| Ενότητα 18

Συνεχ.

23. $f(x) = \begin{cases} x\ln x + \alpha x - \beta, & x>0 \\ 1, & x=0 \\ e^{\frac{1}{x}} \ln(-x) + \alpha, & x<0 \end{cases}$

• $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \left[\underbrace{\cancel{x\ln x}}_{0} + \alpha x - \beta\right] = -\beta$

→ $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{DLH}{=}} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to 0^+} -\dfrac{x^{\cancel{2}}}{\cancel{x}} = 0$

• $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \underbrace{\cancel{e^{\frac{1}{x}} \ln(-x)}}_{0} + \alpha = \alpha$

→ $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} \ln(-x) = \lim\limits_{x\to 0^-} \dfrac{\ln(-x)}{\frac{1}{e^{1/x}}} = \lim\limits_{x\to 0^-} \dfrac{\ln(-x)}{e^{-1/x}}$

$$= \lim_{x \to 0^-} \frac{\frac{1}{-x}(-1)}{e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right)} =$$

$$= \lim_{x \to 0^-} \frac{\frac{1}{x}}{e^{-\frac{1}{x}}\,\frac{1}{x^2}} = \lim_{x \to 0^-} \frac{x^{\cancel{2}}}{\cancel{x}\, e^{-1/x}} =$$

$$= \lim_{x \to 0^-} x \cdot \frac{1}{e^{-\frac{1}{x}}} = 0 \cdot 0 = 0$$

$$f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$1 = -B = \alpha$$

$$B = -1 \qquad \alpha = 1$$

29. $f(x)=\begin{cases} \ln x+\alpha-1, & x>1 \\ e^{x-1}+\beta x-\beta, & x\leq 1 \end{cases}$

παρ/μη στο $x_0=1$

$$\left.\begin{aligned} &\lim_{x\to 1^-} f(x) = 1+\beta-\beta = 1 \\ &\lim_{x\to 1^+} f(x) = \alpha-1 \end{aligned}\right\} \quad \alpha-1=1 \iff \alpha=2$$

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{e^{x-1}+\beta x-\beta-1}{x-1} =$$

$$= \lim_{x\to 1^-} \frac{e^{x-1}+\beta}{1} =$$

$$= 1+\beta$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\ln x+\cancel{1}-\cancel{1}}{x-1}$$

$$= \lim_{x\to 1^+} \frac{1}{x} = 1 \qquad 1+\beta=1 \qquad \beta=0$$

27. $f: \mathbb{R} \to \mathbb{R}$ συνεχής

$$e^x\left(f(x) - x\right) = f(x).$$

(α) $e^x f(x) - x e^x = f(x)$

$$e^x f(x) - f(x) = x e^x$$

$$f(x)\left(e^x - 1\right) = x e^x$$

$$\boxed{f(x) = \frac{x e^x}{e^x - 1} \quad x \neq 0}$$

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x e^x}{e^x - 1} =$$

$$= \lim_{x \to 0} \frac{\cancel{e^x} + x\cancel{e^x}}{\cancel{e^x}} = \lim_{x \to 0} 1 + x = 1.$$

$$f(x) = \begin{cases} \dfrac{xe^x}{e^x-1}, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

B) i) $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to+\infty} \dfrac{\dfrac{xe^x}{e^x-1}}{x} =$

$$= \lim_{x\to+\infty} \frac{\cancel{x}e^x}{\cancel{x}(e^x-1)} = \lim_{x\to+\infty} \frac{e^x}{e^x-1}$$

$$= \lim_{x\to+\infty} \frac{\cancel{e^x}}{\cancel{e^x}} = 1.$$

ii) $\lim\limits_{x\to+\infty} f(x) - x = \lim\limits_{x\to+\infty} \left(\dfrac{xe^x}{e^x-1} - x \right)$

$$= \lim_{x\to+\infty} \frac{\cancel{xe^x} - \cancel{xe^x} + x}{e^x-1} = \lim_{x\to+\infty} \frac{1}{e^x} = 0.$$

(γ) $y - f(0) = f'(0)(x-0)$

$y - 1 = \frac{1}{2}(x-0)$ $\quad y = \frac{1}{2}x + 1$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{xe^x}{e^x-1} - 1}{x}$$

$$= \lim_{x\to 0} \frac{xe^x - e^x + 1}{x(e^x-1)} = \lim_{x\to 0} \frac{\cancel{e^x} + xe^x - \cancel{e^x}}{e^x - 1 + xe^x}$$

$$= \lim_{x\to 0} \frac{xe^x}{e^x - 1 + xe^x} = \lim_{x\to 0} \frac{\cancel{e^x} + x\cancel{e^x}}{\cancel{e^x} + \cancel{e^x} + x\cancel{e^x}}$$

$$= \lim_{x\to 0} \frac{1+x}{1+1+x} = \frac{1}{2}.$$

26. $f(x) = \begin{cases} \dfrac{x\ln x}{x-1}, & 0<x\neq 1 \\ 1, & x=1 \\ 0, & x=0 \end{cases}$ συνεχής

(α) $\lim\limits_{x\to 0^+} f(x) =$

$= \lim\limits_{x\to 0^+} \dfrac{x\ln x}{x-1} = \dfrac{0}{-1} = 0$

$\to \boxed{\lim\limits_{x\to 0^+} x\ln x = 0}$

$f(0) = 0$

γιατί $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to 0^+} -\dfrac{x\cancel{^2}}{\cancel{x}} = 0.$

Αφού $f(0) = \lim\limits_{x\to 0^+} f(x)$

Συνεχής στο 0.

$$\lim_{x\to 1} \frac{x\ln x}{x-1} = \lim_{x\to 1} \frac{\ln x + 1}{1} = 1$$

$$f(1) = 1$$

Συνεχής στο 1.

(B) $y - f(1) = f'(1)(x-1)$

$y - 1 = \frac{1}{2}(x-1)$

$$y = \frac{1}{2}x + \frac{1}{2}$$

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\frac{x\ln x}{x-1} - 1}{x-1}$$

$$= \lim_{x\to 1} \frac{x\ln x - x + 1}{(x-1)^2} = \lim_{x\to 1} \frac{\ln x + \cancel{1} - \cancel{1}}{2(x-1)}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{2} = \frac{1}{2}$$

(f) . i) $\lim_{x \to 0} f(x) = \lim_{x \to 0^+} \frac{x \ln x}{x-1} = \frac{0}{-1} = 0$

ii) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x \ln x}{x-1} = \lim_{x \to +\infty} \frac{\ln x + 1}{1}$

$= +\infty$.

25. $f:(0,+\infty)\longrightarrow R$ συνεχή.

$x f(x) - \ln x = f(x) \qquad \forall x>0.$

ⓐ NSo $f(x)=\begin{cases}\dfrac{\ln x}{x-1}, & 0<x\neq 1\\ 1, & x=1\end{cases}$

$x f(x) - f(x) = \ln x$

$(x-1) f(x) = \ln x$

$$\boxed{f(x)=\frac{\ln x}{x-1} \qquad x\in(0,1)\cup(1,+\infty)}$$

$f(1)=\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} = 1$

$f(x)=\begin{cases}\dfrac{\ln x}{x-1} & x\in(0,1)\cup(1,+\infty)\\ 1, & x=1\end{cases}$

(β) $y - f(1) = f'(1)(x-1)$

$y - 1 = -\frac{1}{2}(x-1) \Rightarrow \boxed{y = -\frac{1}{2}x + \frac{3}{2}}$

$\rightarrow f'(1) = \lim\limits_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1} \frac{\frac{\ln x}{x-1} - 1}{x-1}$

$= \lim\limits_{x\to 1} \frac{\ln x - x + 1}{(x-1)^2} =$

$= \lim\limits_{x\to 1} \frac{\frac{1}{x} - 1}{2(x-1)} = \lim\limits_{x\to 1} \frac{-\frac{1}{x^2}}{2} = -\frac{1}{2}$

(γ) i) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0^+} \frac{\ln x}{x-1} = \lim\limits_{x\to 0^+} \ln x \frac{1}{x-1}$

$= -\infty \cdot (-1)$

$= +\infty.$

ii) $\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} \frac{\ln x}{x-1} = \lim\limits_{x\to +\infty} \frac{1}{x} = 0.$

24. $f(x) = \begin{cases} \dfrac{e^x-1}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$

Συνεχής.

(α) $\lim\limits_{x\to 0} f(x) = f(0)$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{e^x-1}{x} = a$

$\lim\limits_{x\to 0} \dfrac{e^x}{1} = \boxed{1 = a}$

(β) $y - f(0) = f'(0)(x-0)$

$y - 1 = \frac{1}{2}x \rightarrow y = \frac{1}{2}x + 1$

• $\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{\frac{e^x-1}{x} - 1}{x}$

• $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2} = \lim\limits_{x\to 0} \dfrac{e^x-1}{2x} = \lim\limits_{x\to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

(γ) i) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{e^x - 1}{x} = 0$

ii). $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{e^x - 1}{x} =$

$= \lim\limits_{x \to +\infty} \dfrac{e^x}{1} = +\infty.$

Ενότητα 24

69. $f(x) = x + \dfrac{3x}{x^2+1}, \quad x \in R$

ⓐ $f'(x) = 1 + \dfrac{3(x^2+1) - 3x \cdot 2x}{(x^2+1)^2}$

$f'(x) = 1 + \dfrac{3x^2+3-6x^2}{(x^2+1)^2}$

$f'(x) = \dfrac{(x^2+1)^2 - 3x^2+3}{(x^2+1)^2}$

$f'(x) = \dfrac{x^4+2x^2+1-3x^2+3}{(x^2+1)^2}$

$\Delta < 0$

$f'(x) = \dfrac{x^4 - x^2 + 2}{(x^2+1)^2} > 0$

$f \nearrow$

β) $f\left(\dfrac{x^3+4x}{x^2+1}-\dfrac{5}{2}\right)=0$

$f\left(\dfrac{x^3+4x}{x^2+1}-\dfrac{5}{2}\right)=f(0)$

f 1-1

$\dfrac{x^3+4x}{x^2+1}-\dfrac{5}{2}=0$

$f(x)=\dfrac{5}{2}$

$f(x)=f(1)$

f 1-1

$x=1$

γ) νδο $f(x^2-\text{συν}x+1)>0 \quad \forall x>0$

$f(x^2-\text{συν}x+1)>f(0)$

$x^2-\text{συν}x+1>0 \quad \forall x>0$

$\oplus \quad \oplus$

(8) i) $\lim\limits_{x\to 0} \dfrac{\ln x}{f(x)} = \lim\limits_{x\to 0^+} \ln x \cdot \dfrac{1}{f(x)} =$

$= -\infty \cdot (+\infty) = -\infty$

x		0	
$f(x)$	$\nearrow$ $-$		$+$ $\nearrow$

ii) $\lim\limits_{x\to 0^+} \dfrac{\eta\mu x}{x\cdot\left(f(2x)-f(x)\right)} = \lim\limits_{x\to 0^+} \dfrac{\eta\mu x}{x} \cdot \dfrac{1}{\underset{+}{f(2x)-f(x)}}$

$\cdot\ x < 2x \Rightarrow f(x) < f(2x)$

$f(2x)-f(x) > 0$

$= 1 \cdot (+\infty) = +\infty$

70. • $f : (0, +\infty) \to \mathbb{R}$

• $f(1) = 1$

• $x^2 f'(x) + x f(x) = 1 \quad \forall x > 0.$

(a) Νδο $f(x) = \dfrac{1 + \ln x}{x}$

$$\underbrace{x f(x) - 1 - \ln x}_{g(x)} = 0$$

$$g'(x) = f(x) + x f'(x) - \frac{1}{x} = \frac{x f(x) + x^2 f'(x) - 1}{x} = 0$$

$$g(x) = c \Rightarrow x f(x) - 1 - \ln x = c$$

$\underline{\underline{x = 1}}$

$$f(1) - 1 = c$$

$$1 - 1 = c$$

$$c = 0$$

$$x f(x) - 1 - \ln x = 0$$

$$f(x) = \frac{1 + \ln x}{x}$$

(B) $$f'(x) = \frac{\frac{1}{x}\,x - (1+\ln x)}{x^2} = \frac{-\ln x}{x^2}$$

$$\rightarrow f'(x) = 0 \Rightarrow \frac{-\ln x}{x^2} = 0 \Rightarrow -\ln x = 0$$

$$\underline{\underline{x = 1}}$$

x	0		1	
f'		$+$	0	$-$
f		$\nearrow$		$\searrow$

$f(x) \le f(1)$

$$\underline{\underline{f(x) \le 1}}$$

γ) Νδο $f\left(\frac{1}{x^2}\right) < f\left(\frac{1}{x}\right)$ $\forall x \in (1, +\infty)$

$\forall x > 1$ $\quad x^2 > x \Rightarrow \frac{1}{x^2} < \frac{1}{x} \in (0,1)$

$$f\left(\frac{1}{x^2}\right) < f\left(\frac{1}{x}\right).$$

(δ) $(x^2+1)^2 = e^{x^2}\left(1 + \ln(x^2+1)\right)$

$$\frac{1}{e^{x^2}} = \frac{1+\ln(x^2+1)}{(x^2+1)^2}$$

$$\frac{x^2+1}{e^{x^2}} = f(x^2+1)$$

$$f(e^{x^2}) = f(x^2+1)$$

- $x^2 \geq 0 \Rightarrow e^{x^2} \geq e^0 \Rightarrow e^{x^2} \geq 1$
- $x^2+1 \geq 1$

$$e^{x^2} = x^2+1$$

$$x^2 = 0$$

$$x = 0.$$

Επομενο Μαθημα

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