{\sqrt{2}}^{1} = -\frac{1}{3}\left(1 - \sqrt{2}^3\right)$$

(ε) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx \overset{(*)}{=}$

$\sqrt{x+1} = t$ (Σηκωνω στο τετραγωνο

$x+1 = t^2$ να φυγει η ριζα.)

$x = t^2 - 1$ (λυνω ως προς x)

$1 \cdot dx = 2t\,dt$

παραγωγιζω ως προς x

παραγωγιζω ως προς t

$$\overset{(*)}{=} \int_1^2 \frac{t^2-1}{\cancel{t}}\,2\cancel{t}\,dt = 2\int_1^2 t^2 - 1\,dt$$

$$= 2\int_1^2 t^2\,dt - 2\int_1^2 1\,dt$$

$$= \frac{2}{3}\left(t^3\right)_1^2 - 2\left(t\right)_1^2 =$$

$$= \frac{14}{3} - 2$$

Μην ξεχνας την αντικαταστασ η στα ακρα

$\sqrt{x+1} = t$

$x=3 \longrightarrow t=2$

$x=0 \longrightarrow t=1.$

17. (B) $\int_{-1}^{0} x^2(x-1)^4\,dx$ $\underline{\underline{(*)}}$

Θέτω τη μεγαλύτερη δύναμη.

Θέτω $x-1=t$

λύνω ως προς x

$x=t+1$

παραγωγίζω ως προς x

παραγωγίζω ως προς t

$1\cdot dx = 1\cdot dt$

$dx = dt$

$x=0 \longrightarrow t=-1$

$x=-1 \longrightarrow t=-2$

Έχουμε $x-1=t$

$\underline{\underline{(*)}}\ \int_{-2}^{-1} (t+1)^2 t^4\,dt =$

$\int_{-2}^{-1} (t^2+2t+1)t^4\,dt =$

$= \int_{-2}^{-1} t^6+2t^5+t^4\,dt = \frac{1}{7}\left(t^7\right)_{-2}^{-1} + \frac{2}{6}\left(t^6\right)_{-2}^{-1} + \frac{1}{5}\left(t^5\right)_{-2}^{-1}$

4. (α) $f(x) = \dfrac{e^x}{x^2}, \quad x > 0$

Ενοτητα 37

$$f'(x) = \frac{e^x x^2 - 2x e^x}{x^4}$$

$$f'(x) = \frac{e^x x - 2 e^x}{x^3} = \frac{e^x (x-2)}{x^3}$$

x	0	2
f'	$-$	$+$
f	$\searrow$	$\nearrow$

$f(x) \geq f(2)$

$$\boxed{f(x) \geq \frac{e^2}{4}}$$

(β) Νδο $\dfrac{e^2}{2} < \displaystyle\int_2^4 f(x)\,dx < \dfrac{e^4}{8}$.

$2 < x < 4$

$f \uparrow$

$f(2) < f(x) < f(4)$

$$\frac{e^2}{4} < f(x) < \frac{e^4}{16}$$

$$\int_2^4 \frac{e^2}{4}\,dx < \int_2^4 f(x)\,dx < \int_2^4 \frac{e^4}{16}\,dx$$

$$\frac{e^2}{4}(x)_2^4 < \int_2^4 f(x)\,dx < \frac{e^4}{16}(x)_2^4$$

$$\frac{e^2}{4}\cdot 2 < \int_2^4 f(x)\,dx < \frac{e^4}{16}\,2$$

$$\boxed{\frac{e^2}{2} < \int_2^4 f(x)\,dx < \frac{e^4}{8}}$$

① Ν δ ο $\int_1^5 f(x)\,dx > e^2$

Γνωρίζω ότι $f(x) \geq \frac{e^2}{4}$

Άρα $\int_1^5 f(x)\,dx > \int_1^5 \frac{e^2}{4}\,dx$

$$\int_1^5 f(x)\,dx > \frac{e^2}{4}(x)_1^5$$

$$\int_1^5 f(x)\,dx > e^2 \checkmark$$

Σχόλιο

Αν $f(x) \leq g(x)$

τότε

$$\int_\alpha^\beta f(x)\,dx < \int_\alpha^\beta g(x)\,dx$$

Επόμενο Μάθημα

35

2 α β

3 α β

4 α β

5 α β

6 α .

16 α β δ

17 α

18 α β δ